Asymptotic stability in a traffic flow model

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The ARZ model

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$u(x,t) \in \mathbb{R}$: speed of cars.

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Constant solution: $(\rho_0, U(\rho_0)), \rho_0 \in]0,1[$.

The linear problem

$$\left(\partial_t + \begin{pmatrix} U(\rho_0) & \rho_0 \\ 0 & U(\rho_0) - \rho_0 h'(\rho_0) \end{pmatrix} \partial_x + \frac{1}{\tau} \begin{pmatrix} 0 & 0 \\ U_f & 1 \end{pmatrix}\right) \begin{pmatrix} \rho \\ u \end{pmatrix} = 0.$$

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Define $q(x,t) = \rho(x + \lambda_* t, t), v(x,t) = u(x + \lambda_* t, t)$. Then

$$\partial_t^2 q + (\lambda_0^1 + \lambda_0^2)\partial_{xt}^2 q + \lambda_0^1 \lambda_0^2 \partial_x^2 q + \frac{1}{\tau}\partial_t q = 0$$

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This has an explicit solution !

Explicit solution

The solution to
$\partial_t^2 f + (\lambda_1+\lambda_2)\partial_{xt}^2 f + \lambda_1\lambda_2\partial_x^2 f + \delta\partial_t f = S, f(.,0) = f_0, \partial_t f(.,0) = f_1$ is

$$
\begin{aligned}
f(x,t) &= \int_{\lambda_2 t}^{\lambda_1 t} V(y,t)(\delta f_0 + (\lambda_1+\lambda_2)f_0' + f_1)(x-y)dy \\
&+ \int_{\lambda_2 t}^{\lambda_1 t} \partial_t V(y,t) f_0(x-y)dy \\
&+ \lambda_1 \frac{e^{\frac{-\delta\lambda_1 t}{\lambda_1-\lambda_2}}}{\lambda_1-\lambda_2} f_0(x-\lambda_1 t) - \lambda_2 \frac{e^{\frac{\delta\lambda_2 t}{\lambda_1-\lambda_2}}}{\lambda_1-\lambda_2} f_0(x-\lambda_2 t) \\
&+ \int_0^t \int_{\lambda_2(t-s)}^{\lambda_1(t-s)} V(y,t-s)S(x-y,s)dyds,
\end{aligned}
$$

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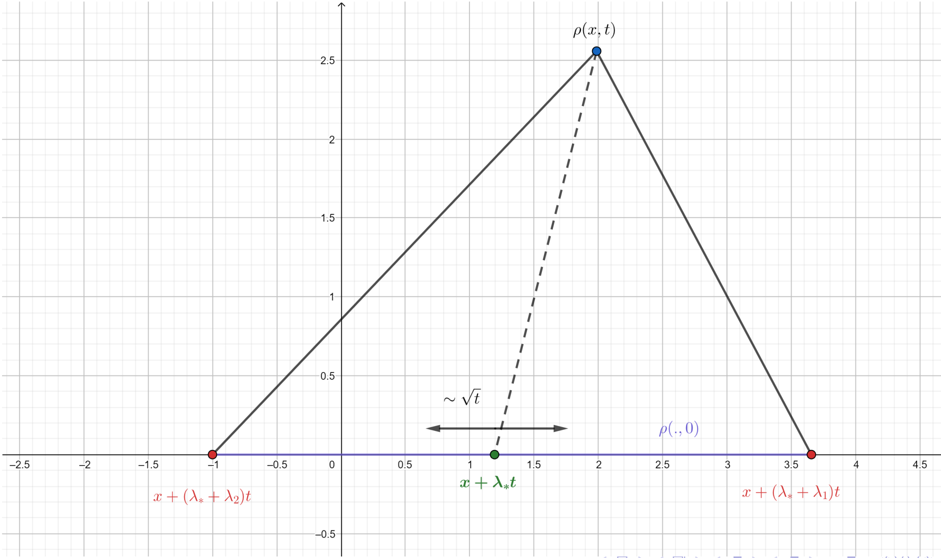
V(y,t) := \frac{e^{-\frac{2\delta\left(-\lambda_1\lambda_2 t + \frac{\lambda_1+\lambda_2}{2}y\right)}{(\lambda_1-\lambda_2)^2}}}{\lambda_1-\lambda_2} I_0\left(\frac{2\delta\sqrt{-\lambda_1\lambda_2}}{(\lambda_1-\lambda_2)^2}\sqrt{-(y-\lambda_1 t)(y-\lambda_2 t)}\right).
$$

How to compute $\rho(x, t)$



Linear stability

Proposition

The solution of the linear problem around $(\rho_0, U(\rho_0))$ with initial data (ρ_i, u_i) of (ARZ) satisfies

$$
\begin{aligned}
& |u(x-\lambda_* t, t)| + |\rho(x-\lambda_* t, t)| \\
\leqslant \quad & \frac{K}{(1+t)^{\frac{1}{2}}} \int_{\lambda_2^0 t}^{\lambda_1^0 t} e^{-a_L \frac{y^2}{1+t}} (|u_i(x-y)| + |\rho_i(x-y)|) dy \\
+ \quad & K e^{-b_L t} \left((|\rho_i| + |u_i|)(x-\lambda_1^0 t) + (|\rho_i| + |u_i|)(x-\lambda_2^0 t) \right)
\end{aligned}
$$

for all $x \in \mathbb{R}$, $t \geqslant 0$ for some $a_L, b_L > 0$.

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Question : Nonlinear stability

Difficulties in the nonlinear case

The equation satisfied by $v(x,t) = u(x+\lambda_* t, t)$ is

$$\partial_t^2 v + (\lambda_1 + \lambda_2)\partial_{xt}^2 v + \lambda_1\lambda_2\partial_x^2 v + \frac{1}{\tau}\partial_t v + \Omega_1 \partial_x v = 0,$$

where $\lambda_1 = \rho_0 U_f + v, \lambda_2 = \rho_0 U_f - (\rho_0 + q)h'(\rho_0 + q) + v$ and $\Omega_1 = \partial_t \lambda_2 + \lambda_1 \partial_x \lambda_2 + \frac{1}{\tau}(v - U_f q)$.

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- Ω_1 is small but not 0. λ_1, λ_2 are not constants.
- We can make two of these quantities constant by change of variable but not all three.
- Slow decay in the linear problem ($t^{-1/2}$) can lead to growth coming from the nonlinearity.

Estimators

for $w_i = |\rho_i| + |u_i|, a, b, \gamma, \delta, \mu_1, -\mu_2 > 0$ we define

$$F^{\mu_1,\mu_2}_{a,\gamma}(x,t) := \frac{1}{(1+t)^\gamma}\int_{\mu_2 t}^{\mu_1 t} e^{-a\frac{y^2}{1+t}} w_i(x-y)dy.$$

$$G^{\mu}_{b,\delta}(x,t) := e^{-bt} \sup_{y\in[-\delta t,\delta t]} w_i(x-\mu t+y).$$

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For the linear problem,

$$\begin{aligned} & |u(x-\lambda_* t,t)| + |\rho(x-\lambda_* t,t)| \\ \leqslant \quad & K\left(F_{a_L,\frac{1}{2}}^{\lambda_1^0,\lambda_2^0} + G_{b_L,0}^{\lambda_1^0} + G_{b_L,0}^{\lambda_2^0}\right)(x,t). \end{aligned}$$

Nonlinear stability

Theorem (Ghoul-Masmoudi-P.)

If $s_{cc}(\rho_0) > 0$, there exists $\varepsilon_0 > 0$ such that if

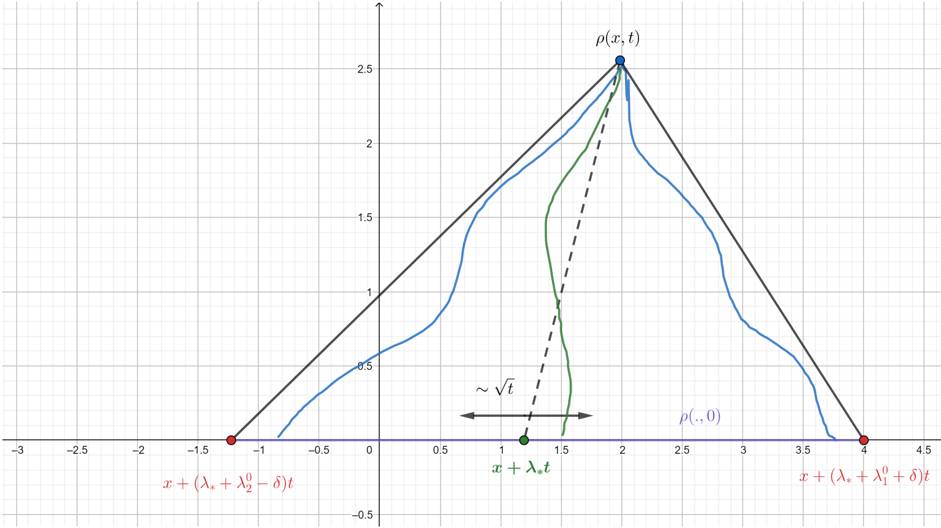
$$\|\rho_i\|_{W^{2,1}(\mathbb{R})} + \|u_i\|_{W^{2,1}(\mathbb{R})} + \|\rho_i\|_{C^2(\mathbb{R})} + \|u_i\|_{C^2(\mathbb{R})} = \varepsilon \leqslant \varepsilon_0,$$

then for $w_i = \sum_{j\in\{0,1,2\}} |\rho_i^{(j)}| + |u_i^{(j)}|$ and $\delta = o_{\varepsilon\to 0}(1)$, there exists $a, b > 0$ such that for the nonlinear problem,

$$\begin{aligned} & |u(x-\lambda_* t, t)| + |\rho(x-\lambda_* t, t)| \\ \leqslant \quad & K_{n,k}(F_{a,1/2}^{\lambda_1^0+\delta,\lambda_2^0-\delta} + G_{b,\delta}^{\lambda_1^0} + G_{b,\delta}^{\lambda_2^0})(x,t) \end{aligned}$$

for all $x \in \mathbb{R}$, $t \geqslant 0$.

Nonlinear stability



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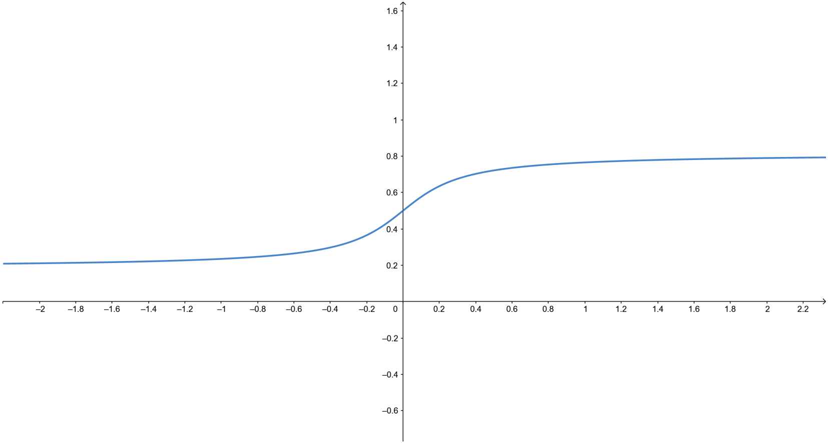
- They spray like the heat equation, at least not faster.
- The result hold for perturbations in $W^{2,1} \cap C^2$ (in particular no need for localisation).
- Perturbations decay in L^∞ like $\frac{1}{\sqrt{t}}$.

Graph of a travelling wave



Stability of travelling waves

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Goal: Improve the stability result using the stability of constant flows.

- We want to keep information on the localisation of the perturbation
- We want to give a more precise criteria for the stability.

Overview of the arguments

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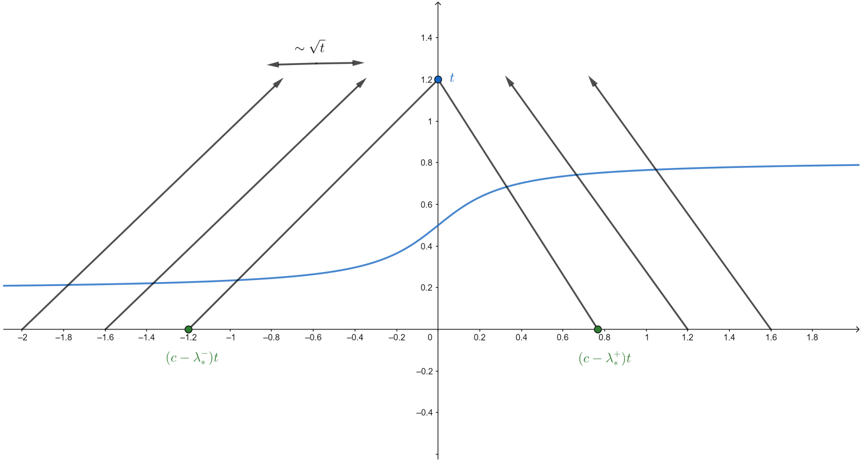
We do the linear setting, then the nonlinear one.

Summary in a drawing (WIP)



Eternal oscillations and large movements (WIP)