Oneset of nonlinear instabilities in monotonic viscous boundary layers

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Incompressible Navier-Stokes equation

- The 2D incompressible Navier Stokes equations in an half plane

$$\partial_t u^\nu + (u^\nu \cdot \nabla) u^\nu - \nu \Delta u^\nu + \nabla p^\nu = f^\nu, \quad (1)$$

$$\nabla \cdot u^\nu = 0, \quad (2)$$

together with the Dirichlet boundary condition

$$u^\nu = 0 \qquad \text{for} \qquad y = 0, \quad (3)$$

where $\nu > 0$ denotes the viscosity, i.e., the inverse of the Reynolds number Re.

- We are interested in the stability of a shear layer profile $U(y) = (U_s(y), 0)$.
- Note in particular that we consider the profiles like the exponential profile $U_s(y) = 1 - \exp(-\delta y)$ where δ is a positive constant, Blasius profile, but not Couette's profile (Bedrossian, Germain, Masmoudi, Annals of Math, 2017, Vicol, Zhao......).

Background and Motivation

The classical question is to describe the behavior of sequences of solution u^ν to (1)-(2) as the viscosity goes to 0, and in particular to investigate whether these solutions converge, and in which sense, towards solutions u^E of the incompressible Euler equations

$$\partial_t u^E + (u^E \cdot \nabla) u^E + \nabla p^E = f^E, \quad \nabla \cdot u^E = 0. \tag{4}$$

The main problem is that the boundary condition changes in this limiting process, going from $u^\nu = 0$ to

$$u^E \cdot n = 0 \qquad \text{on} \qquad \partial\Omega, \tag{5}$$

where n is an unit vector normal to the surface.

- This question has been intensively studied in the physical literature since the 19^{th} century, starting with Lord Rayleigh, then later by Prandtl, Orr, Sommerfeld, Tollmien, Schlichting, C.C. Lin.

Background and Motivation

In particular, in 1904, L. Prandtl introduced the concept of "boundary layer", namely an area of fast transition near the boundary which allows the transition between (3) and (5).

- A formal analysis shows that this boundary layer should be of size $O(\sqrt{\nu})$, and that the flow in this layer should satisfy

$$\partial_t u_H + (u_H \cdot \nabla_H) u_H + u_V \cdot \partial_Z u_H - \partial_{ZZ} u_H = f \tag{6}$$

$$\nabla_H \cdot u_H + \partial_Z u_V = 0, \tag{7}$$

  with $u_H = 0$ and $u_V = 0$ as $Z = 0$, where u_H is the component of the flow parallel to the boundary, u_V its normal component, $Z = z/\sqrt{\nu}$, and f is a given forcing term (linked to p^E).

- Following Prandtl's analysis, as ν goes to 0, u^ν should be of the form

$$u^\nu(t,x,y,z) = u^E(t,x,y,z) + u^P(t,x,y,\nu^{-1/2}z) + O(\sqrt{\nu}). \tag{8}$$

Background and Motivation

The main problem with Prandtl's equations is that there is no prognostic equation on u_V. The vertical velocity is only recovered through the vertical integration of the horizontal divergence

$$u_V(t,x,y,Z) = -\int_0^Z \nabla_{X,Y} \cdot u_H(t,x,y,z)dz,$$

leading to the loss of one space derivative.

- Positive results
  - Under a monotonicity assumption (seminal work of Oleinik 1963, 1999), existence of solutions.
  - Alexandre-Wang-Xu-Yang, JAMS, 2015; Masmoudi-Wong, CPAM, 2015, constructed Oleinik solution by energy method.
  - In the energy space $L^2(\Omega)$, obtained by T. Kato in 1984, namely u^ν converges to u^E in $L^\infty([0,T],L^2)$ provided

$$\nu \int_0^T \int_{0\leq z\leq \nu} \|\nabla u^\nu\|^2 dxdt \to 0. \tag{9}$$

  - In the recent years, all results for analyticity or Gevrey regularity, or assuming that there is no vorticity in a vicinity of the boundary, see the works of leading mathematicians like M. Sammartino, R.E. Caflisch, N. Masmoudi, Y. Maekawa, V. Vicol, D. Gérad-Varet, Z.P. Xin, L.Q. Zhang, P. Zhang, Z.F. Zhang......

Background and Motivation

- Negative results: For Sobolev regularity of the solutions
  - W. E and B. Engquist, Prandtl layer may blow up in finite time, CPAM 1997.
  - D. Gérard-Varet, E. Dormy, Prandtl equation is linearly ill-posedness, JAMS 2010.
  - E. Grenier, Y. Guo, T. T. Nguyen, linear instability of boundary layer, Adv Math, Duke Math J, 2016.
  - E. Grenier, T. T. Nguyen, Prandtl expansion is false for particular initial data in L^∞ sense, Ann. PDE 2019.
- Along in this direction: we consider the stability of shear layer profile $(U_s(y), 0)$. Note that this shear layer profile is a stationary solution of Navier Stokes equations provided we add the forcing term $f^\nu = (-\nu\Delta U_s, 0)$. We assume that U_s is a smooth function, with $U_s(0) = 0$, $\partial_y U_s(0) \neq 0$ and that U_s converges at $+\infty$ to some constant U_+.

Setup

Let L be the linearized Navier Stokes operator near the shear layer profile U, namely

$$Lv = (U \cdot \nabla)v + (v \cdot \nabla)U - \nu \Delta v + \nabla q, \tag{10}$$

with $\nabla \cdot v = 0$ and Dirichlet boundary condition. We want to study the resolvant of L, namely to study the equation

$$(L + \lambda)v = f \tag{11}$$

where f is a given forcing term and λ a complex number. We look for solutions of the form

$$v = \nabla^\perp \Big(e^{i\alpha(x-ct)} \psi(y) \Big).$$

Taking the curl of (11), we get the classical Orr Sommerfeld equations

$$Orr_{\lambda,\alpha,\nu}(\psi) = (U_s - c)(\partial_y^2 - \alpha^2)\psi - U_s''\psi - \frac{\nu}{i\alpha}(\partial_y^2 - \alpha^2)^2\psi = i\frac{\nabla \times f}{\alpha} \tag{12}$$

with $\nabla \times (f_1, f_2) = i\alpha f_2 - \partial_y f_1$, and the Dirichlet boundary condition:

$$\psi(0) = \partial_y \psi(0) = 0. \tag{13}$$

Two cases

- When $\nu = 0$, OS reduces to Rayleigh equations, which are resolvant of Euler equation

$$Ray_{c,\alpha}(\psi) = (U_s - c)(\partial_y^2 - \alpha^2)\psi - U_s''\psi \tag{14}$$

  which is a second order operator, together with the boundary condition $\psi(0) = 0$.
- Two cases appear:
  - There exists an eigenvalue c with $\Im c > 0$
    - "Euler unstable profile" U_s
    - There exists a growing mode for linearised Euler equations
    - According to Rayleigh's criterium, such instabilities only occur for profiles U_s with inflection points.
  - There is no eigenvalue c with $\Im c > 0$
    - "Euler stable profile" U_s
    - It is the case for monotonic profiles U_s with $U_s'' < 0$.
    - For instance exponential or Blasius profiles.

Euler unstable profiles

- Rayleigh: eigenvalue c and eigenvector ψ, $\Im c > 0$.
- Orr Sommerfeld: for ν close to 0, there exists c_ν, ψ_ν
- $c_\nu \to c$ as $\nu \to 0$.
- $\psi_\nu = \psi(x, y) +$ boundary layer $(x, \nu^{-1/2} y) + O(\nu^{1/2})$
- Time scale $O(1)$
- Spatial scales $O(1)$ and $O(\nu^{1/2})$
  - Nonlinear instability for Euler equations: Grenier (CPAM 2000).
  - Nonlinear instability for Navier Stokes equations: Grenier and Nguyen (Ann PDE 2019).
  - Gives a counter example for Prandtl's analysis

Euler stable profiles

- Rayleigh: no exponentially growing mode
- Orr Sommerfeld: exponentially growing mode for small $\nu > 0$.
  - *Adding viscosity destabilises the flow*
  - How does such an unstable mode evolve and disappear when for instance small rotation, high compressibility, small stratification, small magnetic field is added? This work is pioneered in [Bian, Grenier, arxiv, 2022].
- Structure of the instability
  - Time scale $O(\nu^{-1/2})$: slow instabilities
  - Horizontal scale $O(\nu^{-1/4})$: "Long wave instabilities"
  - Strong spatial anisotropy since their sizes are of order $O(1)$ in y but of order $O(\nu^{-1/4})$ in x. Moreover they grow very slowly, within time scales of order $O(\nu^{-1/2})$. The corresponding eigenvalue λ has a real part $\Re\lambda$ of order $O(\nu^{1/2})$.
- Linear instability by Grenier, Guo and Nguyen (Duke Math J, Adv Math 2016).
- Nonlinear instability is open: This is our goal.

Construction of the instability: overview

- Orr Sommerfeld has two "fast" solutions $\psi_{f,\pm}$ and two "slow" solutions $\psi_{s,\pm}$. The "-" subscript refers to solutions which go to 0 at infinity and the "+" subscript to solutions diverging as y goes to $+\infty$.
- "Slow solutions" $\psi_{s,\pm}$: have bounded derivatives, approximated by solutions of Rayleigh equations
- "Fast solutions" $\psi_{f,\pm}$: very large gradients and higher derivatives. For these solutions Orr Sommerfeld may be approximated by its higher order derivatives, namely by

$$(U_s - c)\partial_y^2\psi - \frac{\nu}{i\alpha}\partial_y^4\psi = 0 \tag{15}$$

$$\theta = \partial_y^2\psi$$

then we recover the modified Airy equations

$$(U_s - c)\theta = \frac{\nu}{i\alpha}\partial_y^2\theta.$$

Construction of the instability: Rayleigh

- solutions for $\alpha = 0$: $U_s - c$ and another one

$$\psi_{+,0}(y) = (U_s(y) - c)\int_0^y \frac{1}{(U_s(z)-c)^2}dz. \qquad (16)$$

- By a perturbative argument, solutions for small α: $(U_s - c)e^{-\alpha y} + O(\alpha)$, another one, $\psi_{+,\alpha}$, which diverges at $+\infty$ and which has a $(y - y_c)\log(y - y_c)$ singularity at y_c.
- Using an iterative scheme, treating $(\partial_z^2 - \alpha^2)^2\psi$ as a perturbation, it is possible to construct genuine solutions of the Orr operator, which are close to $\psi_{\pm,\alpha}$.
- In particular it can be proven that $\psi_{s,-}$ satisfies

$$\psi_{s,-}(0) = U_s(0) - c + \alpha\frac{U_+^2}{U_s'(0)} + O(\alpha c), \qquad (17)$$

$$\partial_y\psi_{s,-}(0) = U_s'(0) + O(\alpha), \qquad (18)$$

$$\partial_y^2\psi_{s,-}(0) = O(1), \qquad (19)$$

where $U_+ = \lim_{y\to+\infty} U_s(y)$.

Construction of the instability: Airy

- Expanding U near y_c at first order, the modified Airy operator may be approximated by

$$-\varepsilon\partial_y^2\psi + U_s'(y_c)(y - y_c)\psi = 0, \tag{20}$$

  which is the classical Airy equation.
- A first solution to (20) is given by

$$A(y) := Ai(\gamma(y - y_c)), \tag{21}$$

  where Ai is the classical Airy function, solution of $Ai'' = xAi$, and where $\varepsilon\gamma^3 = U_s'(y_c)$, namely $\gamma = \left(\frac{i\alpha U_s'(y_c)}{\nu}\right)^{1/3}$.
- Another independent solution to (20) is given by $Ci(\gamma(y - y_c))$ where $Ci = i\pi(Ai + iBi)$, with $Bi(\cdot)$ being the other classical Airy function.

Construction of the instability: Airy

- By Langer's transformation (which is a transformation of the phase and amplitude of the solution), we can get the solution of modified Airy equation.
- To go back to (15) we then have to integrate twice these solutions. As a consequence, fast solutions to the Orr Sommerfeld equation may be expressed in terms of second primitives $Ai(2,.)$ and $Bi(2,.)$ of the classical Airy functions Ai and Bi.
- Finally, it can be proven that $\psi_{f,-}$ satisfies

$$\psi_{f,-}(0) = Ai(2, -\gamma y_c) + O(\alpha), \tag{22}$$

$$\partial_y \psi_{f,-}(0) = \gamma Ai(1, -\gamma y_c) + O(1), \tag{23}$$

$$\partial_y^2 \psi_{f,-}(0) = \gamma^2 Ai(-\gamma y_c) + O(\alpha^{-1}). \tag{24}$$

Construction of the instability: dispersion relation

- An eigenmode of Orr Sommerfeld equation is a combination of these particular solutions which goes to 0 at infinity and which vanishes, together with its first derivative, at $y = 0$. As an eigenmode must go to 0 as y goes to infinity, it is a linear combination of $\psi_{f,-}$ and $\psi_{s,-}$ only. There should exist nonzero constants a and b such that

$$a\psi_{f,-}(0) + b\psi_{s,-}(0) = 0,$$

$$a\partial_y\psi_{f,-}(0) + b\partial_y\psi_{s,-}(0) = 0.$$

  Equivalently the determinant

$$E = \begin{vmatrix} \psi_{f,-}(0) & \psi_{s,-}(0) \\ \partial_y\psi_{f,-}(0) & \partial_y\psi_{s,-}(0) \end{vmatrix}$$

  must vanish.

- The dispersion relation is thus

$$\frac{\psi_{f,-}(0)}{\partial_y\psi_{f,-}(0)} = \frac{\psi_{s,-}(0)}{\partial_y\psi_{s,-}(0)} \tag{25}$$

  or

$$\alpha\frac{U_+^2}{U_s'(0)^2} - \frac{c}{U_s'(0)} = \gamma^{-1}\frac{Ai(2, -\gamma y_c)}{Ai(1, -\gamma y_c)} + O(\nu^{1/4}). \tag{26}$$

Construction of the instability: dispersion relation

We will focus on the particular case where α and c are both of order $\nu^{1/4}$. It turns out that this is an area where instabilities occur, and we conjecture that this is the region where the most unstable instabilities may be found. We rescale α and c by $\nu^{1/4}$ and introduce

$$\alpha = \alpha_0 \nu^{1/4}, \quad c = c_0 \nu^{1/4}, \quad Z = \gamma y_c.$$

Introduce the Tietjens function, of the real variable y

$$Ti(y) = \frac{Ai(2, ye^{-5i\pi/6})}{ye^{-5i\pi/6} Ai(1, ye^{-5i\pi/6})}. \tag{27}$$

At first order the dispersion relation becomes

$$\alpha_0 \frac{U_+^2}{U_s'(0)} = c_0 \Big[1 - Ti(-Ze^{5i\pi/6}) \Big]. \tag{28}$$

Construction of the instability: eigenvector

Let us now detail the linear instability. Its stream function ψ_{lin} is of the form

$$\psi_{lin} = b\psi_{s,-} + a\psi_{f,-}.$$

Choosing $b = 1$ we see that $a = O(\nu^{1/4})$, hence

$$\psi_{lin}(z) = U_s(z) - c + \alpha \frac{U_+^2}{U_s'(0)} + aAi\Big(2, \gamma(z - z_c)\Big) + O(\nu^{1/2}). \quad (29)$$

The corresponding velocity and vorticity u_{lin} and w_{lin} are given by

$$u_{lin} = \nabla^\perp \psi_{lin} \sim \begin{pmatrix} U_s' e^{-\alpha y} + a\gamma Ai(1, \gamma(y - y_c)) \\ -i\alpha U_s e^{-\alpha y} - i\alpha a Ai(2, \gamma(y - y_c)) \end{pmatrix}, \quad (30)$$

$$\omega_{lin} = \nabla \times u_{lin} = -(\partial_y^2 - \alpha^2)\psi_{lin}^+ \sim -U_s'' e^{-\alpha y} - \gamma^2 a Ai(\gamma(y - y_c)). \quad (31)$$

Main idea

We will solve nonlinear instability (longwave instability): construct an instability of the form

$$v = \nabla^{\perp}\left(e^{i\alpha x+\lambda t}\psi(y)\right), \ \lambda = -i\alpha c.$$

- A simple ordinary differential equation's model would be, taking into account symmetries, the classical Hopf's bifurcation, namely

$$\dot{\phi} = \lambda\phi + A|\phi|^2\phi + O(\phi^5) \tag{32}$$

  where $\Re\lambda = O(\nu^{1/2})$. The stability of $\phi = 0$ depends on the sign of $\Re A$.
  - If $\Re A < 0$ then (32) has a stable periodic solution, of size $O(\nu^{1/4})$. In this case, starting from a small initial data, ϕ grows and reaches a magnitude $O(\nu^{1/4})$.
  - If $\Re A > 0$ then the corresponding periodic solution is unstable, and ϕ may grow for ever, and in particular reach $O(1)$.
- The aim of this work is to evaluate the coefficient A for particular shear layer profiles.

Main idea

- The idea is to construct an approximate solution up to the third order, in the form

$$u_{app} = U + \nu^N e^{\lambda t} u_{lin} + \nu^{2N} e^{2\lambda t} u_q + \nu^{3N} e^{3\lambda t} u_c + c.c. \tag{33}$$

  ($c.c.$ meaning complex conjugate), and to understand the geometry of u_c with respect to u_{lin}, in order to know whether cubic terms enhance or stop the linear instability.
- Following a classical strategy in bifurcation theory we compare (33) and (32) to compute the coefficient A.

Difficulties

- The construction of an approximate solution is delicate since the solution has three spatial scales, namely $O(1)$ (that of the shear layer itself), $O(\nu^{1/4})$ (size of the so called "critical layer"), and $O(\nu^{-1/4})$ (horizontal instability size, and also recirculation size). There is no explicit formula for u_q and u_c, thus we have to approximate them numerically.
- Numerical computations are also very difficult. First the vorticity is concentrated in a critical layer of size $O(\nu^{1/4})$ and has a singular behavior as $\nu \to 0$. Next, linearised Euler and Navier Stokes equations (namely Rayleigh and Orr Sommerfeld equations) degenerate in the critical layer, leading to singularities.
- We will mix a precise asymptotic analysis with numerical computations to study A as $\nu \to 0$.

Numerical setting

We have to describe the various functions away from the boundary, near the boundary and in the critical layer.

- Away from the boundary, namely for $\sigma \le y \le Y_0$ we use a grid, with step h. We approximate Orr Sommerfeld equations by Rayleigh equations, thus neglecting the diffusion of the vorticity in this area and we numerically solve Rayleigh equations using an Euler scheme.
- Close to the boundary we first invert Rayleigh equations by looking for solutions under the form

$$\psi_{Ray} = \sum_{n \ge 1} d_n Y^n \log Y + \sum_{n \ge 0} e_n Y^n \tag{34}$$

  where $Y = y - y_c$. Doing this we make an error $Diff(\psi_{Ray})$, which is very large and which will be corrected in the next step. Note that the corresponding vertical velocity has the same form, whereas the corresponding horizontal velocity has an additional term in $\log Y$ and the vorticity has terms in $\log Y$ and in Y^{-1}.

Numerical setting

- In the critical layer, for $0 \leq y \leq \Theta \nu^{1/4}$ (with some large Θ), we use a grid with a step h_c of order $\nu^{1/4}$ and approximate Orr Sommerfeld equations by $\mathcal{A}$, with source term $Diff(\psi_{Ray})$. This leads to an error $ErrAiry(\mathcal{A}^{-1} Diff(\psi_{Ray}))$.

Notes for numerical setting

- Each function of the construction is thus described by its numerical values on two meshes, one ranging from σ to Y_0 with step h, one ranging from 0 to $\Theta\nu^{1/4}$ with step h_c, by three series, one in $y^n \log y$, one in $y^n \log^2 y$, and one in y^n, used for $0 \le y \le \sigma$.
- Far away from the boundary, namely for $y > Y_0$, we note that the vorticity decays exponentially fast, whereas the stream function has a slow decay, like $e^{-\alpha y}$. We will also match ψ_{Ray} and its first derivative at $y = \sigma$, and take care of the boundary conditions at $y = 0$.

Definition of the adjoint

- We will consider the classical L^2 product between two stream functions ψ_1 and ψ_2, namely $(\psi_1, \psi_2) = \int \psi_1 \bar{\psi}_2 dx$. We have

$$(Orr(\psi_1), \psi_2) = (\psi_1, TOrr(\psi_2)),$$

where

$$TOrr(\psi) = (\partial_y^2 - \alpha^2)(U_s - \bar{c})\psi - U_s''\psi + \frac{\nu}{i\alpha}(\partial_y^2 - \alpha^2)^2\psi.$$

- Taking the complex conjugate we define the adjoint of Orr Sommerfeld equation to be

$$Orr^t_{c,\alpha,\nu}(\psi) := (\partial_y^2 - \alpha^2)(U_s - c)\psi - U_s''\psi - \frac{\nu}{i\alpha}(\partial_y^2 - \alpha^2)^2\psi, \quad (35)$$

with boundary conditions $\psi(0) = \partial_y\psi(0) = 0$.

- We also introduce the corresponding adjoint of the Rayleigh operator

$$Ray^t = (\partial_y^2 - \alpha^2)(U_s - c)\psi - U_s''\psi. \quad (36)$$

- The adjoint Airy operator

$$\mathcal{A} = (\partial_y^2 - \alpha^2)\, Airy. \quad (37)$$

Construction of adjoint modes

We have constructed $\phi_{s,-}^{t,app}$ in our previous paper [arxiv 2022]

$$\phi_{s,-}^{t,app}(y) = e^{-\alpha y} - f_1(y_c)\psi_3(y) - g_1(y) + \cdots \tag{38}$$

where $\psi_3(y)$, $f_1(y_c)$ and $g_1(y)$ satisfy

$$Airy(\psi_3) = (U_s - c + \varepsilon\alpha^2)\psi_3 - \varepsilon\partial_y^2\psi_3 = e^{-\alpha y},$$

$$f_1(y) = Ray_\alpha^{-1}\Big(-2\alpha(U_s - c)U_s' e^{-\alpha y}\Big)$$

and

$$g_1(y) = \frac{f_1(y) - f_1(y_c)}{U_s(y) - c}.$$

Computation of A

Let us focus on the mode $+\alpha$. The approximate solution on this mode is

$$\psi_\alpha^{app}(t) = \nu^N \psi_{lin} e^{\Re\lambda t + i\Im\lambda t} + \nu^{3N}\psi_c^1 e^{3\Re\lambda t + i\Im\lambda t} + O(\nu^{5N} e^{5\Re\lambda t}). \quad (39)$$

Let

$$\phi(t) = (\psi_\alpha^{app}, \psi_{lin}^t)_v.$$

We expand the solution ϕ of (32), which gives

$$\phi(t) = \nu^N \phi_0 e^{\lambda t} + \frac{A\nu^{3N}|\phi_0|^2\phi_0}{2\lambda} e^{3\lambda t} + O(\nu^{5N} e^{5\lambda t}). \quad (40)$$

It remains now to compute A by identifying the various terms. First

$$\phi_0 = (\psi_{lin}, \psi_{lin}^t)_v.$$

Moreover,

$$A\frac{|\phi_0|^2\phi_0}{2\lambda} = (\psi_c^1, \psi_{lin}^t)_v, \quad (41)$$

where the right hand side is given by

$$\int \nabla_\alpha \psi_c^1 \cdot \nabla_\alpha \bar{\psi}_{lin}^t = \frac{i}{\alpha\hat{c}} \int Q_2^1 \bar{\psi}_{lin}^t \quad (42)$$

where $\nabla_\alpha = (\partial_y, i\alpha)$.

Conjecture

We fulfil these computations in the particular case of an exponential profile $U_s(y) = 1 - \exp(-\delta y)$ and of Blasius profile, however they can be repeated on any given profile. We find that $\Re A < 0$, namely that cubic interactions tend to "tame" the linear instability.

Conjecture

We thus conjecture that linear instabilities of an exponential profile or of Blasius profile grow until they reach a size $O(\nu^{1/4})$. Then cubic interactions tend to saturate the linear instability.

Thank You !