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Homogenization error of unsteady flow ruled by Darcy's law

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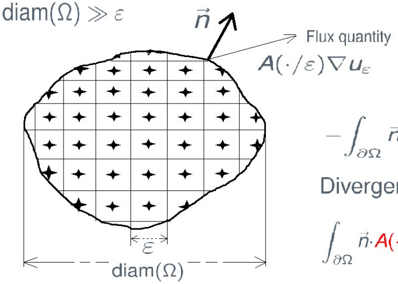
Part 1: Introduction

Introduction

——Start from elliptic homogenization

1. Strongly nonhomogeneous case (Fourier's law).

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) := -\nabla \cdot A(\cdot/\varepsilon)\nabla u_\varepsilon = F & \text{in } \Omega; \\ u_\varepsilon = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$



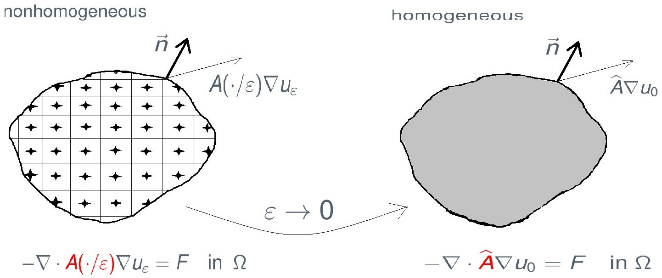
$$-\int_{\partial\Omega} \vec{n}\cdot A(\cdot/\varepsilon)\nabla u_\varepsilon dS = \int_\Omega F dx$$

Divergence theorem

$$\int_{\partial\Omega} \vec{n}\cdot A(\cdot/\varepsilon)\nabla u_\varepsilon dS = \int_\Omega \nabla\cdot A(\cdot/\varepsilon)\nabla u_\varepsilon dx$$

A. Compensated compactness (**Tartar, 1977**.) A is elliptic, periodic.

$$\begin{cases} u_\varepsilon \rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega); \\ A(\cdot/\varepsilon)\nabla u_\varepsilon \rightharpoonup \widehat{A}\nabla u_0 \quad \text{weakly in } L^2(\Omega;\mathbb{R}^d), \end{cases} \quad \text{as } \varepsilon \to 0. \tag{2}$$



B. Two-scale convergence, (**Nguetseng, 89**; **Allaire, 90**.)

How to compute

$\widehat{A}$?

$$\boxed{\widehat{A} = \fint_{Y=[0,1)^d} \left(A + A\nabla \underbrace{\Phi}_{corrector} \right) dy} \qquad (3)$$

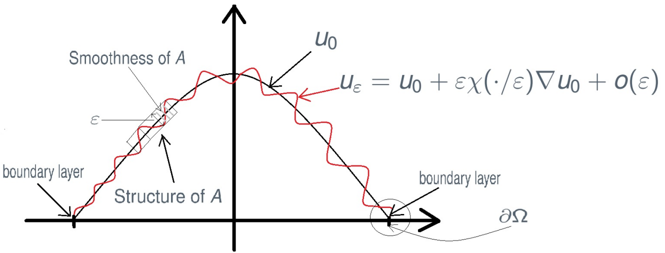
Let $x \in \Omega$ and $y = x/\varepsilon$. Let $\mathcal{L}_\varepsilon = \nabla_x A(x/\varepsilon)\nabla_x$.

$$
\begin{aligned}
u_\varepsilon(x) &= u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots ;\\
\mathcal{L}_\varepsilon &= \big\{\nabla_x + \varepsilon^{-1}\nabla_y\big\} \cdot A(y)\big\{\nabla_x + \varepsilon^{-1}\nabla_y\big\}
\end{aligned}
$$

Recall $\mathcal{L}_\varepsilon(u_\varepsilon) = F$.

$$
O(\varepsilon^{-2})A(u_0) + O(\varepsilon^{-1})B(u_0,u_1) + O(1)C(u_0,u_1,u_2) + O(\varepsilon) = F.
$$

- $A(u_0) = 0 \quad \Longrightarrow \quad u_0(x,y) = u_0(x)$.
- $B(u_0,u_1) = 0 \quad \Longrightarrow \quad \begin{cases} \nabla \cdot A(y)\nabla(\chi_k + y_k) = 0 \quad \text{in } \mathbb{R}^d; \\ \qquad\qquad u_1(x,y) = \chi(y)\cdot\nabla u_0(x). \end{cases}$
- $C(u_0,u_1,u_2) = F \quad \Longrightarrow \quad \widehat{A} = \fint_Y \big(A + A\nabla\chi\big)dy$.



Fix a **bounded** domain Ω, and $\chi \in H^1_{\text{per}}(Y)$.

$$
\begin{aligned}
\|\chi(\cdot/\varepsilon)\nabla u_0\|_{H^{1/2}(\partial\Omega)} &\lesssim^{\text{trace theorem}} \|\chi(\cdot/\varepsilon)\nabla u_0\|^{1/2}_{L^2(\Omega)}\|\chi(\cdot/\varepsilon)\nabla u_0\|^{1/2}_{H^1(\Omega)} \\
&= O(\varepsilon^{-\frac{1}{2}});
\end{aligned}
$$

This gives

$$
\varepsilon\|\sum_{k=1}^{d}\chi_k(\cdot/\varepsilon)\partial_k u_0\|_{H^{1/2}(\partial\Omega)} = O(\varepsilon^{1/2}). \tag{4}
$$

A. Large-scale estimates (“quenched” type estimates).

- Compactness methods (**Avellaneda-Lin, 1987**).
- Campanato’s iteration + homogenization error
  - **Armstrong-Kuusi-Mourrat-Shen, 2014**
  - **Gloria-Neukamm-Otto, 2014**

B. Homogenization error.

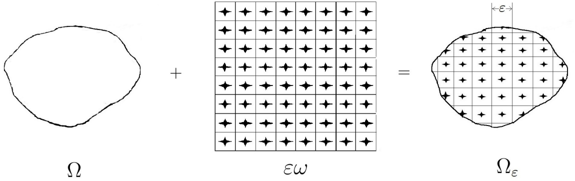
- algebra formula (flux corrector, **Jikov-Kozlov-Oleinik**, 90s)
- duality (seek for a sharp error) (**Aubin-Nitsche’s methods, 1960s; Kenig-Lin-Shen-Suslina, 2012; Shen-Xu, 2016**)

C. Boundary layers (higher-order expansions’s error).

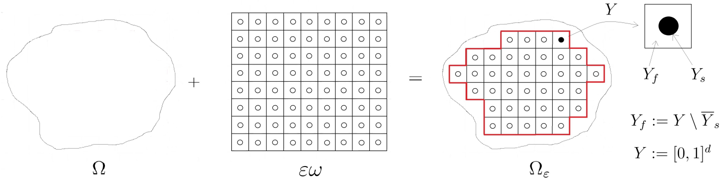
- **Gérard-Varet-Masmoudi, 2012.**
- **Armstrong-Kuusi-Mourrat-Prange, 2017.**
- **Shen-Zhuge, 2018.**

Bella-Duerinckx-Fisher-Giunti-Otto, 2016-now for higher-order expansions.

- A perforated domain Ω_ε of type (I) domains.



- A perforated domain Ω_ε of type (II) domains.



[1] Oleinik-Shamaev-Yosifian: Mathematical Problems in Elasticity and Homogenization. Studies in Mathematics and its Applications, (1992).

1. **Assumption:**
   - Let Ω_ε be the perforated domain of type (II) with smooth boundary;
   - $0 < \varepsilon \ll 1$ and $0 < T \leq \infty$.
2. **Equations:**

$$\begin{cases} \dfrac{\partial u_\varepsilon}{\partial t} - \varepsilon^2 \mu \Delta u_\varepsilon + \nabla p_\varepsilon = f & \text{in} \quad \Omega_\varepsilon \times (0, T]; \\ \nabla \cdot u_\varepsilon = 0 & \text{in} \quad \Omega_\varepsilon \times (0, T]; \\ u_\varepsilon = 0 & \text{on} \quad \partial\Omega_\varepsilon \times (0, T]; \\ u_\varepsilon\big|_{t=0} = 0 & \text{on} \quad \Omega_\varepsilon, \end{cases} \tag{5}$$

   - Velocity: u_ε;
   - Pressure: p_ε;
   - Density of forces: f;
   - Viscosity: μ.

Remark: assume $\mu = 1$ throughout for simplicity.

Theorem 1 (Lions, 81; Allaire-Mikelić, 90s)

Let $0 < T \leq \infty$*. There exists an extension* $(\tilde{u}_\varepsilon, \tilde{p}_\varepsilon)$ *of the solution* $(u_\varepsilon, p_\varepsilon)$*, which* *weakly converges* *in* $L^2(0,T;L^2(\Omega)^d) \times L^2(0,T;L^2(\Omega)/\mathbb{R})$ *to the unique solution* (u_0, p_0) *of the homogenized problem*[a]*:*

$$\begin{cases} u_0(x,t) = \int_0^t ds A(s)(f - \nabla p_0)(x, t-s) & \text{in} \quad \Omega \times (0,T); \\ \nabla \cdot u_0 = 0 & \text{in} \quad \Omega \times (0,T); \\ u_0 \cdot \vec{n} = 0 & \text{on} \quad \partial\Omega \times (0,T), \end{cases} \tag{6}$$

where $\vec{n}$ *is the unit outward normal vector of* $\partial\Omega$*. Moreover, there holds*

$$\int_0^T dt \Big\| \tilde{u}_\varepsilon(\cdot,t) - \int_0^t ds W(\cdot/\varepsilon, t-s)(f - \nabla p_0)(\cdot, s) \Big\|^2_{L^2(\Omega)} \to 0, \tag{7}$$

as ε *goes to zero.*

[a] The homogenized problem (6) is referred to as Darcy's law with memory.

1. **Correctors:**[1]

$$\begin{cases} \dfrac{\partial W_j}{\partial t} - \Delta W_j + \nabla \pi_j = 0 & \text{in} \quad \omega \times (0,\infty); \\ \nabla \cdot W_j = 0 & \text{in} \quad \omega \times (0,\infty); \\ W_j = 0 & \text{on} \quad \partial\omega \times (0,\infty), \\ W_j|_{t=0} = e_j & \text{on} \quad \omega. \end{cases} \tag{8}$$

where $e_i = (0, \cdots, 1, \cdots, 0)$ with 1 in the i^{th} place.

2. **Permeability:**

$$A_{ij}(t) = \int_{Y_f} dy W_j(y,t) \cdot e_i, \tag{9}$$

- symmetry;
- positive definite;
- exponential decay.

[1] We take Mikelić's definition, which slightly differs from that given by Allarie.

1. **Homogenization error:**

$$\|u_\varepsilon - W(\cdot/\varepsilon) * (f - \nabla p_0)\|_{L^2(\Omega_{\varepsilon,T})} + \|\varepsilon\nabla u_\varepsilon - \nabla W(\cdot/\varepsilon) * (f - \nabla p_0)\|_{L^2(\Omega_{\varepsilon,T})} + \|p_\varepsilon - p_0 - c\|_{L^2(\Omega_{\varepsilon,T})} \leq C\varepsilon^{1/2}\|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}; \tag{10}$$

2. **Large-scale regularity estimates:** for any $0 < \beta < \alpha$, there holds

$$\mathrm{Exc}_r(u_\varepsilon, f) \lesssim \left(\frac{r}{R}\right)^{2\beta}\mathrm{Exc}_R(u_\varepsilon, f) \tag{11}$$

for any $\varepsilon \leq r < R$, where

$$\mathrm{Exc}_R(u_\varepsilon, f) := \inf_{E\in\mathbb{R}^d}\int_0^T \Big(\fint_{Q_R^\varepsilon} |\varepsilon\nabla u_\varepsilon - (\nabla W)^\varepsilon * E|^2 + R^{2\alpha}\|f\|^2_{C^{0,\alpha}(Q_R)}\Big).$$

$Q_R^\varepsilon := Q_R \cap \varepsilon\omega$ and $Q_R \subset \mathbb{R}^d$ is a cube.
The notation "$*$" represents revolution w.r.t. time variable.

- Allaire's definition of correctors:

$$\begin{cases} \dfrac{\partial w_j}{\partial t} - \Delta w_j + \nabla \tilde{\pi}_j = e_j & \text{in} \quad \omega \times (0,\infty); \\ \nabla \cdot w_j = 0 & \text{in} \quad \omega \times (0,\infty); \\ w_j = 0 & \text{on} \quad \partial\omega \times (0,\infty); \\ w_j|_{t=0} = 0 & \text{in} \quad \omega. \end{cases} \tag{12}$$

which is connected to Mikelić's by the relationship

$$\partial_t w_j(\cdot,t) = W_j(\cdot,t)$$

in the sense of Stokes semigroup representation, up to a projection.

$$A_{ij}(t) = \int_{Y_f} dy \partial_t w_j(y,t) \cdot e_i.$$

- If changing the initial data of Mikelić's definition into $e_j - \nabla b_j$[2], its solution is no different[3] from W_j in the sense of Stokes semigroup representation, where $\Delta b_j = 0$ in Y_f with $\partial b_j / \partial \nu = n_j$ on ∂Y_s.

[2] This is Sandrakov's way.

[3] But, it will change the value of defined permeability at zero.

- As pointed out by Allaire in stationary cases, **the ratio of the solid obstacle size to the periodic repetition one is crucial** when people study the limit behavior for such problems, and different ratios will lead to Darcy's law, Brinkman's law and Stokes' law, separately.

- In terms of the scaling ε^2 of the viscosity in the equations (5), it is not a simple change of variable as in the stationary case because the density in front of the inertial term has been scaled to 1. **The present scaling in** (5) **will precisely lead to a limit problem depending on time in a nonlocal manner**, which is the critical case (see Sandrakov's work, 97).

- Recently, Shen provides an optimal error estimate (11) for steady Stokes systems on perforated domains. Instead, the present work avoids using boundary correctors defined in his way, since any useful nontangential maximal function estimates employed by Shen turns to be very difficult for unsteady Stokes systems[4].

[4] It is known from Shen that the Rellich type estimates involve pressure terms in a surprising way, and it will lead us to some uneasy estimates on pressures.

Part 2: Nonstandard Expansion

Nonstandard Expansion

——standard expansion & difficulties

- Formal two-scale expansion:

$$\begin{cases} u_\varepsilon = u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \cdots ; \\ p_\varepsilon = p_0(x,y) + \varepsilon p_1(x,y) + \varepsilon^2 p_2(x,y) + \cdots , \end{cases} \qquad y = x/\varepsilon.$$

The known results:

$$u_0(x,y,t) = \int_0^t ds W(y,s) \underbrace{(f - \nabla p_0)}_{F}(x,t-s); \quad p_0(x,y) = p_0(x).$$

- Recall the qualitative result (7), i.e.,

$$\int_0^T \|u_\varepsilon - u_0\|^2_{L^2(\Omega_\varepsilon)} \longrightarrow 0, \quad \text{as} \quad \varepsilon \to 0,$$

which suggests that the error term should be the form of

$$w_\varepsilon^{(1)} = u_\varepsilon - W^\varepsilon * F. \tag{13}$$

- **Difficulties**: (inhomogeneous conditions, i.e.,)

$$\begin{cases} \nabla \cdot w_\varepsilon^{(1)} = -W^\varepsilon *_2 \partial F & \text{in } \Omega_\varepsilon \times (0,T); \\ \qquad w_\varepsilon^{(1)} = -W^\varepsilon * F & \text{on } \partial\Omega \times (0,T). \end{cases}$$

Nonstandard Expansion

——notations

1. ψ_ε is **radial type cut-off function**, while φ_ε is a general cut-off one.
2. Notation for different type correctors:
   - Corrector: (W, π);
   - Flux corrector: Φ;
   - Corrector of Bogovskii's operator: ϕ;
   - Boundary-layer correctors: (ξ, η).
3. Notation for convolutions (involving Einstein's summation convention):

$$a *_1 b(t) := \int_0^t ds a(t-s)\cdot b(s); \qquad A * b(t) := \int_0^t ds A(t-s) b(s);$$

$$C *_2 A(t) := \int_0^t ds C(t-s) : A(s); \qquad C *_3 D(t) := \int_0^t ds C_{ijk}(t-s) D_{ijk}(s),$$

4. Notation for important quantities:

$$J_1 := \nabla\psi_\varepsilon \cdot \big[(W^\varepsilon - A) * G\big];$$

$$J_2 := \nabla\psi_\varepsilon \cdot (A * G) + \varepsilon\nabla\psi_\varepsilon \cdot (\phi^\varepsilon *_2 \partial G) + \psi_\varepsilon \frac{A}{|Y_f|} *_2 \partial G + \varepsilon\psi_\varepsilon\phi^\varepsilon *_3 \partial^2 G,$$

where $G := S_{\frac{\varepsilon}{2}}(\varphi_\varepsilon F)$ with S_δ being a smoothing operator and $F := f - \nabla p_0$.

Let $\phi^\varepsilon(x,t) := \phi(\frac{x}{\varepsilon}, t)$; $W^\varepsilon(x,t) := W(\frac{x}{\varepsilon}, t)$; and $\pi^\varepsilon(x,t) := \pi(\frac{x}{\varepsilon}, t)$.

Nonstandard Expansion

1. **Velocity terms:**

$$w_\varepsilon = u_\varepsilon - \psi_\varepsilon \underbrace{W^\varepsilon * G}_{\text{zero order}} + \underbrace{\hat{\xi} + \hat{\eta}}_{\text{involving boundary-layer}} - \varepsilon\psi_\varepsilon \underbrace{\phi^\varepsilon *_2 \partial G}_{\text{first order}}$$

where

$$(\hat{\xi}, \hat{\eta})(\cdot, t) := \int_0^t ds (\xi, \eta)(\cdot, s), \tag{14}$$

and corrector of Bogovskii's operator[5] is given by (for any $t \geq 0$)

$$\begin{cases} \nabla \cdot \phi_{i,j}(\cdot, t) = -W_{ij}(\cdot, t) + |Y_f|^{-1} A_{ij}(t) & \text{in} \quad \omega; \\ \phi_{i,j}(\cdot, t) = 0 & \text{on} \quad \partial\omega. \end{cases} \tag{15}$$

2. Pressure terms:

$$q_\varepsilon = p_\varepsilon - p_0 - \varepsilon\psi_\varepsilon \pi^\varepsilon *_1 \partial G$$

[5] To the authors' best knowledge, this type corrector was originally imposed by E. Marušić-Paloka, A. Mikelić for stationary cases.

1. Why do we introduce radial type cut-off function ψ_ε?
   - Make the inhomogeneous boundary condition be homogeneous;
   - Structural interest:

$$\nabla\psi_\varepsilon = -|\nabla\psi_\varepsilon|\vec{n}(\tilde{x}) \quad \text{on } O_\varepsilon, \tag{16}$$

   where $O_\varepsilon := \mathrm{supp}\nabla\psi_\varepsilon$, and $\tilde{x} : O_\varepsilon \to \partial\Omega$. By virtue of $\vec{n} \cdot u_0 = 0$ on $\partial\Omega$, there holds

$$\|\nabla\psi_\varepsilon \cdot u_0\|_{L^2(\Omega)} = O(1) \tag{17}$$

2. Why do we impose the corrector of Bogovskii's operator ϕ?
   - It plays an important role in the following improvement

$$W^\varepsilon *_2 \partial G \quad \rightsquigarrow \quad \frac{A}{|Y_f|} *_2 \partial G = \frac{A}{|Y_f|} *_2 \partial(G - F)$$

   by using $A *_2 \partial F = \nabla \cdot u_0 = 0$ in Ω.

3. Why do we impose boundary-layer correctors associated with Bogovskii's operator?

- Let $w_\varepsilon^{(2)} = u_\varepsilon - \psi_\varepsilon(W^\varepsilon * G + \varepsilon\phi^\varepsilon *_2 \partial G)$. There holds

$$\begin{cases} -\nabla\cdot w_\varepsilon^{(2)} = J_1 + J_2 & \text{in } \Omega_\varepsilon \times (0,T); \\ w_\varepsilon^{(2)} = 0 & \text{on } \partial\Omega_\varepsilon \times (0,T), \end{cases} \tag{18}$$

- **The crucial idea** is to introduce a "magical" quantity

$$\sum_i \Big(\fint_{O_\varepsilon^i} J_1\Big) 1_{O_\varepsilon^i},$$

where $\operatorname{supp}(J_1) = O_\varepsilon = \sum_i O_\varepsilon^i$, and $1_{O_\varepsilon^i}$ is the indicator function, such that

$$J_1 + J_2 = \underbrace{J_1 - \sum_i \Big(\fint_{O_\varepsilon^i} J_1\Big) 1_{O_\varepsilon^i}}_{\Pi} + \underbrace{\sum_i \Big(\fint_{O_\varepsilon^i} J_1\Big) 1_{O_\varepsilon^i} + J_2}_{\mathcal{H}}.$$

- Construct solutions (ξ, η) to

$$(1)\begin{cases} \nabla\cdot\xi = \partial_t\mathcal{H} & \text{in } \Omega_\varepsilon; \\ \eta = 0 & \text{in } \partial\Omega_\varepsilon, \end{cases} \quad \text{and} \quad (2)\begin{cases} \nabla\cdot\eta = \partial_t\Pi & \text{in } O_\varepsilon; \\ \eta = 0 & \text{in } \partial O_\varepsilon. \end{cases} \tag{19}$$

Thus, $w_\varepsilon = w_\varepsilon^{(2)} + \hat{\xi} + \hat{\eta}$ leads to all homogeneous conditions.

Part 3: Outline of Proofs

Step 1. Find the formula on the first-order expansion, i.e.,

$$\begin{cases} w_\varepsilon = u_\varepsilon - \psi_\varepsilon \big(W^\varepsilon * G + \varepsilon \phi^\varepsilon *_2 \partial G \big) + \hat{\xi} + \hat{\eta}; \\ q_\varepsilon = p_\varepsilon - p_0 - \varepsilon \psi_\varepsilon \pi^\varepsilon *_1 G. \end{cases} \tag{20}$$

Step 2. Derive the equations that error term $(w_\varepsilon, q_\varepsilon)$ satisfies, i.e.,

$$\begin{cases} \dfrac{\partial w_\varepsilon}{\partial t} - \varepsilon^2 \Delta w_\varepsilon + \nabla q_\varepsilon = I_1 + \varepsilon I_2 + \varepsilon^2 I_3 + \varepsilon^3 I_4, & \text{in} \quad \Omega_\varepsilon \times (0, T); \\ \nabla \cdot w_\varepsilon = 0, & \text{in} \quad \Omega_\varepsilon \times (0, T) \end{cases} \tag{21}$$

with zero initial-boundary data.

Step 3. Energy estimates. Rewriting the right-hand side of (21) as the form of

$$\Theta + \varepsilon \nabla \cdot \Lambda + \varepsilon \psi_\varepsilon \nabla \cdot \Xi,$$

we have

$$\|w_\varepsilon\|_{L^\infty(0,T;L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla w_\varepsilon\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \lesssim \|(\Theta, \Lambda, \psi_\varepsilon \Xi)\|_{L^2(0,T;L^2(\Omega_\varepsilon))}.$$

Then, reduce the error estimates to show:

$$\boxed{\|(\Theta, \Lambda, \psi_\varepsilon \Xi)\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = O(\varepsilon^{1/2}).} \tag{22}$$

Step 4. The desired estimate (22) relies on three type estimates:

- Smoothness of (flux) correctors, i.e.,

$$\|(\partial_t W_j, \nabla\pi_j)\|_{L^1(0,T;L^r(Y_f))} + \|(\partial_t\Phi, \partial_t\phi)\|_{L^1(0,T;W^{1,q}(Y_f))} \lesssim 1; \tag{23}$$

- Well-posedness of the homogenized system, i.e.,

$$\|p_0\|_{L^q(0,T;C^{m+1,\alpha}(\bar{\Omega}))} \lesssim \|f\|_{L^q(0,T;C^{m,\alpha}(\bar{\Omega}))}; \tag{24}$$

- Regularity estimates on boundary-layer correctors, i.e.,

$$\|(\xi,\eta)\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon\|(\nabla\widehat{\xi}, \nabla\widehat{\eta})\|_{L^2(0,T;L^2(\Omega_\varepsilon))} = O(\varepsilon^{1/2}). \tag{25}$$

Step 5. Show the estimate on the inertial term

$$\|\partial_t w_\varepsilon\|_{L^2(\Omega_{\varepsilon,T})} = O(\varepsilon^{1/2}). \tag{26}$$

Step 6. Show the estimate on the pressure term

$$\inf_{c\in\mathbb{R}} \|q_\varepsilon - c\|_{L^2(\Omega_{\varepsilon,T})} = O(\varepsilon^{1/2}).$$

which base upon the estimates (22) and (26), as well as, a duality argument.

Outline of Proofs

——main difficulties:

(D1). If created by a simple cut-off argument, boundary layers would easily "destroy" the desired estimate because of the incompressibility condition[6]

(D2). Concerning correctors' estimates (23), we are required to derive a refined regularity estimate for correctors (W, π) **without compatibility condition** between the initial and boundary data;

(D3). Regarding effective solution's estimate (24), we have to establish well-posedness of the integro-differential equation (6) in general Bochner spaces[7].

[6] It was pointed out by Marušić-Paloka, Mikelić in their cooperated job.

[7] In terms of Hilbert space, the existence had been well established by Lions, Mikelić and Sandrakov through different methods.

Part 4: Correctors' estimates

Proposition 1 (Wang-Xu-Zhang, 22)

Let $0 < T < \infty$ *and* $1 < q < \infty$. *Then,*

$$\|W_j\|_{L^\infty(0,T;L^2(Y_f))} + \|W_j\|_{L^2(0,T;H^1(Y_f))} \lesssim 1, \tag{27}$$

Also, for any $\alpha > (1/3)$, *we have higher regularity estimates:*

$$\begin{aligned} &\|W_j\|_{L^2(0,T;W^{1,q}(Y_f))} + \|t^\alpha \partial_t W_j\|_{L^2(0,T;L^2(Y_f))} + \|t^\alpha \pi_j\|_{L^2(0,T;L^2(Y_f)/\mathbb{R})} \lesssim 1; \\ &\|\partial_t W_j\|_{L^1(0,T;L^q(Y_f))} + \|\nabla^2 W_j\|_{L^1(0,T;L^q(Y_f))} + \|\nabla \pi_j\|_{L^1(0,T;L^q(Y_f))} \lesssim 1, \end{aligned} \tag{28}$$

Moreover, let $\tilde{W}_j$ *be the zero-extension of* W_j. *For any* $t \in [0,T]$, *define* $b_{ij}(\cdot,t) := \tilde{W}_j(\cdot,t)\cdot e_i - A_{ij}(t)$. *Then, there exists* $\Phi_{ki,j} \in H^1_{per}(Y)$ *such that*

$$\nabla_k \Phi_{ki,j} = b_{ij} \quad \text{and} \quad \Phi_{ki,j} = -\Phi_{ik,j}, \tag{29}$$

as well as, the following regularity estimates:

$$\begin{aligned} &\|\partial_t \Phi\|_{L^1(0,T;H^1(Y))} + \|\Phi\|_{L^1(0,T;H^1(Y))} + \|\Phi(\cdot,0)\|_{H^1(Y)} \lesssim 1; \\ &\|\partial_t \Phi\|_{L^1(0,T;W^{1,q}(Y))} + \|\Phi\|_{L^1(0,T;W^{1,q}(Y))} \lesssim 1. \end{aligned} \tag{30}$$

Consequently, we also have $\Phi_{ki,j}, \partial_t \Phi_{ki,j} \in L^1(0,T;C_{per}(Y))$.

Correctors' estimates

——main ingredients:

Lemma 2 (semigroup estimate I)

Let $2 < p < \infty$. Then, for any $t > 0$, there holds

$$\Big(\int_{Y_f} |\nabla W_j(\cdot, t)|^2\Big)^{1/2} \leq C_p t^{-\frac{p}{3p-2}}, \tag{31}$$

in which the constant C_p depends on d, p and the character of Y_f.

Lemma 3 (semigroup estimate II)

Let $p \geq 2$ be sufficiently large, and $\gamma := \frac{p(19p-14)}{(3p-2)(7p-2)}$. Then, for any $r \in (1, \infty)$, there exists a constant $q \in (1, \infty)$, such that $\lambda := \frac{2(q-r)}{r(q-2)}$ satisfying $0 \leq \lambda \leq 1$, and there holds

$$\Big(\int_{Y_f} |\partial_t W_j(\cdot, t)|^r\Big)^{1/r} \lesssim t^{-1+\lambda(1-\gamma)} \tag{32}$$

for any $t > 0$, where the multiplicative constant depends on d, p, q and the character of Y_f.

Correctors' estimates

——semigroup estimate I

The key observation on the estimate (31):

- Caccioppoli type inequality offers a good decay in the interior region, but produces a bad scale factor.
- The semigroup estimate can dominate the region near boundary, owning a relatively bad decay, but creating a good scale factor.

The idea is to bring in a parameter ρ to balance their advantage and disadvantage such that we can "improve" the decay power of semigroup estimates.

Step 1. Decompose the integral domain into two parts: $(Y_f)_\rho$ and $Y_f \setminus (Y_f)_\rho$, where $(Y_f)_\rho := \{y \in Y_f : \mathrm{dist}(y, \partial Y_f) \geq 2\rho\}$ for the parameter $\rho > 0$, which will be fixed later.

$$\begin{aligned}\Big(\int_{Y_f} |\nabla W(\cdot,t)|^2\Big)^{\frac{1}{2}} &\lesssim \Big(\int_{Y_f\setminus(Y_f)_\rho} |\nabla W(\cdot,t)|^2\Big)^{\frac{1}{2}} + \Big(\int_{(Y_f)_\rho} |\nabla W(\cdot,t)|^2\Big)^{\frac{1}{2}} \\ &=: I_1 + I_2.\end{aligned} \tag{33}$$

Step 2. Classical semigroup estimates $+$ Hölder's inequality:

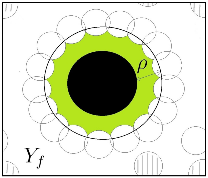
$$I_1 \lesssim \rho^{\frac{1}{2}-\frac{1}{p}}\Big(\int_{Y_f} |\nabla W(\cdot,t)|^p\Big)^{1/p} \lesssim \rho^{\frac{1}{2}-\frac{1}{p}} t^{-\frac{1}{2}}. \tag{34}$$

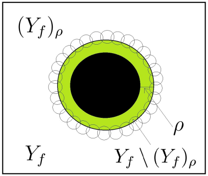
Step 3. Estimate I_2. We start from giving a family of cut-off functions, denoted by $\{\chi_i\}$, which satisfy that $\chi_i(y) = \chi_0(y + y_i)$ and $(Y_f)_\rho \subset \bigcup_i B(y_i, \rho/2) \subset (Y_f)_{\frac{1}{2}\rho}$. It is fine to assume that $y_0 = 0$, and

- $\chi_0 \in C^1_{per}(Y)$ is a cut-off function;
- $\chi_0 = 1$ on $B(0, \rho/2)$ and $\text{supp}(\chi_0) \subset B(0, \rho)$ with $|\nabla \chi_0| \lesssim \frac{1}{\rho}$;
- If $|i - j| > 2$, $\text{supp}\{\chi_i\} \bigcap \text{supp}\{\chi_j\} = \emptyset$.

From the assumptions on $\{\chi_i\}$, it follows that $\text{dist}(\partial Y_s, \text{supp}\chi_i) \geq \rho$. Moreover, we define a family of indicator functions associated with $\{\chi_i\}$ as follows:

$$\tilde{\chi}_i = 1 \text{ in } B(y_i, \rho) \quad \text{and} \quad \tilde{\chi}_i = 0 \text{ outside } B(y_i, \rho).$$





We claim that

$$I_2 \leq \Big(\sum_i \int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 \Big)^{1/2} \lesssim \frac{1}{\rho}. \tag{35}$$

Caccioppoli type estimates: for each i and any $t > 0$, there hold

$$\int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 \lesssim \int_{Y_f} \chi_i^2 |\nabla \times W(\cdot, t)|^2 + \frac{1}{\rho^2} \int_{Y_f} \tilde{\chi}_i |W(\cdot, t)|^2; \tag{36}$$

and

$$\int_{Y_f} \chi_i^2 |\nabla \times W(\cdot, t)|^2 \lesssim \frac{1}{\rho^2} \int_0^T \int_{Y_f} \tilde{\chi}_i |\nabla W|^2. \tag{37}$$

Step 4. Admitting the claims (36), (37) for a moment,

$$\int_{Y_f} \chi_i^2 |\nabla W(\cdot, t)|^2 dy \lesssim \frac{1}{\rho^2} \Big\{ \int_0^T \int_{Y_f} \tilde{\chi}_i |\nabla W|^2 + \sup_{0 \le t \le T} \int_{Y_f} \tilde{\chi}_i |W(\cdot, t)|^2 \Big\}.$$

Thus,

$$I_2 \lesssim \frac{1}{\rho} \Big(\int_0^T \int_{Y_f} |\nabla W|^2 + \sup_{0 \le t \le T} \int_{Y_f} |W(\cdot, t)|^2 \Big)^{1/2} \lesssim^{(27)} \frac{1}{\rho}. \tag{38}$$

As a result, plugging (34) and (38) back into (33), one can acquire

$$\Big(\int_{Y_f} |\nabla W(\cdot, t)|^2 \Big)^{1/2} \lesssim \rho^{\frac{1}{2} - \frac{1}{p}} t^{-\frac{1}{2}} + \rho^{-1} \lesssim t^{-\frac{p}{3p-2}},$$

where we take $\rho = t^{\frac{p}{3p-2}}$ (which requires t to be small).

Step 3-a. To show

$$\int_{Y_f} \chi_i^2 |\nabla \times W(\cdot,t)|^2 \lesssim \frac{1}{\rho^2} \int_0^T \int_{Y_f} \tilde{\chi}_i |\nabla W|^2,$$

we ask for test function to be the form of

$$\nabla \times (\chi_i^2 \nabla \times W),$$

and employ

$$\begin{cases} \nabla \cdot (\nabla \times (\chi_i^2 \nabla \times W)) = 0 \\ \nabla \times (\chi_i^2 \nabla \times W) = \nabla \chi_i^2 \times (\nabla \times W) + \chi_i^2 \nabla \times \nabla \times W \\ \nabla \times \nabla \times W = -\Delta W + \nabla(\nabla \cdot W) = -\Delta W \end{cases} \tag{39}$$

Step 3-b. To show

$$\int_{Y_f} \chi_i^2 |\nabla W(\cdot,t)|^2 \lesssim \int_{Y_f} \chi_i^2 |\nabla \times W(\cdot,t)|^2 + \frac{1}{\rho^2} \int_{Y_f} \tilde{\chi}_i |W(\cdot,t)|^2;$$

we can go back the last two line of (39).

Correctors' estimates

——semigroup estimate II

The key ingredient:

$$\Big(\int_{Y_f} dy|\Delta W(y,t)|^2\Big)^{1/2} \lesssim t^{-\gamma}, \tag{40}$$

with $\gamma := \frac{p(19p-14)}{(3p-2)(7p-2)}$. This implies

$$\|\mathcal{A}W(\cdot,t)\|_{L^2(Y_f)} = \|P(-\Delta)W(\cdot,t)\|_{L^2(Y_f)} \leq \|\Delta W(\cdot,t)\|_{L^2(Y_f)} \lesssim t^{-\gamma},$$

which together with the semigroup estimates: for $1<q<\infty$,

$$\|\mathcal{A}W(\cdot,t)\|_{L^q(Y_f)} \lesssim t^{-1}, \quad \text{and} \quad \partial_t W = \mathcal{A}W,$$

leads to the desired estimate (32).

Arguments for (40)**:**

$$\Big(\int_{Y_f}|\Delta W(\cdot,t)|^2\Big)^{1/2} \leq \Big(\int_{(Y_f)_\rho}|\Delta W(\cdot,t)|^2\Big)^{1/2} + \Big(\int_{Y_f\setminus(Y_f)_\rho}|\Delta W(\cdot,t)|^2\Big)^{1/2}.$$

- Interior estimates (stream function methods + Caccioppoli type estimates):

$$\Big(\int_{(Y_f)_\rho}|\nabla^2 W(y,t)|^2 dy\Big)^{1/2} \lesssim \rho^{-3} t^{-\frac{p}{3p-2}}, \tag{41}$$

- Hölder inequality + L^p-estimates + semigroup estimates:

$$\Big(\int_{Y_f\setminus(Y_f)_\rho} dy|\Delta W(y,t)|^2\Big)^{1/2} \lesssim \rho^{\frac{1}{2}-\frac{1}{p}} t^{-1}. \tag{42}$$

Proposition 2 (corrector of Bogovskii's operator)

Let $2 \leq q < \infty$. Then, there exists at least one weak solution ϕ associated with W and A by

$$\begin{cases} \nabla \cdot \phi_{i,j}(\cdot, t) = -W_{ij}(\cdot, t) + |Y_f|^{-1} A_{ij}(t) & \text{in} \quad \omega; \\ \quad \phi_{i,j}(\cdot, t) = 0 & \text{on} \quad \partial\omega \end{cases}$$

with $t \geq 0$, whose component is 1-periodic and satisfies $\phi_{ki,j}(\cdot, t) \in H^1_{per}(Y_f)$. Moreover, there holds

$$\|\phi\|_{L^1(0,T;W^{1,q}(Y_f))} + \|\phi(\cdot, 0)\|_{L^q(Y_f)} + \|\partial_t \phi\|_{L^1(0,T;W^{1,q}(Y_f))} \lesssim 1, \tag{43}$$

and we concludes that $\partial_t \phi_{ki,j} \in L^1(0, T; C_{per}(Y_f))$.

Lemma 4 (antisymmetry and regularities)

For any $0 < t < T$*, let* $B(\cdot,t) = \{b_{ij}(\cdot,t)\}_{1\leq i,j\leq d}$*. Then,*

$$\text{(i)}\ \ \nabla\cdot B(\cdot,t) = 0, \qquad \text{(ii)}\ \ \int_Y B(\cdot,t) = 0. \tag{44}$$

Moreover, there exists $\Phi(\cdot,t) = \{\Phi_{ki,j}(\cdot,t)\}_{1\leq i,j,k\leq d}$ *with* $\Phi_{ki,j}(\cdot,t) \in H^1_{loc}(\mathbb{R}^d)$ *being 1-periodic, and satisfying*

$$\nabla\cdot\Phi(\cdot,t) = B(\cdot,t) \quad \text{in} \quad Y \tag{45}$$

under the antisymmetry condition (i.e., $\Phi_{ki,j} = -\Phi_{ik,j}$*). Also, for any* $1 < q < \infty$*, we have the regularity estimates:*

$$\|\Phi(\cdot,t)\|_{W^{1,q}(Y)} \lesssim \|B(\cdot,t)\|_{L^q(Y)}, \tag{46}$$

where the up to constant is independent of t*.*

Part 5: Effective equations

Proposition 3 (well-posedness)

Let $1 \leq q \leq \infty$, $m \geq 1$ *and* $\alpha \in (0,1)$. *Given* $0 < T < \infty$, *suppose* $f \in L^q(0,T;C^{m,\alpha}(\bar{\Omega})^d)$ *and* $\partial\Omega \in C^{m+1,\alpha}$. *Then, there exists a unique* $p_0 \in L^q(0,T;C^{m+1,\alpha}(\bar{\Omega}))$ *to the integral-differential equations* (6) *with the condition* $\int_\Omega p_0(\cdot,t) = 0$ *for a.e.* $t \geq 0$. *Moreover, we have*

$$\|p_0\|_{L^q(0,T;C^{m+1,\alpha}(\bar{\Omega}))} \leq C\|f\|_{L^q(0,T;C^{m,\alpha}(\bar{\Omega}))}, \tag{47}$$

where the constant C *depends only* Ω, Y_f *and* T.

Remark 4.1

In terms of temporal variable, it is ***hard*** *to improve the temporal regularity in* (47). *However, if replacing Hölder's norm by Sobolev norm, we have*

$$\|p_0\|_{L^q(0,T;H^{m+1}(\Omega))} \leq C\|f\|_{L^q(0,T;H^m(\Omega))}, \tag{48}$$

with $m \geq 0$, *where we regard* $H^0(\Omega)$ *as* $L^2(\Omega)$.

Lemma 5 (short-time solution)

Assume the same conditions as in Proposition 3. There exists $0 < \delta_0 \ll 1$, depending on $\|\partial_t A\|_{L^1(0,T)}$, such that the integral-differential equation (6) possesses the solution $p_0 \in L^q(0,\delta_0; C^{m+1,\alpha}(\bar{\Omega}))$, satisfying the estimate

$$\|p_0\|_{L^q(0,\delta_0;C^{m+1,\alpha}(\bar{\Omega}))} \lesssim \|f\|_{L^q(0,\delta_0;C^{m,\alpha}(\bar{\Omega}))}, \tag{49}$$

where the up to constant depends only on Ω.

Step 1. Taking $\nabla\cdot$ and ∂_t,

$$\begin{cases} \nabla\cdot A(0)\nabla p_0 + \nabla\cdot A' * \nabla p_0 = \nabla\cdot\big[A' * f + A(0)f\big] & \text{in } \Omega; \\ \vec{n}\cdot A(0)\nabla p_0 + \vec{n}\cdot A' * \nabla p_0 = \vec{n}\cdot\big[A' * f + A(0)f\big] & \text{on } \partial\Omega. \end{cases} \tag{50}$$

Step 2. Introduce $\hat{p}$:

$$\begin{cases} \nabla\cdot\big[A(0)\nabla\hat{p} - A' * \nabla p\big] = 0, & \text{in } \Omega; \\ \vec{n}\cdot\big[A(0)\nabla\hat{p} - A' * \nabla p\big] = 0, & \text{on } \partial\Omega, \end{cases} \quad \int_\Omega \hat{p}(\cdot,t) = 0. \tag{51}$$

and set $\mathcal{L} := \nabla\cdot A(0)\nabla; \quad K_1(p) := \mathcal{L}^{-1}\nabla\cdot A' * \nabla p.$

$$\hat{p} = K_1(p); \quad \Longrightarrow \quad \mathcal{L}\hat{p} = \mathcal{L}(K_1(p)) = \nabla\cdot A' * \nabla p. \tag{52}$$

Step 3. Replacing $\nabla \cdot A' * \nabla p_0$,

$$\begin{cases} \mathcal{L}(p_0 + K_1(p_0)) = \nabla \cdot \left[A' * f + A(0)f\right] \quad \text{in } \Omega; \\ \vec{n} \cdot A(0)\nabla(p_0 + K_1(p_0)) = \vec{n} \cdot \left[A' * f + A(0)f\right] \quad \text{on } \partial\Omega. \end{cases} \tag{53}$$

This implies

$$p_0 + K_1(p_0) = \mathcal{L}^{-1}\nabla \cdot \left[A' * f + A(0)f\right] := \tilde{f}. \tag{54}$$

Step 4. Reduce to a fix point problem: Let

$$\mathcal{T}_1(p) := \tilde{f}_1 - K_1(p) \qquad \forall p \in L^q(0, T_1; C^{m+1,\alpha}(\bar{\Omega})), \tag{55}$$

and verify that the map

$$\mathcal{T}_1 : L^q(0, T_1; C^{m+1,\alpha}(\bar{\Omega})) \to L^q(0, T_1; C^{m+1,\alpha}(\bar{\Omega}))$$

is a **strict contraction**.

Key ingredients:

$$\boxed{\|K_1(p)(\cdot, t)\|_{C^{2,1/2}(\bar{\Omega})} \le C_1 \int_0^t ds |A'(t-s)| \|p(\cdot, s)\|_{C^{2,1/2}(\bar{\Omega})}.} \tag{56}$$

and

$$\begin{aligned} &\|\mathcal{T}_1(p)\|_{L^q(0,T_1;C^{m+1,\alpha}(\bar{\Omega}))} \\ &\lesssim \|A'\|_{L^1(0,T_1)} \left(\|p\|_{L^q(0,T_1;C^{m+1,\alpha}(\bar{\Omega}))} + \|f\|_{L^q(0,T_1;C^{m,\alpha}(\bar{\Omega}))}\right) + \|f\|_{L^q(0,T_1;C^{m,\alpha}(\bar{\Omega}))}. \end{aligned} \tag{57}$$

Lemma 6 (inductions)

Let $0 < \delta_0 \ll 1$ *be given as in Lemma* 5. *Let* $n \geq 2$ *be an arbitrary fixed large integer, and* $T_k = k\delta_0$ *with* $k = 1, \cdots, n$. *Assume that there exists a unique solution* $p_0 \in L^q(0, T_{n-1}; C^{m+1,\alpha}(\bar{\Omega}))$ *to the equations* (6), *satisfying*

$$\|p_0\|_{L^q(0,T_{n-1};C^{m+1,\alpha}(\bar{\Omega}))} \leq C_{n-1}\|f\|_{L^q(0,T_{n-1};C^{m,\alpha}(\bar{\Omega}))}. \tag{58}$$

Then, there exists a unique extension of the solution $p_0 \in L^q(0, T_n; C^{m+1,\alpha}(\bar{\Omega}))$ *to the equations* (6), *and satisfies the estimate*

$$\|p_0\|_{L^q(0,T_n;C^{m+1,\alpha}(\bar{\Omega}))} \leq C_n\|f\|_{L^q(0,T_n;C^{m,\alpha}(\bar{\Omega}))}, \tag{59}$$

where C_n *is monotonically ascending w.r.t.* n.

Remark: This leads to that the multiplicative constant in (48) will rely on a given finite time

For any fixed $n \geq 2$, and for any $t \in [T_{n-1}, T_n]$, we start from considering

$$\begin{cases} \mathcal{L}\left[p_0(\cdot,t) + K_n(p_0)(\cdot,t)\right] = \nabla \cdot \left[A' * f + A(0)f - \int_0^{T_{n-1}} dsA'(t-s)\nabla p_0(\cdot,s)\right] & \text{in } \Omega; \\ \dfrac{\partial}{\partial \nu}\left[p_0(\cdot,t) + K_n(p_0)(\cdot,t)\right] = \vec{n} \cdot \left\{A' * f + A(0)f - \int_0^{T_{n-1}} dsA'(t-s)\nabla p_0(\cdot,s)\right\} & \text{on } \partial\Omega, \end{cases} \tag{60}$$

where the auxiliary function $K_n(p_0)$ is given by:

$$\begin{cases} \nabla \cdot \left[A(0)\nabla K_n(p_0)(\cdot,t) - \int_{T_{n-1}}^{t} dsA'(t-s)\nabla p_0(\cdot,s)\right] = 0 & \text{in } \Omega; \\ \vec{n} \cdot \left[A(0)\nabla K_n(p_0)(\cdot,t) - \int_{T_{n-1}}^{t} dsA'(t-s)\nabla p_0(\cdot,s)\right] = 0 & \text{on } \partial\Omega; \\ \int_\Omega K_n(p_0)(\cdot,t) = 0, \end{cases}$$

with

$$\int_{T_{n-1}}^{T_n} \|K_n(p_0)(\cdot,t)dt\|^q_{C^{m+1,\alpha}(\bar{\Omega})} \lesssim \left(\int_0^{\delta_0} dt|A'(t)|\right)^{\frac{q}{q'}+1} \int_{T_{n-1}}^{T_n} dt\|p_0(\cdot,t)\|^q_{C^{m+1,\alpha}(\bar{\Omega})}.$$

Part 5: Boundary-layer correctors

Proposition 4 (Boundary-layer corrector I)

Let $0 < T < \infty$. Let J_1 and J_2 be given as in (63). Then, for a.e. $t \geq 0$, there exists at least one weak solution to

$$\begin{cases} \nabla\cdot\xi(\cdot,t) = \dfrac{\partial J_2}{\partial t} + \sum_i \big(\fint_{O^i_\varepsilon} \dfrac{\partial J_1}{\partial t}\big) 1_{O^i_\varepsilon}, & \text{in} \quad \Omega_\varepsilon; \\ \xi(\cdot,t) = 0, & \text{on} \quad \partial\Omega_\varepsilon, \end{cases} \tag{61}$$

where $\{O^i_\varepsilon\}$ is a decomposition of $O_\varepsilon := supp(\nabla\psi_\varepsilon)$. Also, we have

$$\|\xi\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon\|\nabla\xi\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \lesssim \varepsilon^{1/2}\|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}, \tag{62}$$

in which the up to constant depends on d, T and the characters of Y_f and Ω.

$$\begin{aligned} J_1 &:= \nabla\psi_\varepsilon \cdot \big[(W^\varepsilon - A) * G\big]; \\ J_2 &:= \nabla\psi_\varepsilon\cdot(A * G) + \varepsilon\nabla\psi_\varepsilon\cdot\big(\phi^\varepsilon *_2 \partial G\big) + \psi_\varepsilon \frac{A}{|Y_f|} *_2 \partial G + \varepsilon\psi_\varepsilon\phi^\varepsilon *_3 \partial^2 G. \end{aligned} \tag{63}$$

Theorem 7 (Bogovskii's operator on perforated domains)

For any $g \in L^2(\Omega_\varepsilon)$ with $\int_{\Omega_\varepsilon} g = 0$, there exists $v_\varepsilon \in H_0^1(\Omega_\varepsilon)^d$ such that $\nabla \cdot v_\varepsilon = g$ in Ω_ε, satisfying

$$\|v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\varepsilon\|\nabla v_\varepsilon\|_{L^2(\Omega_\varepsilon)} \leq C\|g\|_{L^2(\Omega_\varepsilon)}, \tag{64}$$

where the constant C is independent of ε and g.

By virtue of Theorem 7, the existence of the solution ξ to (61) is reduced to verify compatibility condition, while the desired estimate (62) follows from the estimates on the quantities

$$\Big(\int_0^T\int_{\Omega_\varepsilon}\Big|\frac{\partial J_2}{\partial t}\Big|^2\Big)^{1/2} \quad \text{and} \quad \Big(\int_0^T\int_{\Omega_\varepsilon}\Big|\sum_i\Big(\fint_{O_\varepsilon^i}\frac{\partial J_1}{\partial t}\Big)1_{O_\varepsilon^i}\Big|^2\Big)^{1/2}, \tag{65}$$

which will be addressed, separately.

Boundary-layer corrector I

——outline of proofs

1. **Strong norm estimates:**

$$
\begin{aligned}
&\Big(\int_0^T\int_{\Omega_\varepsilon}\Big|\frac{\partial J_2}{\partial t}\Big|^2\Big)^{1/2}\\
&\lesssim \varepsilon^{\frac12}\big(\|\partial_t\phi\|_{L^1(0,T;L^2(Y))}+\|\phi(0)\|_{L^2(Y_f)}\big)\|(\nabla G,\varepsilon^{\frac12}\nabla^2 G)\|_{L^2(0,T;L^\infty(\Omega))}\\
&+\big(\|A'\|_{L^1(0,T)}+|A(0)|\big)\Big\{\varepsilon^{\frac12}\|\nabla F\|_{L^2(0,T;C^0(\bar\Omega))}+\|\nabla(G-F)\|_{L^2(0,T;L^2(\mathrm{supp}\psi_\varepsilon))}\Big\},
\end{aligned}
\tag{66}
$$

whose multiplcative constant is independent of ε.

2. **Weak norm estimates**[8]:

$$
\begin{aligned}
&\Big(\int_0^T\int_{\Omega_\varepsilon}\Big|\sum_i\big(\fint_{O_\varepsilon^i}\frac{\partial J_1}{\partial t}\big)1_{O_\varepsilon^i}\Big|^2\Big)^{1/2}\\
&\lesssim\varepsilon^{\frac12}\|(G,\nabla G)\|_{L^2(0,T;L^\infty(\mathrm{supp}(\psi_\varepsilon)))}\Big\{\|\partial_t\Phi\|_{L^1(0,T;L^\infty(Y))}+\|\Phi(0)\|_{L^\infty(Y)}\Big\}.
\end{aligned}
\tag{67}
$$

[8] This terminology was borrowed from Stochastic homogenization, which is not quite appropriate here. However, we use this terminology to emphasize that the estimate (67) relies on the periodic cancellations deeply.

The key ideas are summarized as follows:

- By imposing **radial cut-off function**, together with the special structure of the effective solution (6) on the boundary, i.e., $\vec{n} \cdot u_0 = 0$ on $\partial\Omega$, it is possible to produce a desired smallness near the boundary, simply by Poincaré's inequality.

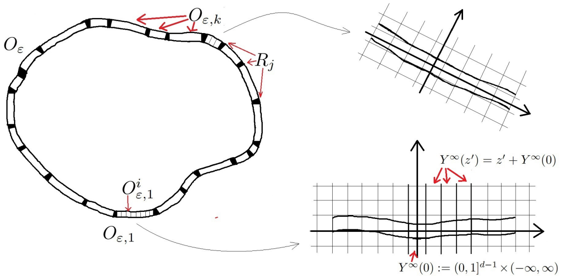
- By **decomposing the boundary layer region**, we can take full advantage of the invariant properties of periodic functions w.r.t. translation and rational rotation[9], which consequently provides us with more cancellations compared to dealing with the estimates on the boundary layer region as a whole.

[9] From E. Schmutz's work, it is known that each orthogonal matrix can be approximated by a rotational matrix with finite denominator, which establishes a theoretical base for dividing boundary layer regions.

Boundary-layer corrector I

——outline of proofs

$$L := \int_0^T \int_{\Omega_\varepsilon} \Big| \sum_i \Big(\fint_{O^i_\varepsilon} \frac{\partial J_1}{\partial t} \Big) 1_{O^i_\varepsilon} \Big|^2 \leq \int_0^T \int_{\Omega_\varepsilon} \Big| \sum_i^{N(\varepsilon)} \Big(\fint_{O^i_\varepsilon} \frac{\partial J_1}{\partial t} \Big) 1_{O^i_\varepsilon} + \sum_{j=1}^{K_0} \Big(\fint_{R_j} \frac{\partial J_1}{\partial t} \Big) 1_{R_j} \Big|^2, \tag{68}$$



$$Q_1(t) := \int_0^t ds \|\partial_t \Phi(t-s)\|_{L^\infty(Y)} \|\nabla G(s)\|_{L^\infty(\mathrm{supp}(\psi_\varepsilon))} + \|\Phi(0)\|_{L^\infty(Y)} \|\nabla G(t)\|_{L^\infty(\mathrm{supp}(\psi_\varepsilon))};$$

$$Q_2(t) := \int_0^t ds \|\partial_t \Phi(t-s)\|_{L^\infty(Y)} \|G(s)\|_{L^\infty(\mathrm{supp}(\psi_\varepsilon))}(t) + \|\Phi(0)\|_{L^\infty(Y)} \|G(t)\|_{L^\infty(\mathrm{supp}(\psi_\varepsilon))}. \tag{69}$$

Part 1. Show the estimates for the regular decomposition part, i.e.,

$$\left|\fint_{O_\varepsilon^i} \frac{\partial J_1}{\partial t}(\cdot,t)\right| \lesssim Q_1(t) + Q_2(t), \tag{70}$$

and then we are also interested in the estimates on the irregular part:

$$\left|\fint_{R_j} \frac{\partial J_1}{\partial t}(\cdot,t)\right| \lesssim \begin{cases} Q_1(t), & d=2; \\ Q_1(t)+\varepsilon^{-1}Q_2(t), & d=3. \end{cases} \tag{71}$$

where the up to constant is independent of i, j and t.

Part 2. Plugging (70) and (71) into the right-hand side of (68),

$$L \le \int_0^T dt \int_{\Omega_\varepsilon} 1_{O_\varepsilon}\left[Q_1(t)+Q_2(t)\right]^2 + (d-2)\sum_{j=1}^{K_0} \varepsilon^{-2}\int_0^T dt\int_{\Omega_\varepsilon} 1_{R_j}\left[Q_2(t)\right]^2.$$

By Fubini theorem, Young's inequality and the facts $|O_\varepsilon| = O(\varepsilon)$,

$$\begin{aligned} L &\lesssim |O_\varepsilon| \int_0^T dt\left[Q_1(t)+Q_2(t)\right]^2 + (d-2)\varepsilon^{d-2}\int_0^T dt\left[Q_2(t)\right]^2 \\ &\lesssim \varepsilon\left\{\|\partial_t\Phi\|^2_{L^1(0,T;L^\infty(Y))}\|\nabla G\|^2_{L^2(0,T;L^\infty(\mathrm{supp}(\psi_\varepsilon)))} + \|\Phi(0)\|^2_{L^\infty(Y)}\|\nabla G\|^2_{L^2(0,T;L^\infty(\mathrm{supp}(\psi_\varepsilon)))}\right\} \\ &+ \varepsilon\left\{\|\partial_t\Phi\|^2_{L^1(0,T;L^\infty(Y))}\|G\|^2_{L^2(0,T;L^\infty(\mathrm{supp}(\psi_\varepsilon)))} + \|\Phi(0)\|^2_{L^\infty(Y)}\|G\|^2_{L^2(0,T;L^\infty(\mathrm{supp}(\psi_\varepsilon)))}\right\}. \end{aligned}$$

Part 1-a. On account of the antisymmetric property of flux corrector Φ, integrating by parts we obtain that

$$\int_{O_\varepsilon^i} \frac{\partial J_1}{\partial t}(\cdot,t) = \int_{O_\varepsilon^i} \frac{\partial}{\partial t}[(W^\varepsilon - A) * G](\cdot,t)\cdot\nabla\psi_\varepsilon = \int_{O_\varepsilon^i} \frac{\partial}{\partial t}\big[(\nabla\cdot\Phi)^\varepsilon * G\big](\cdot,t)\cdot\nabla\psi_\varepsilon$$
$$\overset{(29)}{=} -\varepsilon\int_{O_\varepsilon^i}\frac{\partial}{\partial t}\big[\Phi^\varepsilon *_2 \partial G\big](\cdot,t)\cdot\nabla\psi_\varepsilon + \varepsilon\int_{\partial O_\varepsilon^i} dS\frac{\partial}{\partial t}\big[\Phi^\varepsilon *_3 (G\otimes\nabla\psi_\varepsilon\otimes\vec{n}_s)\big](\cdot,t)$$
$$:= E_1(t) + E_2(t), \tag{72}$$

in which $\vec{n}_s$ is the unit outward normal vector of the boundary ∂O_ε^i.

The term E_1 is easy, and a direct computation leads to

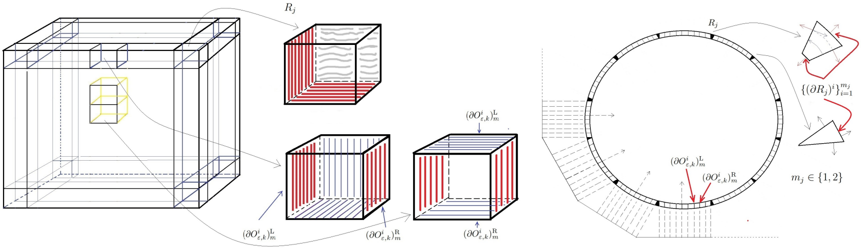
$$|E_1(t)| \lesssim \int_{O_\varepsilon^i}\left|\frac{\partial}{\partial t}[\Phi^\varepsilon *_2 \partial G](\cdot,t)\right| = \int_{O_\varepsilon^i}\left|(\partial_t\Phi^\varepsilon *_2 \partial G)(\cdot,t) + \Phi^\varepsilon(\cdot,0) : \partial G(\cdot,t)\right|$$
$$\le \int_{O_\varepsilon^i}\left|\partial_t\Phi^\varepsilon *_2 \partial G\right|(\cdot,t) + |O_\varepsilon^i|\|\Phi(0)\|_{L^\infty(Y)}\|\nabla G(t)\|_{L^\infty(O_\varepsilon)}.$$

This finally gives us

$$|E_1(t)| \lesssim |O_\varepsilon^i|\Big\{\|\partial_t\Phi\|_{L^\infty(Y)} *_2 \|\partial G\|_{L^\infty(O_\varepsilon)}(t) + \|\Phi(0)\|_{L^\infty(Y)}\|\nabla G(t)\|_{L^\infty(O_\varepsilon)}\Big\}. \tag{73}$$

Boundary-layer corrector I

——the estimate (70)



Part 1-b. Estimate the second term E_2 in (72), which involves more geometrical details on ∂O^i_ε. It concludes that $\nabla\psi_\varepsilon(x) = 0$ on $\partial O^i_\varepsilon \cap \partial O_\varepsilon$. Thus, to complete the estimate of E_2, it suffices to focus on the case $\partial O^i_\varepsilon \cap O_\varepsilon$, which actually appear in pairs

$$\partial O^i_\varepsilon \cap O_\varepsilon = \bigcup_{m=1}^{d-1} \{(\partial O^i_\varepsilon)^{\mathrm{L}}_m \cup (\partial O^i_\varepsilon)^{\mathrm{R}}_m\},$$

where $\{(\partial O^i_\varepsilon)^{\mathrm{L}}_m \cup (\partial O^i_\varepsilon)^{\mathrm{R}}_m\}$ is known as one pair of the lateral boundaries.

Continued with Part 1-b.

- After a εz-translation or a rational rotation for $(\partial O^i_\varepsilon)^{\mathrm{R}}_m$, where $z \in \mathbb{Z}^d$ and $|z| = 1$, its most part can overlap with $(\partial O^i_\varepsilon)^{\mathrm{L}}_m$.
- In this regard, we denote this transformation by T, and $\Gamma^{\mathrm{R}}_m := (\partial O^i_\varepsilon)^{\mathrm{R}}_m(\mathrm{T}\cdot)$. Thus, which allow us to have

$$\big[(\partial O^i_\varepsilon)^{\mathrm{L}}_m \cup \Gamma^{\mathrm{R}}_m\big] = \underbrace{\big[(\partial O^i_\varepsilon)^{\mathrm{L}}_m \cap \Gamma^{\mathrm{R}}_m\big]}_{\text{intersection part}} \bigcup \underbrace{\big[(\partial O^i_\varepsilon)^{\mathrm{L}}_m \triangle \Gamma^{\mathrm{R}}_m\big]}_{\text{difference part}}. \tag{74}$$

- For the ease of the statement, it is fine to assume $(\partial O^i_\varepsilon)^{\mathrm{L}}_m \subset \Gamma^{\mathrm{R}}_m$ in later computations. On account of the smoothness of $\partial\Omega$, we have the estimate of the difference part[10]:

$$\big|(\partial O^i_\varepsilon)^{\mathrm{L}}_m \triangle \Gamma^{\mathrm{R}}_m\big| \lesssim \varepsilon^d. \tag{75}$$

[10] It is reduced to study the variance in derivatives of boundary functions along one direction. Since the scale of decomposed cubes is ε, the variance would be $O(\varepsilon^2)$. The desired estimate follows from "the variance" $\times\ O(\varepsilon^{d-2})$.

Boundary-layer corrector I

——the estimate (70)

Continued with Part 1-b.

1. By virtue of the notation defined above, we can rewrite E_2 as

$$E_2(t) = \varepsilon \sum_{m=1}^{d-1} \Big\{ \int_{(\partial O^i_\varepsilon)^{\mathrm{L}}_m} + \int_{(\partial O^i_\varepsilon)^{\mathrm{R}}_m} \Big\} dS \frac{\partial}{\partial t}\big[\Phi^\varepsilon *_3 (G \otimes \nabla\psi_\varepsilon \otimes \vec{n}_s)\big](\cdot, t) := \varepsilon \sum_{m=1}^{d-1} E_2^m(t),$$

2. For any fixed i and m, denoted εz-translation or rational rotation transformation by T^i_m, it follows from the equality (74) together with the notation therein that

$$\Gamma^{\mathrm{R}}_m = (\partial O^i_\varepsilon)^{\mathrm{L}}_m \bigcup \big[(\Gamma^{\mathrm{R}}_m \setminus (\partial O^i_\varepsilon)^{\mathrm{L}}_m\big]. \tag{76}$$

3. Important facts:
   - According to the **periodicity** of flux corrector,

$$\Phi^\varepsilon = \Phi^\varepsilon((\mathrm{T}^i_m)^{-1}\cdot). \tag{77}$$

   - the unit outward normal vector $\vec{n}_s$ of ∂O^i_ε takes **the opposite direction** on $(\partial O^i_\varepsilon)^{\mathrm{L}}_m$ and $(\partial O^i_\varepsilon)^{\mathrm{R}}_m$.

Continued with Part 1-b.

$$
\begin{aligned}
E_2^m(t) &= \int_{(\partial O_\varepsilon^i)_m^{\mathrm{L}}} dS \frac{\partial}{\partial t}\Big\{\Phi^\varepsilon *_3 \Big[\big(G - G((\mathrm{T}_m^i)^{-1}\cdot)\big) \otimes \nabla\psi_\varepsilon \otimes \vec{n}_s\Big](\cdot,t)\Big\} \\
&+ \int_{(\partial O_\varepsilon^i)_m^{\mathrm{L}}} dS \frac{\partial}{\partial t}\Big\{\Phi^\varepsilon *_3 \Big[G((\mathrm{T}_m^i)^{-1}\cdot) \otimes \big(\nabla\psi_\varepsilon - \nabla\psi_\varepsilon((\mathrm{T}_m^i)^{-1}\cdot)\big) \otimes \vec{n}_s\Big](\cdot,t)\Big\} \\
&- \int_{\Gamma_m^{\mathrm{R}} \backslash (\partial O_\varepsilon^i)_m^{\mathrm{L}}} dS \frac{\partial}{\partial t}\Big\{\Phi^\varepsilon *_3 \Big[G \otimes \nabla\psi_\varepsilon \otimes \vec{n}_s\Big]((\mathrm{T}_m^i)^{-1}\cdot, t)\Big\} \\
&:= E_{2,1}^m(t) + E_{2,2}^m(t) + E_{2,3}^m(t).
\end{aligned} \tag{78}
$$

Arguments for $E_{2,2}^m(t)$. Since the translation or rotation transformation preserves the distance, we have

$$
\mathrm{dist}(\cdot, \partial\Omega) = \mathrm{dist}((\mathrm{T}_m^i)^{-1}\cdot, \partial\Omega) \quad \text{on} \quad (\partial O_\varepsilon^i)_m^{\mathrm{L}}.
$$

This leads to

$$
\nabla\psi_\varepsilon - \nabla\psi_\varepsilon((\mathrm{T}_m^i)^{-1}\cdot) = 0 \quad \text{on} \quad (\partial O_\varepsilon^i)_m^{\mathrm{L}}, \tag{79}
$$

whereupon the second term $E_{2,2}^m$ vanishes.

Continued with Part 1-b.

Arguments for $E^m_{2,3}(t)$. In view of (75), i.e.,

$$\left|\Gamma^{\mathrm{R}}_m \setminus (\partial O^i_\varepsilon)^{\mathrm{L}}_m\right| \lesssim \varepsilon^d,$$

as well as, Fubini's theorem, we simply obtain that

$$|E^m_{2,3}(t)| \lesssim |O^i_\varepsilon| Q_2(t). \tag{80}$$

Arguments for $E^m_{2,1}(t)$. From the differential mean value theorem, it follows that

$$\sup_{x\in O_\varepsilon} |G(x,t) - G((\mathrm{T}^i_m)^{-1}x,t)| \lesssim \varepsilon \|\nabla G(t)\|_{L^\infty(O_\varepsilon)}.$$

This estimate coupled with Fubini's theorem leads to

$$|E^m_{2,1}(t)| \lesssim \varepsilon |(\partial O^i_\varepsilon)^{\mathrm{L}}_m| Q_1(t) \lesssim |O^i_\varepsilon| Q_1(t).$$

As a result,

$$|E^m_2(t)| \leq |E^m_{2,1}(t)| + |E^m_{2,3}(t)| \lesssim |O^i_\varepsilon| \Big\{ Q_1(t) + Q_2(t) \Big\}. \tag{81}$$

By noting the relationship $E_2(t) := \varepsilon \sum_{m=1}^{d-1} E^m_2(t)$, we finally obtain

$$|E_2(t)| \overset{(81)}{\lesssim} |O^i_\varepsilon| \Big\{ Q_1(t) + Q_2(t) \Big\},$$

Part 1-c. The irregular part:

$$\Big|\fint_{R_j}\frac{\partial J_1}{\partial t}(\cdot,t)\Big| \lesssim \begin{cases} Q_1(t), & d=2;\\ Q_1(t)+\varepsilon^{-1}Q_2(t), & d=3.\end{cases}$$

Using the same idea as given for (72) in Part 1-a, we have

$$\begin{aligned}\int_{R_j}\frac{\partial J_1}{\partial t}(\cdot,t) &\overset{(29)}{=} -\varepsilon\int_{R_j}\frac{\partial}{\partial t}\big[\Phi^\varepsilon *_2 \partial G\big](\cdot,t)\cdot\nabla\psi_\varepsilon \\ &+\varepsilon\int_{\partial R_j} dS\frac{\partial}{\partial t}\big[\Phi^\varepsilon *_3 (G\otimes\nabla\psi_\varepsilon\otimes\vec{n}_s)\big](\cdot,t) := E_3(t)+E_4(t),\end{aligned} \tag{82}$$

We can deal with the term E_3 as we did for E_1 in (73) in Part 1-b,

$$|E_3(t)| \lesssim |R_j|Q_1(t). \tag{83}$$

The main difference between the terms E_4 and E_2 is that the geometry of R_j is not as the same as that of O^i_ε. The main challenge is that we have to find a “smallness” in the same level but from the different geometry facts. **The ideas** are following: (I) using dimensional condition to increase a “smallness” for the case $d=3$; (II) employing the invariant property of flux corrector in terms of rotation (requiring all its components to be rational number) for the case $d=2$.

Boundary-layer corrector II

——main results

Proposition 5 (Boundary-layer corrector II)

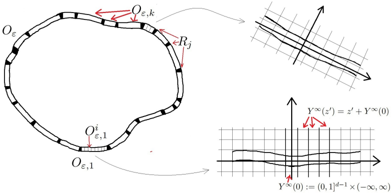
Let J_1 be given as in (63). Then, for a.e. $t \geq 0$, there exists at least one weak solution to

$$\begin{cases} \nabla \cdot \eta(\cdot, t) = \dfrac{\partial J_1}{\partial t} - \sum_i \left(\fint_{O^i_\varepsilon} \dfrac{\partial J_1}{\partial t} \right) 1_{O^i_\varepsilon}, & \text{in} \quad O_\varepsilon; \\ \eta(\cdot, t) = 0, & \text{on} \quad \partial O_\varepsilon, \end{cases} \tag{84}$$

satisfying the following estimate

$$\|\eta\|_{L^2(0,T;L^2(\Omega_\varepsilon))} + \varepsilon \|\nabla \eta\|_{L^2(0,T;L^2(\Omega_\varepsilon))} \lesssim \varepsilon^{1/2} \|f\|_{L^2(0,T;C^{1,1/2}(\bar{\Omega}))}, \tag{85}$$

where the up to constant depends on d, T and the characters of Y_f and Ω.



For each O_ε^i, we can get a η_i which satisfies the following equation:

$$\begin{cases} \nabla \cdot \eta_i = \dfrac{\partial J_1}{\partial t} - \left(\fint_{O_\varepsilon^i} \dfrac{\partial J_1}{\partial t}\right) 1_{O_\varepsilon^i}, & \text{in} \quad O_\varepsilon^i; \\ \eta_i = 0, & \text{on} \quad \partial O_\varepsilon^i, \end{cases} \tag{86}$$

Moreover, we have the following estimate

$$\|\nabla \eta_i\|_{L^2(O_\varepsilon^i)} \leq C \left\| \frac{\partial J_1}{\partial t} - \left(\fint_{O_\varepsilon^i} \frac{\partial J_1}{\partial t}\right) 1_{O_\varepsilon^i} \right\|_{L^2(O_\varepsilon^i)} \leq C \|\partial_t J_1\|_{L^2(O_\varepsilon^i)}, \tag{87}$$

where the constant C **does not** depend on ε and i.
Let

$$\eta := \sum_i \eta_i,$$

and it is not hard to observe that for a.e. $t \geq 0$,

$$\|\nabla \eta(t)\|_{L^2(O_\varepsilon)}^2 = \sum_i \|\nabla \eta_i(t)\|_{L^2(O_\varepsilon^i)}^2.$$

This together with (87) leads to

$$\|\nabla \eta(t)\|_{L^2(O_\varepsilon)}^2 \lesssim \|\partial_t J_1(t)\|_{L^2(O_\varepsilon)}^2. \tag{88}$$

$$\int_0^T\int_{O_\varepsilon}\Big|\frac{\partial J_1}{\partial t}\Big|^2\lesssim\varepsilon^{-2}\int_0^T\int_{O_\varepsilon}\Big|\frac{\partial}{\partial t}[(W^\varepsilon-A)*G]\Big|^2$$
$$\lesssim\varepsilon^{-2}\Big(\|\partial_t W^\varepsilon-A'\|^2_{L^1(0,T;L^2(O_\varepsilon))}+\|W^\varepsilon(0)-A(0)\|^2_{L^2(O_\varepsilon)}\Big)\|G\|^2_{L^2(0,T;L^\infty(O_\varepsilon))}. \tag{89}$$

By a rescaling argument used for $\partial_t W^\varepsilon$ and using its periodicity, we note that

$$\begin{aligned}\|\partial_t W^\varepsilon-A'\|_{L^1(0,T;L^2(O_\varepsilon))}&=\int_0^T dt\Big(\int_{O_\varepsilon}dx|\partial_t W(x/\varepsilon,t)-A'(t)|^2\Big)^{1/2}\\&\lesssim\varepsilon^{1/2}\int_0^T dt|A'(t)|+\int_0^T dt\Big(\int_{O_\varepsilon}dx|\partial_t W(x/\varepsilon,t)|^2\Big)^{1/2}\\&=\varepsilon^{1/2}\Big\{\|A'\|_{L^1(0,T)}+\|\partial_t W\|_{L^1(0,T;L^2(Y_f))}\Big\};\end{aligned}$$

By the same token, we have

$$\|W^\varepsilon(0)-A(0)\|_{L^2(O_\varepsilon)}\lesssim\varepsilon^{1/2}.$$

Inserting the above two estimates back into (89), and then together with (88), there holds

$$\varepsilon^2\int_0^T\int_{O_\varepsilon}|\nabla\eta|^2\lesssim\varepsilon\Big\{\|A'\|_{L^1(0,T)}+\|\partial_t W\|_{L^1(0,T;L^2(Y_f))}+1\Big\}^2\|G\|^2_{L^2(0,T;L^\infty(O_\varepsilon))}.$$

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Remark: We refer the reader to Wang-Xu-Zhang's job for more references therein.

Thank you for your attention!