Multichannel Scattering Theory

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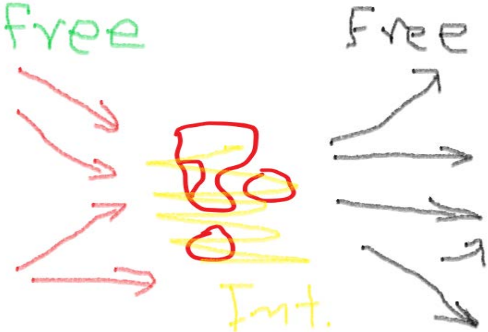
General Nonlinear Schrödinger Equations

Scattering!!



We consider global solutions to the nonlinear Schrödinger equation

$$\begin{cases} i\partial_t \phi + \Delta\phi = \mathcal{N}(|\phi|, x, t)\phi, \\ \qquad \phi(0,x) = \phi_0 \in H^1_{rad}(\mathbb{R}^3) \cap L^2_{rad}(\mathbb{R}^3, |x|^{\frac{1}{2}} dx), \end{cases}$$

with solutions that satisfy the global H^1 bound

$$\sup_{t\in[0,\infty)} \|\phi\|_{H^1(\mathbb{R}^3)} < \infty.$$

We impose the uniform decay assumption on the interaction: for any $\alpha \in (\frac{1}{3}, 1)$ and $F(\lambda)$ being a smooth characteristic function of the interval $[1, +\infty)$, there exists constants $\beta_0 = \beta_0(\alpha, F) > 1$ and $C = C(\alpha, F)$ such that

$$\left| F(\frac{|x|}{t^\alpha} \geq 1)\mathcal{N}(|\phi|, x, t) \right| \leq Ct^{-\beta_0}, \quad \forall t \geq 1.$$

For the sharp estimates on the structure of the localized solution, we need in some cases the assumptions that the interaction is analytic in ϕ, ϕ^* and that $(x \cdot \nabla)^N V(x, t) \in L^\infty$; $N = 0, 1, ...$

Key Questions:

Is it true? How to prove it?

Describe properties of the solution.

The classical starting Point: Prove the solution spreads- Local Decay Estimates.

Previous Results

Examples

Two body Scattering:

$$\mathcal{N}(|\phi|, x, t) = V(x); < x >^{\sigma} V(x) \lesssim 1.$$
$$\mathcal{N}(|\phi|, x, t) = V(x, t); < x >^{\sigma} V(x, t) \lesssim 1.$$

Time independent case initiated by Moller (1945), Friedrichs (1948) Wave Operators, Ekstein(1956), Berezin-Fadeev-Minlos (1961) S matrix. Then, huge amount of work was done by resolvent estimates, going back to Cook(1957), Kato, Birman, Agmon,... *Time dependent methods* approach: Enss (1978), Mourre (1979). L^p Theory: Journé- Sogge-S (1990), Yajima, Rodnianski-Schlag,... **Time Dependent Potentials** The results are only in special cases: L^1 decay in time by Howland, Time periodic by reduction to Floquet operator, some generalizations. L^p Theory: (X. Wu-S. 2020).

Wave Operators, Scattering Matrix

$$H = H_0 + V$$
$$s - \lim_{t \to \infty} e^{iH_0 t} e^{-iHt} P_c(H)\psi(0) = (\Omega^-)^* \psi(0)$$
$$S = (\Omega^-)^* \Omega^+.$$

Multichannel Scattering- Linear

By resolvent methods: 3-body Scattering Faddeev (1963), N-body: Sigal(1982). Many spectral assumptions.
By Time Dependent methods: Enss Method 3-body, Enss (1983); Mourre Theory and Propagation Estimates N-body Sigal-S (1986), Graf (1990), Derezinski(1993)...

Channels

In multichannel Scattering we need to construct, for each channel of asymptotic state, the corresponding **channel wave operator.**

$$s - \lim_{t\to\infty} e^{iH_a t} J_a e^{-iHt} P_c(H)\psi(0) = \Omega_a^* \psi(0),$$

$$\sum_a J_a = Id$$

$$J_0 = F(|x| \le t^{\alpha}), \quad \alpha < 1$$

$\Omega_0^* = 0$, a consequence of Local Decay.

Preliminary Remarks: In the time independent case, we only need these limits for a dense set of initial data.
$\Omega_0^* = 0$ **does not hold** in general, for the time dependent Interaction case.

Nonlinear/General Multichannel Scattering

In this case we need first to find **the possible asymptotic states!**
Fortunately, for many relevant physical systems we have details:
Besides the free wave, there are localized **coherent structures**:
Solitons, Dark Solitons, Breathers, Kinks, Vortices, Black-holes, Topological Solitons, Skyrmions, Hedgehogs, Peakons, Monopoles;
what about other?
Known results in the **Integrable case**:
Recent works (following tools from Deift-Zhou (1993))
* R. Jenkins, J. Liu, P. Perry and C. Sulem (The derivative NLS equations). * G. Chen - J. Liu- B.Lu (2020) proved all asymptotic states of NL Sine Gordon equation are made of solitons, breathers, kinks and wobbly kinks, and free waves. * Chen-Liu (2019) for the modified KdV equation.

Nonintegrable Multichannel Scattering

The first class of results in this direction were focused on **Asymptotic Stability** of the coherent structures: small perturbation of a coherent structure typically propagates to a nearby coherent structure and a free wave. This was initiated in the works of MI Weinstein-S (1988-2004), for NLS and NLKG with a localized potential term. Since then, there has been a huge amount of work to this day. The problems of Kinks and Black-holes for example, are intensely studied at this time. But besides solitons and nonlinear bound states (in the presence of a localized potential), much to be desired for all coherent structures. **Asymptotic Completeness** however, requires us to move away from the case of small perturbation.

In recent years AC was established for various classes of Nonlinear equations. Most notably, the method initiated by Kenig and Merle(2005) and Duyckaerts-Kenig-Merle(2012-2021) allowed the solution of the NLWE with *critical power nonlinearity* problem, with and without a localized potential. Their method is based on bubble decomposition of the solution, Concentrated compactness, Virial estimates, and recently more refined estimates that hold near the light-cone; further notable works: Jia-Kenig(2017), H. Jia, B. Liu and G. Xu.(NLW with potential-radial),Jia-Kenig-Liu-Schlag,...

The Nonlinear Schrödinger equation

For NLS equation the situation is different. There are no results of the above type, with the exception of the work of Tao: A potential term and a *defocusing* nonlinearity, in very high dimension. In this case, building on a method to be discussed below, Tao proved all solutions evolve to a localized part plus a free wave:
Tao(2004-2008), Killip-Oh-Pocovnicu-Visan(2017), T. Roy(2017) bi-Laplacian NLS.
The main results of Tao are similar to ours: the solution of NLS equations (in the inter-critical cases) converges in H^1 to a weakly localized part plus a free wave. Furthermore, the localized part is smooth, up to a part that converge to zero in $\dot{H}^1$. Tao's work is based studying the behavior of the interaction (Duhamel) term. Weak convergence and compactness methods, decomposition to incoming/outgoing waves and microlocalization via Fourier integral methods. As Tao points out, the novelty in the tools used is the decomposition into incoming and outgoing waves.

New Results for Schrödinger type equations

Our main result is the following Theorem:
Let $\phi(t)$ be a global solution to the NLS equation satisfying our conditions, then we have the following asymptotic decomposition

$$\lim_{t\to+\infty} \|\phi(t) - e^{i\Delta t}\Omega_f^* \phi_0 - \phi_{wl}(t)\|_{H^1(\mathbb{R}^3)} = 0.$$

Here Ω_f^* **is the bounded nonlinear scattering wave operator**, mapping the initial data to the asymptotic free wave; ϕ_{wl} is the weakly localized part of the solution with the following properties

1. It is localized in the region $|x| \leq t^{\frac{1}{2}}$, in the following sense

$$(\phi_{wl}, |x|\phi_{wl}) \lesssim t^{\frac{1}{2}}.$$

2. It is smooth, and for $k \geq 1$,

$$\|(x \cdot \nabla_x)^k \phi_{wl}\|_{L^2_x} \lesssim 1.$$

   $k \leq K$, K depends on the regularity of the potential term, if present.

3. If the solution $\phi(t)$ is time periodic, then

$$\|x\phi_{wl}\|_{L^2_x} \lesssim 1.$$

All the estimates hold uniformly in time for $t \geq 0$.
In the special case when the interaction term is a time-independent potential, one gets a new direct proof of Asymptotic Completeness; Radial symmetry is not assumed in this case.

New Results for Localized Part of Schrödinger type Equations

The localized (asymptotic) solution breaks into two parts: Localized around the origin, and smooth.
A second part is a HALO, which has its own propagation set. This Halo part has non-zero mass (L^2 norm) for all times, but carries *zero energy* to infinity. As such it acts like a **Zombie state,** (ignoring the gravitational energy due to the non-zero mass.)
Further property of the halo is its localization in the phase-space. We show that it is localized where

$$\bigcup_{0<\alpha\leq 1/2} \{|x|/t^{\alpha} \sim 1; |\gamma| \sim t^{-\alpha}\}.$$

So, this is a function with the property that the radial derivative bring a factor of $t^{-\alpha}$ at the point in space $|x| \sim t^{\alpha}$. This is exactly how a self similar function behaves! So one expects that we can approximate the solution by a function $t^{-\frac{d}{2}\alpha} S(|x|/t^{\alpha}, t)e^{iEt}$, with S regular and sharply localized. d the dimension.

Looking for a special solution for such S, it shows up as a solution of an approximate Elliptic Equation, with the leading Nonlinear term for large t as the interaction part. Such solutions are the familiar Solitons. A recent result pointing in this direction is in Killip-Oh-Pocovnicu-Visan (2017). When the nonlinear terms are rational functions of ϕ only, then the leading equation will be a monomial. Such solutions in general blow-up in three dimension (or vanish), but for a general interaction, other terms which depend on time (or space) may take over to stop the blow-up. Of course it is also possible that there exist solutions with time-dependent envelopes $S(x, t)$. For example solutions which live on different scales for large times, and for large x.

Non-radial cases and Klein-Gordon type equations

In recent followup works some parts of the above theorem were extended to include non-radial data. Furthermore, it was also extended to the case of Klein-Gordon equations. Some of the above results hold in arbitrary dimension. These are joint works with Xiaoxu Wu. The new tools are based on new constructions of Channel Wave Operators.

We consider a general class of Klein-Gordon type equations of the form:

$$\begin{cases} (\Box + 1)u = -\mathcal{N}(u,x,t) = -V(x,t)u - \mathcal{N}_0(u)u \\ \vec{u}(0) := (u(x,0), \dot{u}(x,0)) = (u_0(x), \dot{u}_0(x)) \in \mathcal{S} \end{cases}, \ (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tag{1}$$

for a Hilbert space $\mathcal{S} = H^1 \oplus L^2$, with space dimension $n \geq 1$.
Here $\Box := \partial_t^2 - \Delta_x$ and $\mathcal{N}_0(u)$ is real.
(Theorem $n \geq 1$) **Localized time-dependent potential** $\mathcal{N}(u,x,t) = V(x,t)u$, such that either $V(x,t) \in L_t^\infty L_{\delta,x}^2(\mathbb{R}^n \times \mathbb{R})$ or $V(x,t) \in L_t^\infty L_{\delta+n/2,x}^\infty(\mathbb{R}^n \times \mathbb{R})$ for some $\delta > 1$. In addition,

$$\|V(x,t)u(t)\|_{L_t^\infty L_x^2} \lesssim \sup_{t\in\mathbb{R}} \|u(t)\|_{\mathcal{H}_x^1}.$$

Typical example is

$$\mathcal{N}(u,x,t) = V(x,t)u + a(x)u^2 + b(x)u^3, \quad \text{in 1 dimension}$$

provided that we have global existence in $\mathcal{H}^1 \oplus \mathcal{H}^0$.
In higher dimensions typical examples are

$$\mathcal{N}(u,x,t) = V(x,t)u + \lambda u^3 + \lambda' u^4, \quad \text{in 3 or higher dimensions.}$$

More generally, one can control

$$(\Box + 1 + V(x,t))u = f(u)u, \quad 1 + V(x,t) \geq v_0 > 0,$$

with

$$\sup_t \|f(u)\|_{L^2_x} < \infty.$$

Here $V(x,t)$ can be of general charge transfer type, that is,
$V(x,t) = \sum_{j=1}^{N} V_j(x - g_j(t)v_j, t)$.

The existence of self similar weakly localized states

$$\begin{cases} i\partial_t\psi = H_0\psi + \langle t\rangle^{-2\epsilon}V(\frac{x}{\langle t\rangle^{\epsilon}})\psi + \mathcal{N}(|\psi|)\psi \\ \psi(x,t_0) = \psi_s(x) + e^{-i\epsilon D\ln(\langle t_0\rangle)}\psi_b(x) \in \mathcal{H}^1_x(\mathbb{R}^n) \\ \sup_{t\in\mathbb{R}} \|\psi(t)\|_{H^1_x} \lesssim_{\psi(t_0)} 1 \\ \|\mathcal{N}(|f(x)|)\|_{L^2_x\cap L^p_x} \lesssim \|f(x)\|_{H^1_x} \\ \text{Both } \psi(t_0) \text{ and } V(x) \text{ are radial in } x \end{cases}, (x,t)\in\mathbb{R}^n\times\mathbb{R} \tag{2}$$

for some $p \in (1,2)$, when $n \geq 5$ and $\epsilon \in (0,1)$ will be chosen small. Here $V(x), \psi_b(x)$ satisfy $H\psi_b = E\psi_b$,($H := H_0 + V(x)$) and

$$\begin{cases} \|D\psi_b(x)\|_{L_x^\infty} \lesssim 1 \\ \|\frac{1}{\lambda - H} P_c\|_{L_x^\infty \to L_x^\infty} \lesssim 1 \end{cases}, \tag{3}$$

and $\psi_s(x)$ is a soliton of

$$i\partial_t \phi = H_0 \phi + \mathcal{N}(|\phi|)\phi, \tag{4}$$

We show that the weakly localized part of (2) has at least two bubbles: a non-trivial self-similar part and a non-trivial localized part near the origin.
If $\mathcal{N}, V(x), H$ satisfy (2), then when $n \geq 5$, $\epsilon > (2/n, 1/2)$,

$$\tilde{A}(\infty) := \lim_{t\to\infty} e^{i\lambda t}\tilde{a}(t) \tag{5}$$

exists and

$$\psi_{w,l}(x,t) = c(t)e^{-i\epsilon D\ln(\langle t\rangle)}\psi_b(x) \oplus \psi_c(x,t) \tag{6}$$

$$c(t) := (e^{-i\epsilon D\ln(\langle t\rangle)}\psi_b(x), \psi_{w,l}(x,t))_{L^2_x}, \tag{7}$$

$$(e^{-i\epsilon D\ln(\langle t\rangle)}\psi_b(x), \psi_c(x,t))_{L^2_x} = 0, \tag{8}$$

$c(t) \geq 1/10$ for all t large enough .

Furthermore, for $\delta = \frac{\epsilon}{2}$,

$$\liminf_{t\to\infty} \|\chi(|x| \le t^{\delta})\psi(t)\|_{L^2_x} \ge c' \tag{9}$$

for some $c' > 0$. Moreover, the $\langle t\rangle^{\epsilon}$-self-similar channel wave operator

$$\Omega^*_{\langle t\rangle^{\epsilon}}\psi(0) := w\text{-}\lim_{s\to\infty} e^{isH} e^{\epsilon iD \ln(\langle T^{-1}(s)\rangle)}\psi(T^{-1}(s)) \tag{10}$$

exists in L^2_x and

$$\Omega^*_{\langle t\rangle^{\epsilon}}\psi(0) = \tilde{A}(\infty)\psi_b(x). \tag{11}$$

New Approach to Scattering: Exterior Propagation Estimates

The proof is based on the following starting point: We do not need local decay; instead, we localize far enough from the interaction support, and then prove the solution satisfies **Propagation Estimates** which are similar to that of the free wave. We can then subtract the exterior part, and be left with the localized part. In particular, we conclude that the free channel wave operator exists with partition $J_a = F(|x|/t^{1/2} \geq 1)$. Once this step is accomplished, we find that the localized part can spread, sub-ballistically. We then prove propagation estimates for the spreading part as well. Secondly, we develop ways to estimate the nonlinear terms by phase-space methods.

Exterior Propagation Estimates

The **Propagation Estimates** we prove are a-priory estimates on the solution of the equation; they imply that the solution vanishes at some explicit rate in regions of phase-space, which exclude the propagation set. The propagation set for the free flow is given by:

$$|x|/t = v, \quad v = \nabla_p \omega(p), \quad p = -i\nabla_x.$$
$$\omega(p) = p \cdot p = -\Delta.$$

The propagation estimates are proven by microlocal positive commutator methods. We construct (many) Propagation Observables, in short PROB, which are bounded (or bounded above) operators, such that the **the expectation** of these operators, which may depend on time, is monotonic increasing up to integrable corrections. That is, $B(t)$ is a PROB if

$$(\phi(t), B(t)\phi(t)) \lesssim 1$$
$$\partial_t(\phi(t), B(t)\phi(t)) \geq (\phi(t), C^* C\phi(t)) + \mathcal{O}(L^1_t)$$
$$B(t) = B^*(t), \quad \mathrm{D}(B(t)) \supset H^1.$$

Constructing the Initial Decomposition

The proof is broken to three steps. The first is to show that the solution breaks into parts, one of which is localized in the region $|x| \lesssim t^{1/2}$.
We let

$$B_1 = F_1(\frac{|x|}{t^\alpha} \geq 1)\gamma_g F_1(\frac{|x|}{t^\alpha} \geq 1).$$

Using the notation $\langle f, h \rangle$ to denote scalar product in L^2, we have

$$\partial_t \langle \psi(t), B_1 \psi(t) \rangle = \langle \psi(t), i[-\Delta + \mathcal{N}, B_1]\psi(t) \rangle + \left\langle \psi(t), \frac{\partial B_1}{\partial t} \psi(t) \right\rangle =$$

$$\left\langle \psi(t), t^{-\alpha} G_1(\frac{|x|}{t^\alpha} = 1)[\gamma_g^2 - \gamma_g \alpha |x|/t] G_1(\frac{|x|}{t^\alpha} = 1)\psi(t) \right\rangle +$$

$$\left\langle \psi(t), t^{-3\alpha} \tilde{G}_1(\frac{|x|}{t^\alpha} = 1)\psi(t) \right\rangle + \Re \left\langle \psi(t), t^{-1-0} \mathcal{O}(1)\gamma_g \psi(t) \right\rangle.$$

$\alpha m \equiv 1 + 0.$

Here G and $\tilde{G}$ stand for smooth characteristic functions supported in a neighborhood of the derivative of F_1. The first identity above is the Heisenberg derivative of the operator B_1 with respect to the time flow of the nonlinear Schrödinger equation. The second term comes from symmetrization of the commutator with the Laplacian. This symmetrization involves two more commutations, and that is why we get a factor of $t^{-3\alpha}$. We are assuming that the solution is bounded in H^1 for all times, so both sides of this equation are well defined.

$$2\gamma_g = p \cdot \nabla g + \nabla g \cdot p, \;\; p = -i\nabla, \quad g = g(|x|).$$

$\cdot$ stands for scalar product. Such construction is classical, and appeared in many fields and applications: Morawetz estimate, Lavine's work on N-body repulsive potentials, Multipliers used in mathematical general relativity, proof of Asymptotic Completeness for N-body Quantum Mechanics, by Sigal-S and more. In these works γ_g is a multiplier, with non negative commutator with the Laplacian, up to higher order terms. Here our first choice and use is different. In fact we choose the potential function g so that the commutator of γ_g with the radial Laplacian is zero in the exterior domain. This is done by choosing $g = r \equiv |x|, \;\; r > 2$.

We now get our first *propagation estimate*, since (on the RHS) the first term is nonnegative, being localized on the boundary of F_1, and the other two terms are integrable over time, to infinity. Therefore, integrating both sides of the above identity, we get (since the integral of the LHS is uniformly bounded) the propagation estimate

$$\int_1^T \|G_1(\frac{|x|}{t^\alpha} = 1)\gamma_g \psi(t)\|_{L^2}^2 \frac{dt}{t^\alpha} \leq c\|\psi\|_{H^{1/2}}^2,$$

$$\lim_{t\to\infty} \langle \psi(t), B_1\psi(t)\rangle \equiv \Gamma \quad \textbf{exists}.$$

To proceed we note that $\Gamma < 0$ corresponds to incoming waves coming from infinity for all large times. This can be shown to be impossible, as it would imply the blowup after finite time, in contradiction to out assumption that the solution is global in H^1. To see the reason for that, we note that the Heisenberg derivative of the function $|x|\tilde{F}_1(\frac{|x|}{t^\alpha} \geq 1)$ is given by B_1. Therefore the above estimates would imply that $|x|\tilde{F}_1(\frac{|x|}{t^\alpha} \geq 1)$ will be zero after finite time. The case when $\Gamma > 0$ corresponds to the presence of a free wave at infinity, but the state can also have a localized part. This is the generic situation.

When $\Gamma = 0$ there is no free wave. This corresponds to a (weakly) localized state.
One can then use further PROBs to prove that the weakly localized state have the property that

$$\langle \psi(t), |x| \psi(t) \rangle \leq c t^{1/2}, \quad t >> 1.$$

This concludes part A of the proof.

To this end, we would like to prove the existence of the following strong limits in L^2 and then extend it to H^1.

$$s-\lim_{t\to\infty} e^{-i\Delta t} F_1(\frac{|x|}{t^\alpha}\geq 1)\psi(t)\equiv \Omega_F^*\psi(0).$$

For this we use Cook's argument, and write this expression as an integral over time of the derivative w.r.t. time. We then immediately see that we need a propagation estimate on the boundary of F_1. Specifically, we need an estimate of the type:

$$\int_1^\infty \|F_1'|\gamma_g|^{1/2}\psi(t)\|^2\frac{dt}{t^\alpha}\leq c<\infty. \quad F_1'=\partial_r F_1(r).$$

$\alpha\leq 1/2.$

This is proved by a series of (many) propagation estimates, each one is used to cover a part of the phase-space supported in $\frac{|x|}{t^\alpha}\sim 1$. We first focus on the case $\alpha=1/2$, which is sufficient to prove the main result.

First, we use the PROB B_2 given by

$$B_2 = F_1(\frac{|x|}{t^\alpha} \geq 1) F_2(t^\beta \gamma_g \geq c) F_1(\frac{|x|}{t^\alpha} \geq 1)$$
$$\beta < \alpha, \quad \text{or}$$
$$\beta > \alpha \quad \text{or}$$
$$\beta \sim \alpha.$$

Now, for $\alpha = 1/2$, the contribution to the Heisenberg derivative from the interaction term is integrable in time, if the interaction term vanishes faster than $1/(1 + |x|^2)$ at infinity. So, we concentrate of the contribution of the commutator of the Laplacian with B_2.

It is easy to see that the Heisenberg derivative (w.r.t. Laplacian) of the operator F_2 is positive:

$$D_H F_2 = \beta t^{\beta-1}\gamma_g F_2' \approx c t^{-1} F_2' \geq 0.$$

The Heisenberg derivative contribution of F_1 gives

$$t^{-\alpha} G_1(\frac{|x|}{t^\alpha} \sim 1)[\gamma_g - \alpha\frac{|x|}{t}]F_2 G_1(\frac{|x|}{t^\alpha} \sim 1) + \mathcal{O}(t^{-3\alpha+\beta}) \geq$$
$$t^{-\alpha} G_1(\frac{|x|}{t^\alpha} \sim 1)[c t^{-\beta} - \alpha t^{\alpha-1}]F_2 G_1(\frac{|x|}{t^\alpha} \sim 1) + \mathcal{O}(t^{-3\alpha+\beta}).$$

Several arguments were used to derive the above estimate. We can not use standard pseudo-differential calculus here, since functions of γ_g are not ΨDO. We use **commutator expansion lemmas**, Sigal-S(1987) Hunziker-Sigal-S(1999), Helffer-Sjostrand(1989), Amrein-Boutet de Monvel-Georgescu(1996)

The remainder term comes from symmetrization as before, and involves two more commutators. That gives the factor $t^{-3\alpha}$. However, we have one commutator (with x) hits the operator F_2, which gives a positive power t^{β}. Note that the functions of γ_g are defined in terms of the spectral theorem for self-adjoint operators, and therefore can not be defined for the radial derivative, without smoothing near the origin. Finally, to get the positivity and the resulting Propagation Estimate, we need to impose

$$ct^{\beta} > \alpha t^{\alpha-1}.$$

So, in fact to cover all of the relevant phase-space, it is necessary to inductively choose neighborhoods that are covering all possibilities, by making many choices of
of the constants localizing $\frac{|x|}{t^\alpha}, t^\beta \gamma_g$ together, while keeping the symmetrization terms integrable. The difficulty in getting this estimate follows from the fact that in principle the solution can propagate inside the set $\frac{|x|}{t^\alpha} = a$, **and** $\gamma_g = at^{\alpha-1}$, which is nothing but the *propagation set* for the free flow where $x = vt + c$, with v the velocity. The way to prove propagation estimate in this region, is to reduce it to an estimate on the boundary of an operator that contains this set, **and** such that the boundary is **away** from the propagation set.

Incoming/Outgoing Wave Decompositions- a Remark

Goes back to Engquist-Majda(1977)(wave equation), Enss (1978)(space and momentum), Mourre(1979)(Spectrum of the Dilation operator) , Sigal-S (functions of operators) (1986),Tao (Fourier-IO) (2004), Stucchio-S(Coherent state beams)(2007), S(2011)(Analytic functions), Beceanu-Deng-Y.Wu-S (FIO)(2018) and more! Also : Greengard-Stucchio(2008-9).
What was before that in Scattering Theory? Sommerfeld Radiation Condition(1912) and similar.

Here we use the analytic functions of $2A \equiv x \cdot p + p \cdot x$:
Let the outgoing projection

$$P_M^+(A) \equiv \frac{1}{2}(1 + tanh\frac{A - M}{R}), \qquad \text{for } M \gg 1, 2 \leq R \lesssim M^{1-0}.$$

Similarly we define the incoming projection

$$P_M^-(A) \equiv \frac{1}{2}(1 - tanh\frac{A + M}{R}), \qquad \text{for } M \gg 1, 2 \leq R \lesssim M^{1-0}$$

These operators project on A positive or negative, provided that $|A| > M$, up to exponentially small corrections.

The crucial properties of such projections are

$$[-i\Delta, P_M^{\pm}(A)] = \pm G_M(A)(-\Delta)G_M(A)$$

with $G_M(A)$ is an explicit and localized function around $A \approx M$. Furthermore, we notice that

$$P_M^{+}(A)\chi(|x| \leq K) =$$
$$P_M^{+}(A)\langle A\rangle^{-2m}(1+A^2)^m\chi = O(M^{-2m})P_M^{+}(A)O(K^{2m})O(p^{2m})$$

Therefore, the commutator of $P_M^{\pm}(A)$ with a localized smooth interaction term, and a smooth function, is higher order in powers of M.

The Weakly Localized part

The intricate result we have is that the propagation estimates show that, such solutions are also living in a very thin set in the phase space. But, while the free wave concentrates on the propagation set defined by $|x|/t \sim k; p \sim k \quad \forall k \in R^3 \setminus 0$, the localized part concentrates on the set $\{|x|/t^{\alpha} \sim c; p \sim c^{-1}t^{-\alpha} \mid 0 < \alpha \leq 1/2; c > 0\}$. In particular, it points to self-similarity of the localized part for large times, since self-similar functions have this property. This concentration phenomena in the phase-space can be understood by noting that it saturates the uncertainty principle $xp \sim 1$. The PROBs we use now, have to take into account the localization property of the localized state.

The first class of PROBs we use are of the form: Let $M \geq 100$ be a constant, we consider the PROB

$$B = F_1(\frac{\langle x \rangle}{M} \geq 1)\gamma F_1(\frac{\langle x \rangle}{M} \geq 1)$$

for $M \gg 1$.
By assuming $|F_1(\frac{\langle x \rangle}{M} \geq 1)\mathcal{N}(\phi)| \leq M^{-k}$, we have $(B\phi, \mathbf{N}(\phi)) = O(M^{-k})$, k large ($k > 3$ *is sufficient*). This holds true if we have saturated nonlinearity with high power, or time dependent potential that decays fast. We will use this to get a PRES (propagation estimate) the same as before, except that now, we need to use the fact that it acts on a localized state. To do this, we integrate over time twice, and use the fact the the integral over time of (the expectation of) velocity is position. Since the position is bounded by $t^{1/2}$, we get new estimates, on the double integral. from this it follows that there exists a sequence of time $t_n \to +\infty$, such that

$$\|A\phi_{wb}(t_n)\|_{L^2(|x| \leq \sqrt{t_n})} \lesssim 1.$$

We begin with some preliminary comments. We know that the localized part of the solution in supported in $|x| \lesssim t^{\alpha}$. We are interested in the part of the solution with large frequency. So we will restrict attention to the region of phase-space where the solution is localized in space and have frequency larger than $K > 1$. This will be done by estimating the quantity $J_K \phi(t)$, with J_K microlocalizing the solution in the desired region.
We know that there exists a sequence of times along which A^2 has a bounded expectation in the region where $|x| \lesssim t^{1/2}$. Therefore on the support of J_K also the expectation of r^2 will be bounded on a sequence, since $A^2 = r^2 D^2$ plus lower order terms, and $K > 1$. This leads us to consider the following propagation estimate: For each $M > M_0$ with M_0 large, we use the PROB

$$B_{KM} = J_K(r^2 P_M^+ + P_M^+ r^2) J_K. \tag{12}$$

We use that there exists a sequence of times going to infinity, such that the LHS is uniformly bounded. If there is a subsequence with the LHS going to infinity, we consider the time intervals around each maxima. At the maximum, the derivative is zero, which gives a new Virial estimate, that we use to control the solution.

Similarly, if the derivative is negative, we get a favorable Virial estimate. It remains to understand the interval where the derivative is positive. This is done by taking the derivative wrt time of the leading term on the RHS, and estimating the resulting expression. Another complicated contribution comes form the commutator of the Laplacian with the phase-space partition of unity, J_K. For this we use the following estimate:

To complete the microlocalization of the localized state, we prove that For all $\alpha > 0$, the localized state vanishes in the region $|x|/t^{\alpha} = c; \quad p \geq t^{-\beta}, \quad \beta < \alpha.$
We let the PROB be
$t^{-\eta} A_4 = t^{-\eta} F_1(\frac{\langle x \rangle}{t^{\alpha}} \geq 1) F_4(\gamma t^{\beta} > c_0) F_1(\frac{\langle x \rangle}{t^{\alpha}} \geq 1)$. We will choose $c_0 > 0$ large enough for the iteration procedure to apply. We will choose η_n successively smaller, until it is zero. Furthermore, after each derivation of a PRES from the above PROB, we will derive a companion PRES using:

$$t^{-\eta} A_{4,l} = t^{-\eta} \ln^{-1} \langle x \rangle F_1(\frac{\langle x \rangle}{t^{\alpha}} \geq 1) F_4(\gamma t^{\beta} > c_0) F_1(\frac{\langle x \rangle}{t^{\alpha}} \geq 1) \ln^{-1} \langle x \rangle.$$

The resulting positive term, which is controlled by the PRES is then:

$$\int_1^T t^{-\eta-\alpha} \left(\sqrt{\tilde{F}_1'(\frac{\langle x \rangle}{t^{\alpha}} \geq 1)} \gamma F_4(\gamma t^{\beta} > c_0) \sqrt{\tilde{F}_1'(\frac{\langle x \rangle}{t^{\alpha}} \geq 1)} \right) dt \lesssim$$
$$C + Sym + INT,$$

where Sym, INT stand for the symmetrization and Interaction terms and the constant comes from the LHS.

Since the SYM term is smaller than the leading term, though they are not in the same phase-space, by iteration we can remove the effect of SYM.

Next we need to show that we can also iterate the interaction term. We would like to proceed in a way similar to the above, but the problem is, that commuting F_4 through the interaction term is not possible. Even if the interaction term is smooth, we still need to gain a factor of $t^{-\alpha}$ (or equivalently a factor of x^{-1}). This means that the Interaction term should be smooth and such that $(r\partial_r)^n\mathcal{N}$ is bounded for all n.

We will instead introduce a different approach, which is a new *iterative high-low phase-space estimates* . By this we mean that the high-low decomposition will be w.r.t. the spectrum of pseudo-differential operators, rather than the derivative operator. Another complication is that the Interaction term has F_1 in it, and not G_1.

We therefore need to upgrade the basic PRES. This is done by using the companion estimate with the extra *log* factor.

The extra *log* factor gives another estimate, in which the leading PRES is a bound on

$$\int_1^T \| \ln^{-3/2} \langle x \rangle F_1(\frac{\langle x \rangle}{t^\alpha} \geq 1) \langle x \rangle^{-1/2} \sqrt{\gamma F_4} \phi(t) \|^2 t^{-\eta} dt \lesssim C + Sym + INT.$$

With this bound we can control the interaction term, since it is decaying faster than this term. But it is necessary to bring a factor of F_4 to the right place.

Iterative High-Low estimates

Consider the generic term in the interaction and decompose it as :

$$\sum \phi \ \ldots \phi \ V(x) = \sum (F_{4,\beta'}\phi + \bar{F}_{4,\beta'}\phi)^{k+1}(F_{4,\beta'}V(x) + \bar{F}_{4,\beta'}V(x)) =$$
$$\sum (\bar{F}_{4,\beta'}\phi \ )^{k+1}(\bar{F}_{4,\beta'}V(x)) + \sum (\bar{F}_{4,\beta'}\phi \ )^{k} V(x) F_{4,\beta'}\phi +$$
$$\sum (\bar{F}_{4,\beta'}\phi \ )^{k+1} F_{4,\beta'}V(x).$$

ϕ stands for either ϕ or its complex conjugate. $\sum$ stands for summing over all terms of this type. All terms which contain a factor with high γ acting on ϕ, are controlled by the leading term in an iteration, using PROBs with β'. Since the interaction term decays in time faster than the leading term, this extra term is controlled by a PRES with η **larger** than the previous step. Therefore after finitely many such iterations η becomes larger than 1, and then there is nothing to prove.

The term of the form low-low is annihilated by the action of the operator $\sqrt{F}_4 F_1$. This is due to a phase-space estimate of the type

$$F_4(\gamma > K)[\bar{F}_4(\gamma < K/4)\phi]^2 \lesssim K^{-l}.$$

Similar estimate holds with A replacing γ.

The above propagation estimates imply by Cook's method that any part of the solution with $|x| > t^{\alpha}, \quad p > t^{-\beta}, \alpha > \beta$
will converge to a free wave, since the corresponding channel wave operator converges in the strong sense. In other regions of the momentum, the solution converges to zero except when $\alpha = \beta$.

THE END

THANK YOU!