A LIOUVILLE-TYPE THEOREM IN A HALF-SPACE AND ITS APPLICATIONS TO THE GRADIENT BLOW-UP BEHAVIOR FOR SUPERQUADRATIC DIFFUSIVE HAMILTON-JACOBI EQUATIONS

Philippe SOUPLET

LAGA, Université Sorbonne Paris Nord

$$
\text{(DHJ)} \qquad \begin{cases} u_t - \Delta u = |\nabla u|^p, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}
$$

MOTIVATION

1) Stochastic control problem

Controled stochastic dynamical system

$$\boxed{dX_s = \alpha_s ds + dW_s, \quad s > 0, \qquad \text{with } X_0 = x \in \Omega}$$

$$\begin{aligned}
(W_s)_{s>0} &= \text{Brownian Motion with values in } \mathbb{R}^n \\
(X_s)_{s\geq 0} &= \text{position of the particle (stochastic process)} \\
(\alpha_s)_{s>0} &= \text{control (the controler can choose the } \textit{velocity} \text{ of } X)
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$(X_s)_{s\geq 0}$ = position of the particle (stochastic process)

$(\alpha_s)_{s>0}$ = control (the controler can choose the *velocity* of X)

$u_0 \in C_0(\overline{\Omega})$ = spatial distribution of rewards, i.e.:

At a given time horizon $s = t > 0$, the **final reward** is

$$\begin{cases} u_0(X_t), & \textit{provided } X \text{ stays in } \Omega \text{ until time } t \text{ (i.e. } \tau := \text{first exit time} > t) \\ 0, & \text{otherwise} \end{cases}$$

The **cost** of the control at each time s is $|\alpha_s|^{p/(p-1)}$ (as long as X_s stays in Ω)

Goal (of the controler): maximize the net gain $$G_t = \chi_{\tau>t} u_0(X_t) - \int_0^\tau |\alpha_s|^{p/(p-1)}\, ds$$

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Theorem: [Barles-Burdeau CPDE 95, Barles-Da Lio JMPA 04] The value function (maximal gain) is given by the unique (continuous) *viscosity solution* of (DHJ), namely:

$$u(x,t) = \sup_{(\alpha_s)_s} \mathbb{E}\left(G_t \,|\, X_0 = x\right)$$

MOTIVATION

2) KPZ model of surface growth

[Kardar-Parisi-Zhang 86] ($p = 2$) and [Krug-Spohn 88] ($p > 1$)

$$u_t = \nu \Delta u + \lambda |\nabla u|^p + \eta(x,t)$$

- u = height of surface, growing by ballistic deposition of dusts (alumine)
- growth term $\lambda |\nabla u|^p$: deposition of new particles on the surface
- diffusion term $\nu \Delta u$: relaxation of the interface by surface tension
- $\eta(x,t)$: noise term

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3) Model case in theory of NL parabolic equations

Among simplest parabolic PDE's with 1st order nonlinearity

Cp. classical problem with zero order nonlinearity (NLH or Fujita equation):

$$u_t - \Delta u = u^p, \quad p > 1$$

BASIC PROPERTIES: LOCAL WELL-POSEDNESS

$p > 1$, $\Omega \subset \mathbb{R}^n$ smooth

$u_0 \in X_+ := \{v \in C^1(\overline{\Omega});\ v \geq 0,\ v_{|\partial\Omega} = 0\}$, with C^1 norm.

• Local existence-uniqueness, maximal **classical** solution

$T = T(u_0) \in (0, \infty]$.

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• Maximum principle estimate:

$$\boxed{0 \leq u(\cdot, t) \leq \|u_0\|_\infty, \quad 0 < t < T}$$

• Blow-up alternative:

If $T < \infty$, then **Gradient Blow-up (GBU)**, i.e.:

$$\boxed{\lim_{t \to T} \|\nabla u(t)\|_\infty = \infty}$$

GLOBAL EXISTENCE vs. GRADIENT BLOW-UP

1) $p \leq 2$: Global existence and C^1-boundedness for all u_0

Consequence of gradient estimates from [Ladyzenskaja-Solonnikov-Ural'seva 50-60's]

2) $p > 2$ and Ω bounded

- Gradient blow-up (GBU) for large data:

$$q \in [1, \infty) \text{ and } \|u_0\|_q \geq {}^{\exists}C(n, p, q, \Omega) > 0 \implies T < \infty$$

- Global existence and decay for small data in C^1

[Ladyzenskaja 56, Filippov 61, Lieberman 86, Alikakos-Bates-Grant 89, Dlotko 91, Alaa 96, S. 02, Benachour-Dabuleanu 03, Hesaaraki-Moameni 04, S.-Zhang 06, ...]

- Any GBU solution can be continued as a unique global *viscosity solution* [Barles-DaLio JMPA 2004]

Rich variety of post GBU phenomena: GBU with or without loss of BC, ultimate regularization, multiple times of GBU and losses and recoveries of BC

[Porretta-Zuazua AIHP 12], [Porretta-S. AIHP 17 & JMPA 19], [Quaas-Rodriguez JDE 18], [Mizoguchi-S. preprints 20', 21']

BEHAVIOR OF GBU SOLUTIONS – QUESTIONS

($p > 2$, Ω bounded assumed throughout)

1. Singular set

2. Time rate

3. GBU spatial profile (and a Liouville theorem)

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3. GBU spatial profile (and a Liouville theorem)

Remark: intensively studied for the Fujita equation (L^∞, or amplitude blow-up)

$$\boxed{u_t - \Delta u = u^p}$$

Detailed theory for description of asymptotic profile near a finite time singularity

[Weissler, Friedman-McLeod, Giga-Kohn, Herrero-Velázquez, Galaktionov-Vázquez, Merle-Zaag, Matano-Merle, ...]

GBU SET

$B(u_0) := \{x_0 \in \overline{\Omega};\ \nabla u \text{ is unbounded near } (x_0, T)\}.$

Prop :

$$\boxed{B(u_0) \subset \partial\Omega}$$

Consequence of gradient estimate [S.-Zhang JAM 06]

(1) $$\boxed{|\nabla u| \le C\delta^{-1/(p-1)}(x) \quad \text{in} \ \ \Omega \times [0,T), \qquad \delta(x) = \text{dist}(x, \partial\Omega)}$$

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Rem:

• Interior GBU occurs for other related equations (e.g. $F(u, \nabla u) = u|\nabla u|^p$)
[Giga 95, Angenent-Fila 96, Asai-Ishimura 98, Fila-Lankeit Math.Ann. 19]

• Similar result for quasilinear case $u_t - \Delta_p u = |\nabla u|^q \quad (q > p > 2)$ [Attouchi JDE 12]

• Estimates of type (1) $\Longrightarrow$ Liouville-type thms for ancient solutions in $(-\infty, 0) \times \mathbb{R}^n$.

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Idea of proof: Local Bernstein-type arguments (cf. [PL Lions JAM 85] in elliptic case)
$\to$ max. principle applied to $z := \eta(x)\varphi(u)|\nabla u|^2$ with suitable φ and cut-off η

$$z_t - \Delta z - b(x,t) \cdot \nabla z \le -z^{(p+2)/2} + C$$

GBU SET (II)

Question: location of GBU points within the boundary ?

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$\Omega \subset \mathbb{R}^n$ bounded, $x_0 \in \partial\Omega$, $\rho > 0 \Longrightarrow$ there exists u_0 such that $T(u_0) < \infty$ and

$$\boxed{B(u_0) \subset B_\rho(x_0) \cap \partial\Omega}$$

Rem:

- True if u_0 is supported near x_0 and suitably concentrated
- $\|u_0\|_\infty$ can be made arbitrarily small (but u_0 has large derivative)

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Idea of proof:

Barriers + Local Bernstein type estimate $\Longrightarrow$ no GBU away from x_0

Rescaling argument $\Longrightarrow T < \infty$

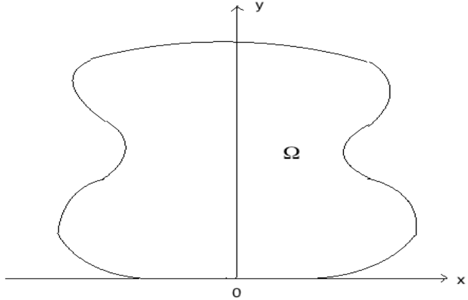
GBU SET (III)

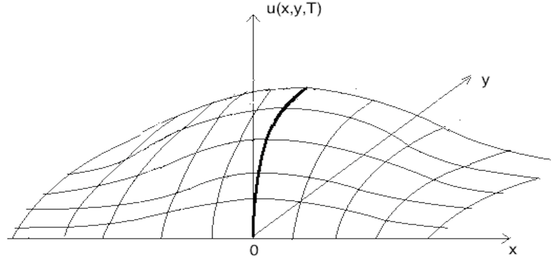
3. Single-point GBU

Theorem. [Li-S. CMP 10] *Assume* $\Omega \subset \mathbb{R}^2$, $0 \in \partial\Omega$, *and* Ω *is either a disk or*

$$\begin{cases} \Omega \text{ is symmetric with respect to the line } x = 0, \\ \Omega \text{ is convex in the } x\text{-direction}, \\ \partial\Omega \text{ is locally flat (contains a segment) near the origin.} \end{cases}$$

Then there exist u_0 *such that* $T(u_0) < \infty$ *and* $B(u_0) = \{0\}$.





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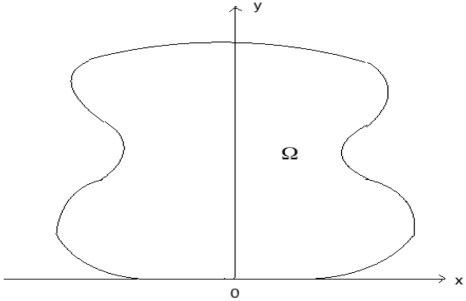

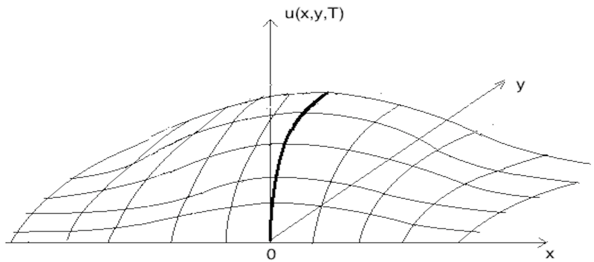

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Rem:

- True if u_0 symmetric decreasing in x, supported near 0 and sufficiently concentrated
- True for more general (nonflat) symmetric domains [Esteve JMPA 19]
- Nonlinear diffusion: $u_t - \Delta_p u = |\nabla u|^q \quad (q > p > 2)$ [Attouchi-S. TAMS 17]
- Possible physical interpretation (KPZ model): the surface tension (diffusion) forces the steep region to become more and more concentrated near a single boundary point

IDEAS OF PROOF:

• Auxiliary function (2D modification of a 1D device of [Friedman-McLeod IUMJ 85])

$$J(x,y,t) = u_x + \lambda x y^{-\gamma} u^q, \quad x,y>0 \text{ small} \quad (q>1,\ 0<\gamma<q-1)$$

• Use maximum principle to show $J \leq 0$
$\to$ long computations using Bernstein type gradient estimate
(+ parabolic version of Serrin's corner Lemma, ...)

• Integration in $x \implies u(x,y,t) \leq C x^{-2/(q-1)} y^{\gamma/(q-1)}, \ \gamma/(q-1) = 1^-$

• Nondegeneracy result (analogue of [Giga-Kohn CPAM 89] for Fujita eqn.)
$\implies$ profile in normal direction near a GBU point (a,T) cannot be $\ll \delta^{(p-2)/(p-1)}$
(where $\delta =$ distance to the boundary).
$\implies$ no GBU at $(x,0)$ for $x>0 \implies B(u_0) = \{0\}$

OPEN PROBLEMS

- Finiteness of $B(u_0)$ for $n = 2$ and nonradial u_0 ([Chen-Matano JDE 89] for Fujita eq.)
- Finiteness of m-Hausdorff measure of $B(u_0)$ ($m = n - 2$?); cf. [Velázquez IUMJ 93]

TIME RATE OF GBU: Lower estimate

• For *any* GBU solution [Porretta-S. JMPA 19]:

$$\boxed{\|\nabla u(t)\|_\infty \geq C(T-t)^{-1/(p-2)}, \qquad 0 < t < T}$$

Previous partial results [Conner-Grant DIE 96, Guo-Hu DCDS 08]

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Idea of proof: Equation for $w := u_t$:

$$\partial_t w - \Delta w = B \cdot \nabla w, \qquad B := p|\nabla u|^{p-2}\nabla u$$

Max principle $\Longrightarrow |u_t| \leq C$

Variation of constants formula + heat semigroup smoothing estimates

$\Longrightarrow \|\nabla u_t(t)\|_\infty \leq C\|B\|_{L^\infty((0,t)\times\Omega)} = C\|\nabla u\|^{p-1}_{L^\infty((0,t)\times\Omega)}$

$\Longrightarrow m(t) := \|\nabla u\|_{L^\infty((0,t)\times\Omega)}$ satisfies $m' \leq Cm^{p-1}$ a.e. $\to$ integrate

+ Additional arguments to improve from $L^\infty((0,t)\times\Omega)$ to $L^\infty(\Omega)$

TIME RATE OF GBU: Lower estimate

- **Consequence:** GBU rate is **never self-similar**

Scale invariance: $u_\lambda(x,t) := \lambda^{-k} u(\lambda x, \lambda^2 t), \quad k = (p-2)/(p-1).$

Backward self-similar solutions would be (in Ω half-space)

$$u(x,t) = (T-t)^{k/2} V\big(x/\sqrt{T-t}\big), \qquad V \in BC^1(\overline{\Omega})$$

Corresponding self-similar rate would be

$$\|\nabla u(t)\|_\infty \sim (T-t)^{-\frac{1}{2(p-1)}}. \qquad \text{But } \frac{1}{p-2} > \frac{1}{2(p-1)} \text{ !!}$$

TIME RATE OF GBU: Upper estimate

Slightly more general problem ($h \in C^1(\overline{\Omega})$, $h \geq 0$, $u_0 \geq 0$)

$$\begin{cases} u_t - \Delta u &= |\nabla u|^p + h(x), \quad x \in \Omega,\ t > 0, \\ u &= 0, \quad x \in \partial\Omega,\ t > 0, \\ u(x,0) &= u_0(x), \quad x \in \Omega. \end{cases}$$

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For time-increasing solutions (sufficient condition: $\Delta u_0 + |\nabla u_0|^p + h \geq 0$), we have

$$C_1(T-t)^{-1/(p-2)} \leq \|\nabla u(t)\|_\infty \leq C_2(T-t)^{-1/(p-2)} \tag{1}$$

provided

- $n = 1$ [Guo-Hu DCDS 08, Porretta-S. JMPA 19]
- $\Omega = B_R$, u_0 radially symmetric [Li-Zhang AMSci 13]
- Ω convex, $2 < p < 3$; or Ω ball or annulus, $2 < p \leq 3$ [Attouchi-S., CVPDE 20]
- Ω bounded with $2 < p \leq 3$ but $(T-t)^{-\frac{1}{p-2}-\varepsilon}$ on the RHS [Attouchi-S., CVPDE 20]

TIME RATE OF GBU: Upper estimate

Ideas of proofs (time-increasing solutions)

Maximum principle applied to auxiliary functions

$n = 1$ $\quad J = u_t - \varepsilon(z+u)e^{-\lambda t}$

$$\text{with } z(x,t) := \Big(1 + \frac{1}{\|u_x(t)\|_\infty^\sigma}\Big)\Big(1 - \frac{u_x}{\|u_x(t)\|_\infty}\Big)$$

$n \geq 2$ $\quad J = u_t - \varepsilon u^{p-1}\delta^{2-p}\big[1+\delta^\kappa\big], \quad \delta(x) := \mathrm{dist}(x, \partial\Omega).$

Other ingredients:

- sharp gradient estimates (see below)
- zero-number arguments on u_t

TIME RATE OF GBU:

Faster rates and complete classification in 1d

$$\begin{cases} u_t - u_{xx} = |u_x|^p, & x \in \Omega = (0,R),\ t > 0 \qquad (0 < R \le \infty) \\ u = 0, & x \in \partial\Omega,\ t > 0 \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Theorem. [Mizoguchi-S., preprint 21']

(a) For *any* $u_0 \in X_+$ with $0 \in \mathcal{B}$, there exist an integer $\ell \ge 1$ and $C > 0$ such that

$$\boxed{\lim_{t\to T}(T-t)^{\frac{\ell}{p-2}}\, u_x(0,t) = C} \qquad (1)$$

(b) For *any* integer $\ell \ge 1$, there exists $u_0 \in X_+$ and $C > 0$ such that (1) holds.

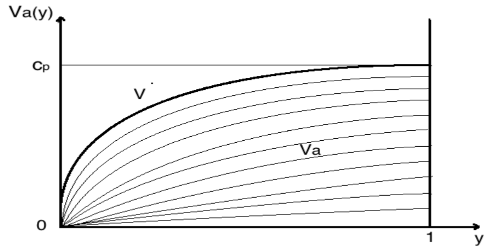
Remark.

$$\sup_{x\in(0,R/2)} |u_x(x,t)| = u_x(0,t), \quad t \to T.$$

TIME RATE OF GBU:

Faster rates and complete classification in 1d (continued)

• **Bubbling space-time behavior**



Define steady states

$V(x) = c_p x^{(p-2)/(p-1)}$ (singular)

$U_a(x) = V(x+a) - V(a), \quad a > 0$ (regular)

In small boundary layer, behavior of u described by

$$u = U_{a(t)}(x) + O(x^2), \qquad \text{as } t \to T_-, \text{ with } a(t) := \beta u_x^{1-p}(0,t) \to 0$$

TIME RATE OF GBU:
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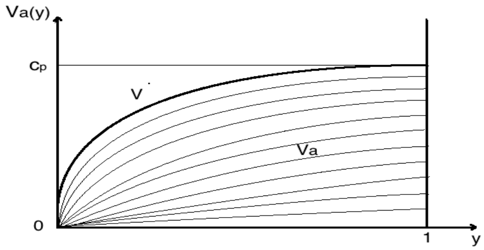

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- **Characterization of ℓ:** number of *vanishing intersections* of $u(\cdot, t)$ with U as $t \to T^-$.

- **Stability of GBU time and GBU rate**
  - T continuous w.r.t. initial data iff ℓ odd
  - rate (and profile) stable iff $\ell = 1$

IDEAS OF PROOFS

Part (b) based on construction of special solutions with precise space-time behavior, by a modification of Herrero-Velázquez' method for NLH "type-II" solutions (1994)

Ingredients:

- similarity variables $y = x/\sqrt{T-t}$, $s = -\log(T-t)$ (Giga-Kohn 1985-89)
- matched asymptotics: inner region (quasi-stationary behavior) and outer region (linearization around singular steady-state $\to$ rates given by eigenvalues !)
- heavy a priori estimates
- topological degree

IDEAS OF PROOFS

Part (a) based on **zero number** and **braid group techniques** to compare 3 solutions u, U, v, where

- U : singular steady state
- v : special sol. with known rate, s.t. $T^*(v) = T^*(u)$ and $v - U$ has same # of vanishing zeros as $u - U$

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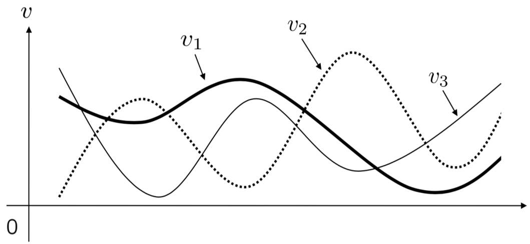

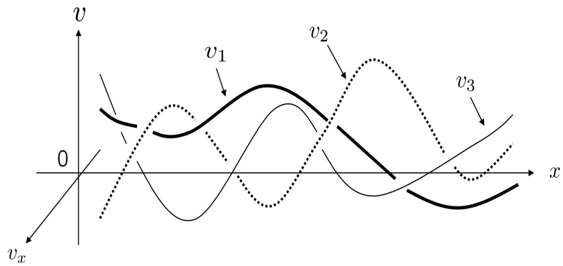

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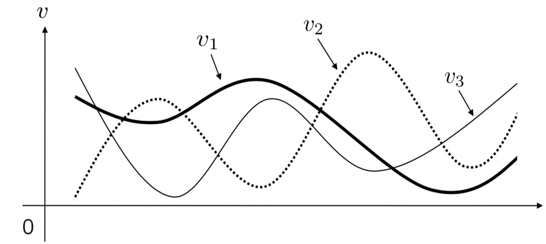
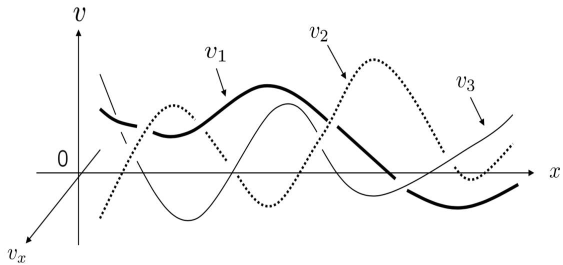
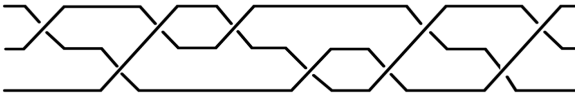






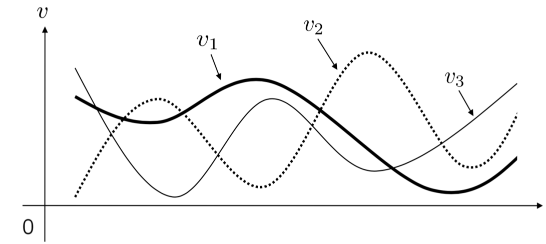
YXY^2X^2YXY

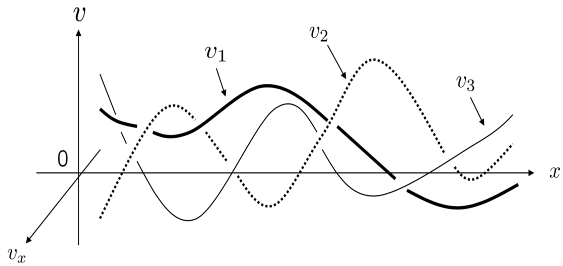
I X Y

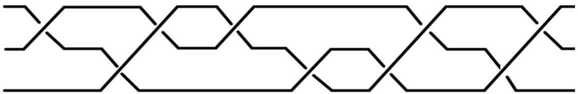
IDEAS OF PROOFS

Part (a) based on **zero number** and **braid group techniques** to compare 3 solutions u, U, v, where

- U : singular steady state
- v : special sol. with known rate, s.t. $T^*(v) = T^*(u)$ and $v - U$ has same # of vanishing zeros as $u - U$







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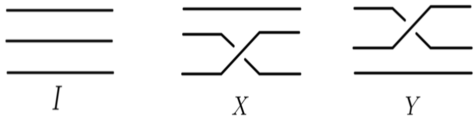


Parabolic reduction principle (Matano): denote $G(t) =$ braid (v_1, v_2, v_3)

$G(t)$ loses finitely many X^2 or Y^2 (up to topological equivalence)

SPACE PROFILE

A Liouville-type theorem.

Related elliptic problem in the half-space $\mathbb{R}^n_+ = \{(x_1, \dots, x_n);\ x_n > 0\}$:

$$(1) \qquad \begin{cases} -\Delta v = |\nabla v|^p, & x \in \mathbb{R}^n_+, \\ v = 0, & x \in \partial\mathbb{R}^n_+ \end{cases}$$

Theorem 3. [Filippucci-Pucci-S. CPDE 2019]
Let $p > 2$ and let $v \in C^2(\mathbb{R}^n_+) \cap C(\overline{\mathbb{R}^n_+})$ be a solution of (1). Then v depends only on the variable x_n.

Remarks

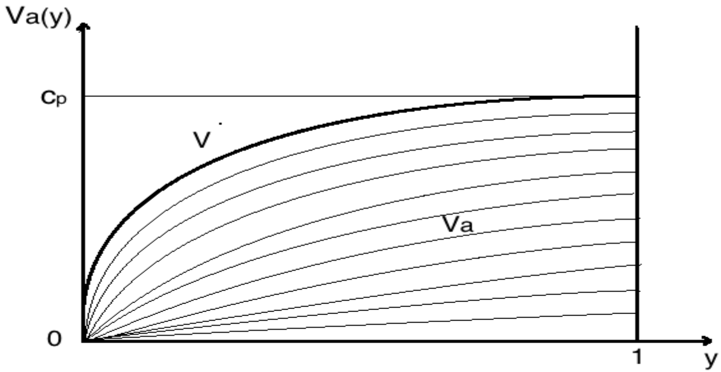
• Whole space case: [PL Lions, JAM 85]
if $p > 1$ and v classical solution of $-\Delta v = |\nabla v|^p$ in $\mathbb{R}^n$, then v is constant

• Elliptic half-space case is important for study of (boundary) GBU (see later)

- Thm 3 also known to be true for $1 < p \leq 2$ provided v is bounded above

[Porretta-Véron, Adv. Nonl. Stud. 06]

- Thm 3 $\implies$ v solves the ODE $-v'' = |v'|^p, \ s > 0$ with $v(0) = 0$



SKETCH OF PROOF OF THEOREM 3

- Write $x = (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty)$ and fix any $h \in \mathbb{R}^{n-1} \setminus \{0\}$. Let

$$z(\tilde{x}, y) = v(\tilde{x} + h, y) - v(\tilde{x}, y), \qquad (\tilde{x}, y) \in \mathbb{R}^{n-1} \times [0, \infty)$$

Goal: show $z \equiv 0$ by contradiction, assuming $\sup_{\mathbb{R}^n_+} z > 0$.

- Local Bernstein estimate [PL Lions 85] $\Longrightarrow$ for all $(\tilde{x}, y) \in \mathbb{R}^{n-1} \times (0, \infty)$:

$$|\nabla v(\tilde{x}, y)| \leq C(n,p) y^{-1/(p-1)}, \quad |v(\tilde{x}, y)| \leq C(n,p) y^{(p-2)/(p-1)},$$

$\Longrightarrow$ supremum of z is finite and localized in a *finite strip*

- Translations parallel to the boundary + compactness procedure

$\Longrightarrow$ supremum of z localized at a *finite point*

- The new function z_∞ satisfies a linear equation with (locally bounded) drift, along with $z = 0$ on $\partial\mathbb{R}^n_+$

$\Longrightarrow$ contradiction with Strong Maximum Principle

APPLICATIONS OF THEOREM 3 TO SPACE PROFILE

[Filippucci-Pucci-S. CPDE 19]

- **Sharp gradient estimate**

$$|\nabla u| \le (1+\varepsilon)d_p\delta^{-\beta} + C_\varepsilon \quad \text{in} \quad \Omega \times [0,T), \qquad \beta = \frac{1}{p-1}, \quad d_p = \beta^{-\beta} \qquad (\forall \varepsilon > 0)$$

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- **Sharp GBU profile in normal direction:** For any **GBU point** $a \in \partial\Omega$,

$$\boxed{\lim_{s\to 0} s^\beta \nabla u(a + s\nu_a, T) = d_p \nu_a} \qquad (\nu_a := \text{inner unit normal vector})$$

In particular

$$|\nabla u(x,T)| \;\sim\; d_p \delta^{-\beta}, \qquad \text{as } x \to a \text{ with } x - a \perp \partial\Omega$$

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Remarks.

a) Early results on GBU profile in 1d [Alikakos-Bates-Grant PRSE 89]

b) Sharp gradient estimate with $\varepsilon = 0$:

$$|\nabla u| \le d_p \delta^{-\beta} \big[1 + C\delta^\alpha\big] \quad \text{in} \quad \Omega \times [0,T)$$

Known for Ω convex and $p < 3$, or $\Omega = B_R$ or annulus [Attouchi-S., CVPDE 20]

APPLICATIONS OF THEOREM 3 (cont'd)

- **More singular tangential behavior:** $\lim_{x \to a,\, x \in \partial\Omega} |x-a|^{\beta} u_{\nu}(x,T) = \infty$

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$$-u_{\nu\nu} \sim (u_\nu)^p, \qquad |u_\tau| \ll u_\nu, \qquad |u_{\tau\tau}| \ll |u_{\nu\nu}|$$

in the region of $(0,T)\times\Omega$ where $|\nabla u| \gg 1$.

Asymptotic scheme: $u_t - \boxed{u_{\nu\nu}} - u_{\tau\tau} = \big(\boxed{u_\nu^2} + u_\tau^2\big)^{p/2}$

Remark. Analogue of [Merle-Zaag CPAM 98]

$u_t \sim u^p$ in $\{u \gg 1\}$ for semilinear heat equation $u_t - \Delta u = u^p \quad (p < p_S)$

Proved by means of Liouville type theorem for ancient solutions

Significant difference: normal spatial direction instead of time direction

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- **Also applications to post GBU behavior** (of global viscosity solution).

MORE ON THE TANGENTIAL GBU PROFILE

Q (in single-point GBU): along $\partial\Omega$, how fast is u_ν damped away from the GBU point ?

1. General result (cf. above): For any GBU point a, the final profile is *more singular* in tangential direction:

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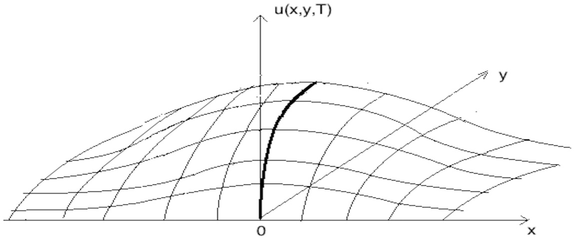
2. Sharp profiles ($n = 2$)

Theorem. [Porretta-S. IMRN 17] *Consider the situation of the single point GBU Theorem, in the locally flat case, with u_0 symm. decreasing in x. Assume $2 < p \le 3$. Then:*

$$\boxed{u_y(x,y,T) \approx d_p\Big[y + C|x|^{2(p-1)/(p-2)}\Big]^{-1/(p-1)}} \qquad \text{for } x, y \text{ small,}$$

with $d_p = (p-1)^{-1/(p-1)}$. Also $|u_x| \le C$.



In particular (final profile of the normal derivative on the boundary):

$$\boxed{u_y(x,0,T) \approx |x|^{-2/(p-2)}}$$

REMARKS (I)

1) Lower estimate true for any $p > 2$. Upper estimate for $p > 3$: open problem

2) GBU profile strongly **non-isotropic**. Different exponents of singularity profile :

- normal direction: $\frac{1}{p-1}$ (self-similar)
- tangential direction: $\frac{2}{p-2}$ (non self-similar)
- time direction (monotone case): $\frac{1}{p-2}$ (non self-similar)

Parabolic relation between time and tangential space directions

3) Comparison with NLH $\quad u_t - \Delta u = u^p \quad (p < (n+2)/(n-2))$

Stable blowup profile is isotropic

$$u(X,T) \sim c(p)|X|^{-2/(p-1)}|\log|X||^{-1/(p-1)} \quad \text{as } X \to 0.$$

HEURISTICS for the exponents $1/(p-2)$ and $2/(p-2)$

1) Maximum principle bounds: $|u_t| \leq C$, $|u_x| \leq C$, $u_{xx} \geq -C$.

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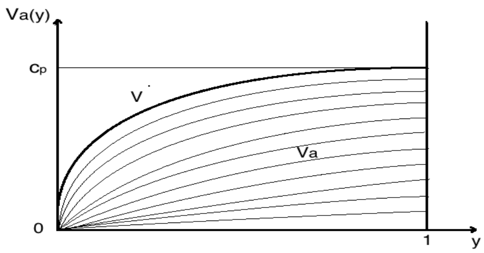
3) Quasi-stationary approximation with family of $1d$ steady-states

$$\begin{cases} -V_a'' = V_a'^p, & y > 0, \\ V_a(0) = 0, & V_a'(0) \in [0, \infty] \end{cases}$$

$V_0(y) = V(y) := c_p y^{(p-2)/(p-1)}$ (singular)

$V_a(y) = V(y+a) - V(a)$, $a > 0$ (regular)



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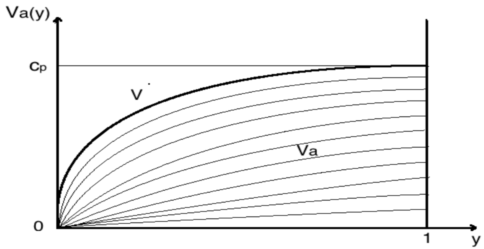

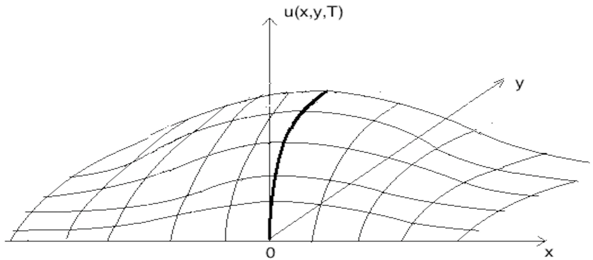

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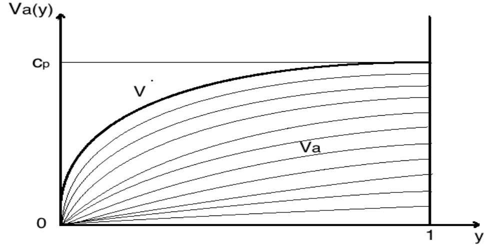

Approximate solution by modulating in a: $\boxed{U(x,y,t) = V(y + a(x,t)) - V(a(x,t))}$

$\boxed{\frac{1}{p-2} \text{ and } \frac{2}{p-2} \textbf{ minimal} \text{ exponents compatible with maximum principle bounds}}$

Warning ! $\exists$ approximate solutions of exponents α and 2α for any $\alpha \ge \frac{1}{p-2}$ ...

TANGENTIAL PROFILE: IDEAS OF PROOF

• Maximum principle bounds

$$|u_t| \le C, \qquad |u_x| \le C, \qquad u_{xx} \ge -C, \qquad u_{xy} \le u_{xx} + C$$

1) Lower estimate of tangential profile

• Direct ODE analysis of the profile on the slice $x = 0$

• Boundary Harnack type elliptic inequalities (at $t = T$)
in scaled boxes centered on a parabola $y = Kx^m$

• Averaging procedure

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2) Upper estimate of tangential profile

• Friedman-McLeod type functional $J = u_x + kxy^{-q(1-\beta)}(1 + Ay)u^q$

• Local regularizing barriers

$$z = z(x, y, t) = c_p\big[(y + \varphi(x, t))^{1-\beta} - \varphi(x, t)^{1-\beta}\big] - \kappa\frac{y^2}{2}$$

with

$$\varphi(x, t) = \eta\, t^{1/(1-\beta)} \left(\frac{r^2 - (x - x_0)^2}{r}\right)^{2/(1-\beta)}, \quad \beta = \frac{1}{p-1}$$

OPEN PROBLEM

- Can one find other profiles ?
- Classification ?