Local Energy Dissipation for Continuous Incompressible Euler Flows

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Outline

- Motivation
  - Onsager's Conjecture and Turbulence
  - Importance of Local energy dissipation
  - Strong Onsager Conjecture
- Brief Survey of Previous Results
- Continuous Euler flows that exhibit local energy dissipation
  - ...
  - ...
  - ...
  - ...
  - ...

Motivation: Weak Solutions to the Euler equations

The incompressible Euler equations for a homogeneous fluid:

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0 \tag{1}$$

$$\nabla_j v^j = 0 \tag{2}$$

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$$\frac{d}{dt} \int_\Omega v^\ell(t,x) dx = \int_{\partial\Omega} p(t,x) n^\ell d\sigma + \int_{\partial\Omega} v^\ell(t,x)(v \cdot n) d\sigma \tag{3}$$

$$\int_{\partial\Omega} (v \cdot n)(t,x) d\sigma(x) = 0 \tag{4}$$

for all Ω with smooth boundary $\partial\Omega$ and interior unit normal n^ℓ.

Motivation: Sufficiently smooth solutions conserve energy

Take the dot product of the Euler equations with v^ℓ

$$
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Then local conservation of energy holds:

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And integration yields conservation of total kinetic energy:

$$\frac{d}{dt} \int_{\mathbb{R}^n} \frac{|v|^2}{2}(t,x)dx = - \int_{\mathbb{R}^n} \text{div} \left[(\frac{|v|^2}{2} + p)v \right] dx = 0$$

Motivation: Onsager's Conjecture (1949)

1. Solutions (v, p) to Euler on $\mathbb{T}^3$ obeying a Hölder estimate

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = 0 \tag{5}$$
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$$|v(t, x + \Delta x) - v(t, x)| \le C|\Delta x|^\alpha \tag{6}$$

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In fact, Onsager's argument leads to an even stronger conjecture on the existence of dissipative Euler flows.

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Kolmogorov (1941): As $\nu \to 0$ for solutions to $3D$ Navier-Stokes:

$$\begin{cases} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nu \Delta v^\ell \\ \nabla_j v^j = 0 \end{cases} \tag{7}$$

the energy dissipation rate remains **strictly positive** as $\nu \to 0$

$$\varepsilon = \lim_{\nu \to 0} \left\langle -\frac{d}{dt} \int \frac{|v_\nu|^2}{2}(t,x)dx \right\rangle > 0.$$

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$$\langle |v(x+\Delta x) - v(x)|^p \rangle^{1/p} \sim \varepsilon^{1/3} |\Delta x|^{1/3}$$

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Onsager considered the case $\nu = 0$; argued that "frequency cascades" may lead to dissipation in the absence of viscosity.

Onsager and Ideal Turbulence

Onsager considered the Euler equations in Fourier series form (which converges for $v \in L^2$)

$$v(x,t) = \sum_k a_k(t) e^{ik\cdot x}$$

$$\frac{da_k}{dt} = i \sum_m a_{k-m} \cdot k \left[-a_m + \frac{(a_m \cdot k)k}{|k|^2} \right]$$

He argued that energy can “cascade” from low wavenumbers to high wavenumbers, and the cascade can happen so rapidly that part of the energy $\sum_k |a_k|^2$ escapes to infinite frequency (i.e. vanishes to small spatial scales) in finite time.

However, only low regularity solutions could behave this way, and he stated that solutions in C^α with $\alpha > 1/3$ must conserve energy.

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By a statistical physics argument, a "typical" turbulent flow should have: $\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ (hence regularity exactly $1/3$).

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Refinements: (Duchon-Robert '00), (Cheskidov-Constantin-Shvydkoy-Friedlander '08): $v \in L_t^3 B_{3,c(\mathbb{N})}^{1/3}$, but not $L_t^\infty C_x^{1/3}$.

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The last part implies the Strong Onsager Conjecture by compactness.

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[GQ. Chen, Glimm, '12] "Weak-K41 implies compactness".

K41 Folklore Conjecture for Navier-Stokes

Conjecture (K41 Folklore Conjecture for N-S)

There is a sequence v_{ν_j} of (regular) solutions to the incompressible Navier-Stokes equations on $I \times \mathbb{T}^3$, I a finite interval, with $\nu_j \to 0$ such that

- **Scaling law:** The norms $\|v_{\nu_j}\|_{L_t^\infty C_x^{1/3}}$ are uniformly bounded
- **Zeroth law:** The sequence exhibits mean rate of energy dissipation independent of viscosity in the sense that

$$\limsup_{j\to\infty} \frac{1}{|I|} \int_I \left[-\frac{d}{dt} \int_{\mathbb{T}^3} \frac{|v_{\nu_j}|^2}{2}(t,x)dx \right] dt \geq \varepsilon > 0 \qquad (12)$$

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Zero viscosity limits dissipate energy locally

Proposition

If v is a strong limit $v_{\nu_j} \to v$ in L^3 of a sequence of classical Navier-Stokes solutions with $\nu_j \to 0$, then v is a weak solution to incompressible Euler that satisfies the **local energy inequality**

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Proof: If v_ν is a classical solution to Navier-Stokes, then the **local energy equality** holds

$$\partial_t \left(\frac{|v_\nu|^2}{2} \right) + \nabla_j \left[\left(\frac{|v_\nu|^2}{2} + p_\nu \right) v_\nu^j \right] - \nu \Delta \frac{|v_\nu|^2}{2} = -\nu |\nabla v_\nu|^2$$

K41 Folklore Conjecture in the inviscid limit

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"Proof" using the K41 Folklore Conjecture: The Scaling Law bounds $\|v\|_{L_t^\infty C_x^{1/3}}$, which guarantees uniformly convergent subsequences by the Aubin-Lions-Simon Lemma.

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(Note: Possible nonuniqueness of solutions.)

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Already difficult question: Are there even incompressible Euler flows that fail to conserve total kinetic energy?

Weak solutions that fail to conserve energy

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- Weak solutions in $L^2_{t,x}(\mathbb{R} \times \mathbb{R}^2)$ with compact support in space and time (Scheffer, '93)
- Weak solutions in $L^2_{t,x}(\mathbb{R} \times \mathbb{T}^2)$ (Shnirelman, '97)
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- Solutions in $L^\infty_{t,x} \cap C_t L^2_x$ with given local energy dissipation

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) = \frac{d}{dt} e(t)$$

(De Lellis, Székelyhidi, '07)

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Solutions are **nowhere continuous** and the argument faces a major difficulty towards obtaining continuous solutions.

C^α weak solutions with given total kinetic energy

- Solutions in $C^\alpha_{t,x}$ for $\alpha < 1/10$ with any prescribed smooth *total kinetic energy* $\frac{1}{2}\int_{\mathbb{T}^3} |v|^2(t,x)dx = e(t)$
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- In $C^\alpha_{t,x}$ for $\alpha < 1/5$ with compact support (I. '12) and prescribed, smooth total kinetic energy $e(t)$ (Buckmaster, De Lellis, Székelyhidi, '13)
- In $L^1_t C^{1/3-\epsilon}_x$ with compact support (Buck., DeL, Szé, '14)
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- Solutions with compact support and

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- Solutions in $C_t H^{1/2-}$ (Buck., Masmoudi, Novack, Vicol) $C_t H^{1/2-} \cap L^{\infty-} \subseteq L^\infty_t B^{1/3-}_{3,\infty}$ (Novack, Vicol)

Significance: Confirmation of energy cascades

Onsager considered the Euler equations in Fourier series form

$$v(x,t) = \sum_k a_k(t) e^{ik\cdot x}$$

$$\frac{da_k}{dt} = i \sum_m a_{k-m} \cdot k \left[-a_m + \frac{(a_m \cdot k)k}{|k|^2} \right]$$

He argued that energy can "cascade" from low wavenumbers to high wavenumbers, and the cascade can happen so rapidly that part of the energy $\sum_k |a_k|^2$ escapes to infinite frequency (i.e. vanishes to small spatial scales) in finite time.

By a statistical physics argument, a "typical" turbulent flow should have: $\sum_{\frac{\lambda}{2} \leq |k| \leq 2\lambda} |a_k|^2 \sim \lambda^{-2/3}$ (hence regularity exactly $1/3$).

Open Problem: Strong Onsager conjecture

Conjecture (Strong Onsager Conjecture)

There is a weak solution v to the incompressible Euler equations of class $v \in L_t^\infty C_x^{1/3}(I \times \mathbb{T}^3)$ that satisfies the local energy inequality

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left[\left(\frac{|v|^2}{2} + p \right) v^j \right] \le 0$$

and does not conserve energy.

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Note: Requiring local energy dissipation is much stronger than requiring dissipation of total kinetic energy. (Consider Burgers.)

Solutions in $L^\infty_{t,x}$ that exhibit local energy dissipation

Theorem (De Lellis, Székelyhidi, 07)

For any smooth $e(t) > 0$ there exist (nowhere continuous) weak solutions solutions (v, p) of class $v \in L^\infty_{t,x} \cap C_t L^2_x$ such that

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Theorem: First result on the Strong Onsager Conjecture

Theorem (I. '17)

For any $\alpha < 1/15$ *and* $d \geq 3$ *there exist solutions of class* $v \in C^{\alpha}_{t,x}(\mathbb{R} \times \mathbb{T}^d)$ *that satisfy the local energy inequality*

$$\mathcal{D}[v,p] := \partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \leq 0$$

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Theorem: Improvement on the Strong Onsager Conjecture

Theorem (De Lellis, Kwon '20)

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Extension to isentropic compressible Euler [Giri, Kwon, '21]
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Idea: Discretize the "Mikado flows" of [Daneri, Székelyhidi, '16]
Open problem: Improve the regularity to $1/3$.

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Outline

- General idea of convex integration
  - Euler-Reynolds flows
  - “One-dimensional” waves
  - Microlocal Lemma
  - Main error terms
- Local energy dissipation ideas
  - Dissipative Euler-Reynolds flows
  - Arrow of time in the construction
  - Trilinear energy cascades (resembles Kraichnan’s LDIA theory)

Continuous Solutions: The Euler-Reynolds Equations

(De Lellis, Székelyhidi): Consider the **Euler-Reynolds equations**

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned} \tag{ER}$$

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Examples: If (v, p) solves the Euler equations then

- $(v_\epsilon, p_\epsilon, R_\epsilon^{j\ell})$, $R_\epsilon^{j\ell} = v_\epsilon^j v_\epsilon^\ell - (v^j v^\ell)_\epsilon$, $v_\epsilon^\ell = \eta_\epsilon * v^\ell$

- **Corollary:** Every continuous incompressible Euler flow (v, p) is the uniform limit of a sequence of C^∞ Euler-Reynolds flows (v_q, p_q, R_q) with $\|R_q\|_{C^0} \to 0$ as $q \to \infty$

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The symmetric tensor $R^{j\ell}$ measures the error from solving Euler. Examples:

- Any v^ℓ that is incompressible and conveserves momentum

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) = U^\ell$$
$$\int_{\mathbb{T}^3} U^\ell(t,x)dx = 0$$
$$\nabla_j R^{j\ell} = U^\ell$$

Continuous Solutions: Convex Integration for Euler

We construct a sequence (v_q, p_q, R_q) indexed by q solving

$$\partial_t v_q^\ell + \nabla_j (v_q^j v_q^\ell) + \nabla^\ell p_q = \nabla_j R_q^{j\ell} \qquad \text{(ERq)}$$
$$\nabla_j v_q^j = 0$$

where $v_{q+1} = v_q + V_q$, $p_{q+1} = p_q + P_q$ solve (ERq+1) with

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$$\textbf{much smaller} \quad |R_{q+1}| \ll |R_q|$$

In the limit as $q \to \infty$, we get continuous solutions

$$\|R_q\|_{C^0} \to 0, \quad |V_q| \sim |R_q|^{1/2}, \quad |P_q| \sim |R_q|$$

Continuous Solutions: Convex Integration for Euler

Start with **any** smooth solution to Euler-Reynolds on $\mathbb{R} \times \mathbb{T}^3$

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned}$$

and add high-frequency corrections

$$\overset{*}{v} = v + V, \qquad \overset{*}{p} = p + P,$$

that are designed to "get rid of" $R^{j\ell}$.

Continuous Solutions: Convex Integration for Euler

Get new solutions $\overset{*}{v} = v + V$, $\overset{*}{p} = p + P$ to Euler-Reynolds

$$\partial_t \overset{*}{v}^\ell + \nabla_j(\overset{*}{v}^j \overset{*}{v}^\ell) + \nabla^\ell \overset{*}{p} = \nabla_j \overset{*}{R}^{j\ell}$$
$$\nabla_j \overset{*}{v}^j = 0$$

with $||\overset{*}{R}||_{C^0_{t,x}}$ **much smaller than** $||R||_{C^0_{t,x}}$.

Continuous Solutions: Convex Integration for Euler

The corrected $\overset{*}{v} = v + V$, $\overset{*}{p} = p + P$ satisfy

$$\begin{aligned} \partial_t \overset{*}{v}{}^\ell + \nabla_j(\overset{*}{v}{}^j \overset{*}{v}{}^\ell) + \nabla^\ell \overset{*}{p} &= \partial_t V^\ell + \ldots + \nabla_j(V^j V^\ell + P\delta^{j\ell} + R^{j\ell}) \\ &= \textbf{ not in the form } \nabla_j \overset{*}{R}{}^{j\ell} \\ \nabla_j \overset{*}{v}{}^j &= 0 \end{aligned}$$

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so we will have to solve a divergence equation:

$$\nabla_j \overset{*}{R}{}^{j\ell} = \partial_t V^\ell + \nabla_j(v^j V^\ell) + \nabla_j(V^j v^\ell) + \nabla_j(V^j V^\ell + P\delta^{j\ell} + R^{j\ell})$$

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The new error $\|\overset{*}{R}\|_{C^0}$ will only be small when V and P are very oscillatory and are designed carefully depending on the given v^ℓ and $R^{j\ell}$.

The Error terms

Let (v, p, R) be a smooth solution to Euler-Reynolds.

$$\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}$$

Then $\overset{*}{v} = v + V$ and $\overset{*}{p} = p + P$ satisfy

$$\begin{aligned}
\partial_t \overset{*}{v}{}^\ell + \nabla_j(\overset{*}{v}{}^j \overset{*}{v}{}^\ell) + \nabla^\ell \overset{*}{p} &= \partial_t V^\ell + \nabla_j(v^j V^\ell) + \nabla_j(V^j v^\ell) \\
&\quad + \nabla_j(V^j V^\ell + P\delta^{j\ell} + R^{j\ell}) \\
\text{want } &= \nabla_j \overset{*}{R}{}^{j\ell} \\
\text{with } \|\overset{*}{R}\|_{C^0} &\lesssim \lambda^{-1}
\end{aligned}$$

where V^ℓ oscillates at large frequency λ.

The Error terms

We name the terms as follows: $\overset{*}{R}^{j\ell} = R_T^{j\ell} + R_S^{j\ell} + R_H^{j\ell}$

Transport term:

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j (v^j V^\ell) + \nabla_j (V^j v^\ell)$$

Stress term:

$$\nabla_j R_S^{j\ell} = \mathsf{LFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell} + R^{j\ell})]$$

High-Frequency Interference terms:

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Each one of R_T, R_S and R_H must be $\|\overset{*}{R}\|_{C^0} \lesssim \lambda^{-1}$.

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(There is also another term involving errors from mollifying $v \mapsto v_\epsilon$ and $R \mapsto R_\epsilon$ that we are neglecting here.)

The High-Frequency Correction

The correction V^ℓ is a high-frequency, divergence free wave. For example, in (I. Vicol, '13), it has the form

$$V^\ell = \sum_I (e^{i\lambda\xi_I} v_I^\ell + \delta V_I^\ell) = \sum_I V_I^\ell$$

$$\nabla_\ell V^\ell = 0 \quad \text{(by choice of small } \delta V_I^\ell\text{)}$$

$$(\partial_t + v_\epsilon^j \nabla_j)\xi_I = 0 \quad (\Rightarrow \text{nonlinear phase functions})$$

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More precisely, $V_I = \mathcal{P}_I[e^{i\lambda\xi_I} v_I]$ where $\mathcal{P}_I$ is a Leray projection localized at frequency of order λ, and v_I includes a cutoff in time to keep $\nabla \xi_I$ within $O(\frac{1}{10})$ of its initial datum.

Microlocal Lemma

Lemma (Microlocal Lemma)

If $U_I^\ell = e^{i\lambda\xi_I} v_I^\ell$*, and for* $k = 0, 1, 2$, $\||h|^k K(h)\|_{L^1(\mathbb{R}^d)} \lesssim \lambda^{-k}$*, then for* $V_I^\ell = \int_{\mathbb{R}^d} U_I^\ell(x-h) K(h) dh$ *we have an expansion*

$$V_I^\ell = e^{i\lambda\xi_I}\left(\widehat{K}(\lambda\nabla\xi_I(x)) v_I^\ell(x) + \delta v_I^\ell\right)$$

where $\delta v_I^\ell \approx \frac{\nabla v_I^\ell}{\lambda} + \frac{\nabla^2 \xi_I v_I}{\lambda}$*. (Special case:* $\widehat{K}(\lambda\nabla\xi_I) = \pi^\perp_{\langle\nabla\xi_I\rangle}$*.)*

Microlocal Lemma

Lemma (Microlocal Lemma)

If $U_I^\ell = e^{i\lambda\xi_I} v_I^\ell$, *and for* $k = 0, 1, 2$, $\||h|^k K(h)\|_{L^1(\mathbb{R}^d)} \lesssim \lambda^{-k}$, *then for* $V_I^\ell = \int_{\mathbb{R}^d} U_I^\ell(x-h) K(h) dh$ *we have an expansion*

$$V_I^\ell = e^{i\lambda\xi_I} \left(\widehat{K}(\lambda \nabla \xi_I(x)) v_I^\ell(x) + \delta v_I^\ell \right)$$

where $\delta v_I^\ell \approx \frac{\nabla v_I^\ell}{\lambda} + \frac{\nabla^2 \xi_I v_I}{\lambda}$. *(Special case:* $\widehat{K}(\lambda \nabla \xi_I) = \pi^\perp_{\langle \nabla \xi_I \rangle}$.*)*

Proof: Write $e^{-i\lambda\xi_I} V^\ell = \int_{\mathbb{R}^d} e^{-i\lambda\xi_I(x)} U^\ell(x-h) K(h) dh$

$$\begin{aligned} e^{-i\lambda\xi_I} V^\ell &= \int_{\mathbb{R}^d} e^{i\lambda(\xi_I(x-h) - \xi_I(x))} u_I^\ell(x-h) K(h) dh \\ &\approx \int_{\mathbb{R}^d} e^{-i\lambda \nabla \xi_I(x) \cdot h} u_I^\ell(x) K(h) dh \\ e^{-i\lambda\xi_I} V^\ell &= \widehat{K}(\lambda \nabla \xi_I(x)) u_I^\ell(x) + \delta u_I^\ell(x) \end{aligned}$$

The High-Frequency Correction

The correction V^ℓ is a high-frequency, divergence free wave. For example, in (I. Vicol, '13), it has the form

$$V^\ell = \sum_I (e^{i\lambda\xi_I} v_I^\ell + \delta V_I^\ell) = \sum_I V_I^\ell$$

$$\nabla_\ell V^\ell = 0 \quad \text{(by choice of small } \delta V_I^\ell\text{)}$$

$$(\partial_t + v_\epsilon^j \nabla_j)\xi_I = 0 \quad (\Rightarrow \text{nonlinear phase functions})$$

$$\nabla \xi_I \cdot v_I = 0 \quad \text{(required for } \nabla_\ell V_I^\ell = 0\text{)}$$

More precisely, $V_I = \mathcal{P}_I[e^{i\lambda\xi_I} v_I]$ where $\mathcal{P}_I$ is a Leray projection localized at frequency of order λ, and v_I includes a cutoff in time to keep $\nabla\xi_I$ within $O(1)$ of its initial datum.

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Recalling the Error terms again

Each one of R_T, R_S and R_H must have size $\|\overset{*}{R}\|_{C^0} \lesssim \lambda^{-1}$, and requires solving a divergence equation:

Transport term:

$$\nabla_j R_T^{j\ell} = \partial_t V^\ell + \nabla_j (v_\epsilon^j V^\ell) + \nabla_j (V^j v_\epsilon^\ell)$$

Stress term:

$$\nabla_j R_S^{j\ell} = \mathsf{LFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell})]$$

High-Frequency Interference terms:

$$\nabla_j R_H^{j\ell} = \mathsf{HFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell})]$$

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$$\nabla_j R_H^{j\ell} = \mathsf{HFreq}[\nabla_j(V^j V^\ell + P\delta^{j\ell})]$$

The Main Error Terms

With this Ansatz the Transport term is under control:
Letting $D_t := (\partial_t + v_\epsilon^j \nabla_j)$ be the "advective derivative" we have

$$\begin{aligned}\partial_t V^\ell + \nabla_j(v_\epsilon^j V^\ell) + \nabla_j(V^j v_\epsilon^\ell) &= (\partial_t + v_\epsilon^j \nabla_j) V^\ell + V^j \nabla_j v_\epsilon^\ell \\ &\approx \sum_I D_t[v_I^\ell e^{i\lambda\xi_I}] + e^{i\lambda\xi_I} v_I^j \nabla_j v_\epsilon^\ell \\ \nabla_j R_T^{j\ell} = \sum_I \underbrace{(D_t[v_I^\ell] + v_I^j \nabla_j v_\epsilon^\ell)}_{\text{low}} \underbrace{e^{i\lambda\xi_I}}_{\text{high}}\end{aligned}$$

Applying div^{-1} yields $\|R_T\|_0 \lesssim \lambda^{-1}$ and $\mathrm{Freq}(R_T) \approx \lambda$.

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Letting $D_t := (\partial_t + v_\epsilon^j \nabla_j)$ be the "advective derivative" we have

$$\begin{aligned}
\partial_t V^\ell + \nabla_j (v_\epsilon^j V^\ell) + \nabla_j (V^j v_\epsilon^\ell) &= (\partial_t + v_\epsilon^j \nabla_j) V^\ell + V^j \nabla_j v_\epsilon^\ell \\
&\approx \sum_I D_t [v_I^\ell e^{i\lambda \xi_I}] + e^{i\lambda \xi_I} v_I^j \nabla_j v_\epsilon^\ell \\
\nabla_j R_T^{j\ell} &= \sum_I \underbrace{(D_t[v_I^\ell] + v_I^j \nabla_j v_\epsilon^\ell)}_{\text{low}} \underbrace{e^{i\lambda \xi_I}}_{\text{high}} + \cdots
\end{aligned}$$

(Used $\nabla_j V^j = 0$.)

The Main Error Terms

The Stress term *should ideally* be controlled as follows:

$$
\begin{aligned}
&\mathsf{LFreq}[\nabla_j (V^j V^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell})] \\
&\qquad = \mathsf{LFreq}\Big[\nabla_j \Big(\sum_{I,J} V_I^j V_J^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell} \Big)\Big] \\
&\qquad = \mathsf{LFreq}\Big[\nabla_j \Big(\sum_{I} V_I^j \overline{V_I^\ell} + P\delta^{j\ell} + R_\epsilon^{j\ell} \Big)\Big] \\
&\qquad := \nabla_j \Big[\sum_I v_I^j \overline{v_I^\ell} + P(t,x)\delta^{j\ell} + R_\epsilon^{j\ell} \Big] \\
&\qquad = \nabla_j [0] = 0
\end{aligned}
$$

Here we solve for the v_I^j at each point subject only to the constraint that $v_I \cdot \nabla \xi_I = 0$ and $\|\nabla \xi_I - \nabla \hat{\xi}_I\|_0 \leq 1/10$.

The Main Error Terms

The remaining High-Frequency Interference term has the form

$$\begin{aligned}\mathsf{HFreq}[\nabla_j(V^jV^\ell)] &= \nabla_j\left[\sum_{J\neq\bar{I}} V_I^j V_J^\ell\right]\\ &\approx \nabla_j\left[\sum_{J\neq\bar{I}} e^{i\lambda(\xi_I+\xi_J)} v_I^j v_J^\ell\right]\\ \nabla_j R_H^{j\ell} &= \lambda\sum_{J\neq\bar{I}} e^{i\lambda(\xi_I+\xi_J)} i(\nabla_j\xi_I+\nabla_j\xi_J) v_I^j v_J^\ell + \mathsf{OK}\end{aligned}$$

The Main Error Terms

The remaining High-Frequency Interference term has the form

$$\mathsf{HFreq}[\nabla_j(V^jV^\ell)] = \nabla_j \left[\sum_{J \neq \bar{I}} V_I^j V_J^\ell \right]$$

$$\approx \nabla_j \left[\sum_{J \neq \bar{I}} e^{i\lambda(\xi_I + \xi_J)} v_I^j v_J^\ell \right]$$

$$\nabla_j R_H^{j\ell} = \lambda \sum_{J \neq \bar{I}} e^{i\lambda(\xi_I + \xi_J)} i(\nabla_j \xi_I + \nabla_j \xi_J) v_I^j v_J^\ell + \mathsf{OK}$$

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&\approx \nabla_j \left[\sum_{J \neq \bar{I}} e^{i\lambda(\xi_I + \xi_J)} v_I^j v_J^\ell \right] \\
\nabla_j R_H^{j\ell} &= \lambda \sum_{J \neq \bar{I}} e^{i\lambda(\xi_I + \xi_J)} i(\nabla_j \xi_I + \nabla_j \xi_J) v_I^j v_J^\ell + \mathrm{OK}
\end{aligned}$$

OK if $v_I \cdot \nabla \xi_J = 0$. So we require that all phase functions align $\nabla \xi_I \approx n_I e_1$ for all I.

The Main Error Terms

The Stress term still has two "components" remaining:

$$\begin{aligned}
&\mathsf{LFreq}[\nabla_j(V^jV^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell})] \\
&\qquad = \mathsf{LFreq}\Big[\nabla_j\Big(\sum_{I,J} V_I^j V_J^\ell + P\delta^{j\ell} + R_\epsilon^{j\ell}\Big)\Big] \\
&\qquad = \mathsf{LFreq}\Big[\nabla_j\Big(\sum_{I} V_I^j \overline{V_I^\ell} + P\delta^{j\ell} + R_\epsilon^{j\ell}\Big)\Big] \\
&\qquad := \nabla_j\Big[\sum_{I} v_I^j \overline{v_I^\ell} + P(t,x)\delta^{j\ell} + R_\epsilon^{j\ell}\Big] \\
&\qquad = \nabla_j[P\delta_{[2]}^{j\ell} + R_{[2]}^{j\ell} + R_{[3]}^{j\ell}] \neq 0
\end{aligned}$$

where $\delta_{[k]}, R_{[k]}(t,x) \in \langle e_k\rangle^\perp \otimes \langle e_k\rangle^\perp$ for $k = 2, 3$

Theorem: First result on the Strong Onsager Conjecture

Theorem (I. '17)

For any $\alpha < 1/15$ *and* $d \geq 3$ *there exist solutions of class* $v \in C^{\alpha}_{t,x}(\mathbb{R} \times \mathbb{T}^d)$ *that satisfy the local energy inequality*

$$\mathcal{D}[v,p] := \partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \leq 0$$

with strict inequality everywhere.

Dissipative Euler Reynolds flow

Idea: Relax the local energy inequality

$$\begin{aligned} \partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p &= \nabla_j R^{j\ell} \\ \nabla_j v^j &= 0 \end{aligned}$$

$$\mathcal{D}[v,p] := \underbrace{\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right)}_{\text{Approximate dissipation}} \leq f$$

Design a scheme such that both R and f tend to zero.

(Not actually going to work.)

Dissipative Euler Reynolds flow

Definition

A dissipative Euler-Reynolds flow is a tuple $(v, p, R, \kappa, \varphi, \mu)$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}, \quad \nabla_j v^j = 0$$

$$\partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right) \leq D_t \kappa + \nabla_j [v_\ell R^{j\ell}] + \nabla_j \varphi^j$$

- Unresolved flux density: κ,
- Unresolved flux current: φ^j
- Dissipation measure: $\mu \geq 0$

$$\mu = -\mathcal{D}[v, p] + D_t \kappa + \nabla_j [v_\ell R^{j\ell}] + \nabla_j \varphi^j$$

Plan of attack

Plan:

Given a dissipative Euler-Reynolds flow $(v, p, R, \kappa, \varphi, \mu)$

$$\partial_t v^\ell + \nabla_j (v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}, \quad \nabla_j v^j = 0$$
$$\mathcal{D}[v,p] \leq D_t \kappa + \nabla_j [v_\ell R^{j\ell}] + \nabla_j \varphi^j$$

Construct $\overset{*}{v} = v + V$, $\overset{*}{p} = p + P$ with (R, κ, φ) *much smaller*, i.e.

$$\mathcal{D}[v+V, p+P] \leq \overset{*}{D}_t \overset{*}{\kappa} + \nabla_j [\overset{*}{v}_\ell \overset{*}{R}^{j\ell}] + \nabla_j \overset{*}{\varphi}^j$$

With all of $(\overset{*}{R}, \overset{*}{\kappa}, \overset{*}{\varphi})$ *much smaller* than (R, κ, φ)

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Given a dissipative Euler-Reynolds flow $(v, p, R, \kappa, \varphi, \mu)$

$$\partial_t v^\ell + \nabla_j(v^j v^\ell) + \nabla^\ell p = \nabla_j R^{j\ell}, \quad \nabla_j v^j = 0$$
$$\mathcal{D}[v,p] \le D_t \kappa + \nabla_j[v_\ell R^{j\ell}] + \nabla_j \varphi^j$$

Construct $\overset{*}{v} = v + V$, $\overset{*}{p} = p + P$ with (R, κ, φ) *much smaller*, i.e.

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With all of $(\overset{*}{R}, \overset{*}{\kappa}, \overset{*}{\varphi})$ *much smaller* than (R, κ, φ)

Approximate dissipation:

Recall that

$$\overset{*}{\mathcal{D}}[v,p] = \partial_t \left(\frac{|v|^2}{2} \right) + \nabla_j \left(\left(\frac{|v|^2}{2} + p \right) v^j \right)$$

is both quadratic and cubic. After adding corrections,

$$\overset{*}{\mathcal{D}}[v+V, p+P] = \mathcal{D}_T + \mathcal{D}_S + \mathcal{D}_H + \mathcal{D}_\kappa + \mathcal{D}_\varphi + \cdots$$

will have linear, bilinear and trilinear terms. The goal is for

$$\overset{*}{\mathcal{D}}[v+V, p+P] \leq \overset{*}{D}_t \overset{*}{\kappa} + \nabla_j [\overset{*}{v}_\ell \overset{*}{R}^{j\ell}] + \nabla_j \overset{*}{\phi}^j$$

The new terms:

$$\overset{*}{\mathcal{D}}[v+V,p+P] = \mathcal{D}_T + \mathcal{D}_S + \mathcal{D}_H + \mathcal{D}_\kappa + \mathcal{D}_\varphi + \cdots$$

$$\mathcal{D}_T = \partial_t(v_\ell V^\ell) + \nabla_j(v_\ell V^\ell v^j) + \nabla_j\left(\left(\frac{|v|^2}{2}+p\right)V^j\right)$$

$$\mathcal{D}_S = \nabla_j\left[v_\ell[\sum_I V_I^j \overline{V_I^\ell} + P\delta^{j\ell} + R^{j\ell}]\right]$$

$$\mathcal{D}_H = \nabla_j\left[v_\ell[\sum_{J\neq \bar{I}} V_I^j V_J^\ell]\right]$$

$$\mathcal{D}_\kappa = \partial_t\left(\frac{|V|^2}{2}+\kappa\right) + \nabla_j\left(\left(\frac{|V|^2}{2}+\kappa\right)v^j\right)$$

$$\mathcal{D}_\varphi = \nabla_j\left[\sum_{I,J,K}(V_I)_\ell V_J^j V_K^\ell + PV^j + \varphi^j\right]$$

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$$\mathcal{D}_\varphi = \nabla_j\left[\sum_{I,J,K}(V_I)_\ell V_J^j V_K^\ell + PV^j + \varphi^j\right]$$

The new terms: The Transport term

The terms linear in V can be treated as follows:

$$\mathcal{D}_T = \partial_t(v_\ell V^\ell) + \nabla_j(v_\ell V^\ell v^j) + \nabla_j\left(\left(\frac{|v|^2}{2} + p\right)V^j\right)$$

The new terms: The Transport term

The terms linear in V can be treated as follows:

$$\mathcal{D}_T = \partial_t(v_{\epsilon\ell}V^\ell) + \nabla_j(v_{\epsilon\ell}V^\ell v_\epsilon^j) + \nabla_j\left(\left(\frac{|v_\epsilon|^2}{2} + p_\epsilon\right)V^j\right)$$
$$+ D_t\overset{*}{\kappa}_T + \nabla_j[\overset{*}{\varphi}{}^j_T]$$

The new terms: The Transport term

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$$= v_{\epsilon\ell}\nabla_j R_T^{j\ell} + V_\ell(\nabla_j R_\epsilon^{j\ell} + \cdots)$$

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&= v_{\epsilon\ell}\nabla_j R_T^{j\ell} + V_\ell(\nabla_j R_\epsilon^{j\ell} + \cdots) \\
&= \nabla_j[v_{\epsilon\ell}R_T^{j\ell}] - \nabla_j v_{\epsilon\ell}\underbrace{R_T^{j\ell} + V_\ell}_{\text{freq. } \approx\lambda}(\nabla_j R_\epsilon^{j\ell} + \cdots)
\end{aligned}$$

The new terms: The Transport term

The terms linear in V can be treated as follows:

$$\mathcal{D}_T = \partial_t(v_{\epsilon\ell}V^\ell) + \nabla_j(v_{\epsilon\ell}V^\ell v_\epsilon^j) + \nabla_j\left(\left(\frac{|v_\epsilon|^2}{2} + p_\epsilon\right)V^j\right)$$

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$$= \nabla_j[v_{\epsilon\ell}R_T^{j\ell}] - \nabla_j v_{\epsilon\ell}\underbrace{R_T^{j\ell} + V_\ell}_{\text{freq. } \approx\lambda}(\nabla_j R_\epsilon^{j\ell} + \cdots)$$

The latter terms can be absorbed into $\overset{*}{\varphi}$ by applying div^{-1}.
The first term is part of $\nabla_j[\overset{*}{v}_\ell \overset{*}{R}^{j\ell}] + \nabla_j \overset{*}{\varphi}^j$.

Getting rid of the Unresolved Flux Current φ

We need to cancel out both the unresolved flux density and current

$$\mathcal{D}_\kappa = \partial_t \left(\frac{|V|^2}{2} + \kappa \right) + \nabla_j \left(\left(\frac{|V|^2}{2} + \kappa \right) v^j \right)$$

$$\mathcal{D}_\varphi = \nabla_j \left[\sum_{I,J,K} (V_I)_\ell V_J^j V_K^\ell + PV^j + \varphi^j \right]$$

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$$\mathcal{D}_\varphi = \nabla_j \left[\sum_{I,J,K} (V_I)_\ell V_J^j V_K^\ell + PV^j + \varphi^j \right]$$

Let's start by canceling out the unresolved flux current φ^j.

Getting rid of the Unresolved Flux Current φ

Recall $V = \sum_I V_I$, $V_I = e^{i\lambda\xi_I} v_I + \dots$

$$\mathcal{D}_\varphi = \mathcal{D}_{\varphi L} + \mathcal{D}_{\varphi H}$$

$$\mathcal{D}_{\varphi L} = \nabla_j \left[\sum_{(I,J,K)\in\mathcal{T}} (V_I)_\ell V_J^j V_K^\ell + \varphi_\epsilon^j \right]$$

$$\mathcal{D}_{\varphi H} = \nabla_j \left[\sum_{(I,J,K)\notin\mathcal{T}} (V_I)_\ell V_J^j V_K^\ell + PV^j + (\varphi^j - \varphi_\epsilon^j) \right]$$

Getting rid of the Unresolved Flux Current φ

Recall $V = \sum_I V_I$, $V_I = e^{i\lambda\xi_I} v_I + \dots$

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The last terms may be absorbed into $\overset{*}{\varphi}{}^j$ after applying div^{-1}.

Getting rid of the unresolved flux density κ

We want our waves to cancel out the unresolved flux density κ

$$\mathcal{D}_\kappa = \partial_t \left(\frac{|V|^2}{2} + \kappa \right) + \nabla_j \left(\left(\frac{|V|^2}{2} + \kappa \right) v^j \right)$$

$$\mathcal{D}_\varphi = \nabla_j \left[\sum_{I,J,K} (V_I)_\ell V_J^j V_K^\ell + P V^j + \varphi^j \right]$$

Getting rid of the unresolved flux density κ

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$$
\begin{aligned}
\mathcal{D}_\kappa &= \mathcal{D}_{\kappa L} + \mathcal{D}_{\kappa H} \\
\mathcal{D}_{\kappa L} &= \overline{D}_t \left[\sum_I \frac{V_I \cdot \overline{V_I}}{2} + \kappa_\epsilon \right] \\
\mathcal{D}_{\kappa H} &= \overline{D}_t \left[\sum_{J \neq \bar{I}} \frac{V_I \cdot V_J}{2} + (\kappa - \kappa_\epsilon) \right] + \cdots
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The highlighted term is positive and cannot cancel out a general κ.

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Choose a non-negative, nonincreasing function $e(t) \geq 0$:

$$\mathcal{D}_{\kappa L} = \overline{D}_t \left[\sum_I \frac{v_I \cdot \overline{v_I}}{2} + \kappa_\epsilon - e(t) \right] + \underbrace{\overline{D}_t e(t)}_{\leq 0}$$

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This point is where the *arrow of time* comes into play.

We can absorb $D_t e(t) \leq 0$ into the new dissipation measure $\overset{*}{\mu}$. (Recall that $\mathcal{D}[v,p] \leq D_t \kappa + \nabla_j [v_\ell R^{j\ell}] + \nabla_j \phi^j$ allows for inequality in the relaxed local energy inequality.)

Now it is possible to cancel out κ_ϵ except...

Getting rid of the unresolved flux density κ

Conflict: We need to simultaneously satisfy two equations

$$\sum_I v_I^j \overline{v_I^\ell} + P(t,x)\delta^{j\ell} + R_\epsilon^{j\ell} = 0$$

$$\sum_I \frac{v_I \cdot \overline{v_I}}{2} + \kappa_\epsilon - e(t) = 0$$

The summation in the second is half the trace of the summation in the first equation.

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The summation in the second is half the trace of the summation in the first equation. However, we can uniquely specify the pressure increment $P(t,x) = (2/3)(-e(t) + \kappa_\epsilon - \mathrm{tr} R_\epsilon/2)$ so that the first equation implies the second equation.

Conflict: Eliminate the Unresolved Flux Current and Stress

Another conflict: Recall that we solved

$$\sum_{(I,J,K)\in\mathcal{T}} v_{I\ell} v_J^j v_K^\ell + \varphi_{\epsilon[1]}^j = 0$$

And need to solve

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To do both, use waves in $\mathcal{I}_\varphi$ to eliminate the current $\varphi_{[1]}$ then

$$\sum_{I\notin\mathcal{I}_\varphi} v_I^j \overline{v_J^\ell} = -P\delta^{j\ell}_{[1]} - R^{j\ell}_{[1]\epsilon} - \sum_{I\in\mathcal{I}_\varphi} v_I^j \overline{v_I^\ell}$$

absorb the $\mathcal{I}_\varphi$ terms as a slightly lower order term.
Ok as long as $\|\varphi_{[1]}\|^{2/3}$ is lower order!

Dangerous terms

The requirement that $\|\phi_{[1]}\|^{2/3}$ be lower order than $\|R_{[1]}\|$ is problematic and threatens the success of the iteration.

It means that for the next stage we need to ensure $\|\overset{*}{\varphi}_{[2]}\|^{2/3}$ is lower order compared to $\|\overset{*}{R}_{[2]}\|$.

For example, $\overset{*}{\varphi}$ contains a dangerous interaction term of the form $V_\ell R_S^{j\ell} = V_\ell(P\delta_{[2]}^{j\ell} + R_{[2]}^{j\ell} + \cdots)$, where $\delta_{[2]} = e_1 \otimes e_1 + (e_3 \otimes e_3)/2$ is a component of $\delta^{j\ell} = \delta_{[1]}^{j\ell} + \delta_{[2]}^{j\ell}$ taking values in $\langle e_2\rangle^\perp \otimes \langle e_2\rangle^\perp$. The vector field $V_\ell P\delta_{[2]}^{j\ell}$ takes values in $\langle e_2\rangle^\perp$ as desired, but has the same order of magnitude as $\|\overset{*}{R}_{[2]}\|^{3/2}$.

Dangerous terms: algebraic cancellation saving the day

What saves us from this difficulty is that the *divergence* of $V_\ell P \delta_{[2]}^{j\ell}$ has a good cancellation:

$$\nabla_j [V_\ell P \delta_{[2]}^{j\ell}] = \sum_I \lambda e^{i\lambda \xi_I} (i \nabla_j \xi_I) v_{I\ell} P \delta_{[2]}^{j\ell} + \cdots$$

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We have that $\nabla \xi_I \overset{\approx}{\in} \langle e_1 \rangle$, $v_I \overset{\approx}{\in} \langle e_2, e_3 \rangle$ and $\delta_{[2]} = e_1 \otimes e_1 + (e_3 \otimes e_3)/2$, so the leading order term approximately vanishes.

We then apply div^{-1} to obtain an acceptable contribution to the new unresolved current $\mathring{\varphi}^j$. Other error terms exhibit similar favorable cancellations or can be directly absorbed into $\mathring{\varphi}^j$.

Thank you!