The role of stochastic differential games of mean-field type in smart cities applications

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Why stochastic differential games?

- Interactive decision-making theory
- Dynamical systems
- Optimization-based systems
- Uncertainty and risk quantification (distribution of the variables of interests)

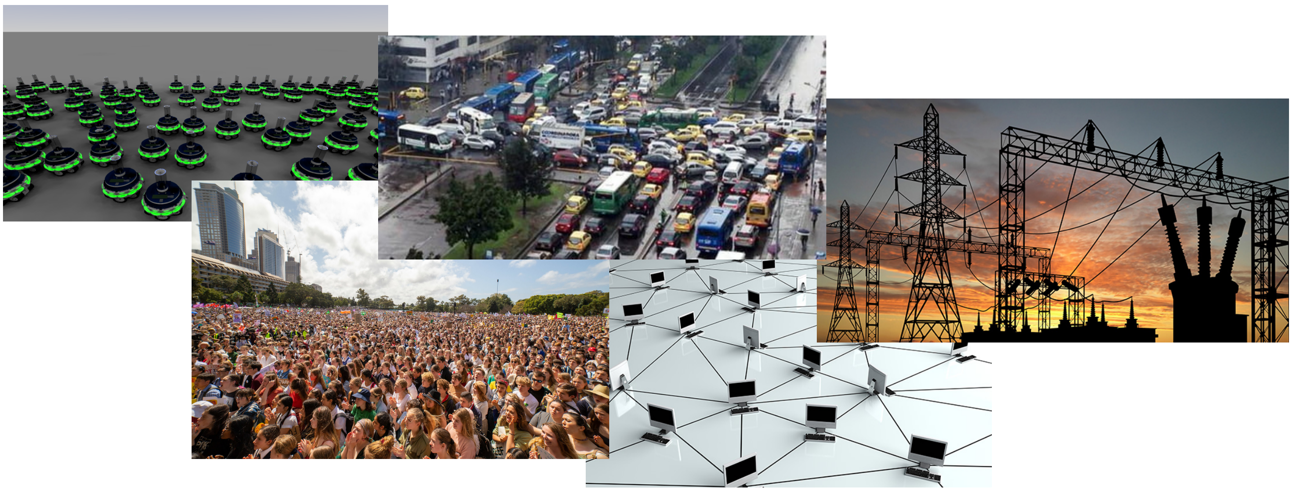

Figure: Engineering applications

Stochastic differential game of mean-field type

Let $\mathcal{I} := \{1, \ldots, I\}$ be decision-makers interacting through the following SDE:

$$dx(t) = \beta(t, x(t), \mu(t, \cdot), u(t))dt + \sigma(t, x(t), \mu(t, \cdot), u(t))dB(t), \ x(0) = x_0 \sim \mu_0, \qquad (1)$$

where $u(t) := (u_1(t), \ldots, u_I(t))$. The accumulative cost for $i \in \mathcal{I}$ is as follows:

$$J_i(u_i, u_{-i}) = \psi_i(T, x(T), \mu(T, \cdot)) + \int_0^T l_i(t, x(t), \mu(t, \cdot), u(t))dt$$

The non-cooperative stochastic differential game of mean-field type consists of

$$\forall i \in \mathcal{I}: \quad \inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}[J_i(u_i, u_{-i})], \text{ s. t. } (1)$$

Best-response strategy

For some given actions $u_{-i}(\cdot)$ made by decision-makers $\mathcal{I} \setminus \{i\}$:

$$\inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}[J_i(u_i, u_{-i}) | x_0 \sim \mu_0, u_{-i}(\cdot) \in \mathcal{U}_{-i}], \text{ s. t. } (1). \qquad (2)$$

A strategy $u_i^*(\cdot)$ that solves Problem (2) is called a best-response of decision-maker $i \in \mathcal{I}$ and the set of best-response strategies is denoted by $\mathrm{BR}_i(u_{-i}(\cdot))$.

Nash equilibria

A strategic profile $u^*(\cdot) := (u_1^*(\cdot), \ldots, u_I^*(\cdot)) \in \mathcal{U}_1 \times \cdots \times \mathcal{U}_I$ is a Nash equilibrium if

$$u_i^*(\cdot) \in \mathrm{BR}_i(u_{-i}^*(\cdot)), \quad \text{for all } i \in \mathcal{I}.$$

Stochastic differential game of mean-field type

If there exits a function v_i, for all $i \in \mathcal{I}$, such that

$$\begin{cases} \dfrac{\partial}{\partial t} v_i = -\inf_{u_i \in U_i} \displaystyle\int_{\mathbb{R}} \Bigg[l_i(t, x, \mu, u) + \left\langle \beta(t, x, \mu, u), \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial \mu} v_i \right\rangle \\ \qquad + \dfrac{1}{2} \left\langle \dfrac{\partial^2}{\partial x^2}\dfrac{\partial}{\partial \mu} v_i \ \sigma(t, x, \mu, u), \sigma(t, x, \mu, u) \right\rangle \Bigg] \mu(t, dx), \\ v_i(T, x(T), \mu(T)) = \displaystyle\int_{\mathbb{R}} \psi_i(T, x(T), \mu(T, \cdot))\mu(T, dx) \end{cases}$$

Then $v_i(0, x_0, \mu_0)$ is the equilibrium cost and u_i^* is the optimal strategy for decision-maker $i \in \mathcal{I}$.

$$H_i(t, x, \mu, p, S) := \inf_{u_i \in U_i} \int_{\mathbb{R}} \left[l_i(t, x, \mu, u) + \langle \beta(t, x, \mu, u), p \rangle + \frac{\langle S\sigma(t, x, \mu, u), \sigma(t, x, \mu, u)\rangle}{2} \right] \mu(t, dx)$$

The PIDE can be re-written as:
$$\begin{cases} \dfrac{\partial}{\partial t} v_i = -H_i\left(t, x, \mu, \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial \mu} v_i, \dfrac{\partial^2}{\partial x^2}\dfrac{\partial}{\partial \mu} v_i \right), \\ v_i(T, x(T), \mu(T)) = \displaystyle\int_{\mathbb{R}} \psi_i(T, x(T), \mu(T, \cdot))\mu(T, dx). \end{cases}$$

Fokker-Plank-Kolmogorov equation: $\frac{\partial}{\partial t}\mu + \frac{\partial}{\partial x}[\beta\mu] + \frac{1}{2}\frac{\partial^2}{\partial x^2}[\sigma^2\mu] = 0, \ \mu(0, \cdot) = \mu_0.$

A. Bensoussan, J. Frehse & S. C. P. Yam. *Mean field games and mean field type control theory.* Springer Briefs in Mathematics, Vol. 1. New York, 2013

H. Tembine. *Mean-Field-Type Games.* AIMS Mathematics, 2(4):706-735, 2017

Non-cooperative LQ case

Consider the following SDE with state-and-input-independent diffusion term:

$$dx(t) = \beta(t, x(t), \mathbb{E}[x(t)], u(t), \mathbb{E}[u(t)])dt + \sigma(t)dB(t), \ x(0) := x_0, \ \mathbb{E}[x(0)] := \mathbb{E}[x_0], \qquad (3)$$

and with linear drift term:

$$\beta(t, \cdot) = b_0(t) + b_1(t)x + \bar{b}_1(t)\mathbb{E}[x(t)] + \sum_{j\in\mathcal{I}} b_{2j}(t)u_j(t) + \sum_{j\in\mathcal{I}} \bar{b}_{2j}(t)\mathbb{E}[u_j(t)]$$

Each decision-maker pursues to minimize the following cost functional:

$$\begin{aligned} J_i(u_i, u_{-i}) &= \frac{1}{2}q_i(T)x(T)^2 + \frac{1}{2}\bar{q}_i(T)\mathbb{E}[x(T)]^2 \\ &\quad + \frac{1}{2}\int_0^T \left(q_i(t)x(t)^2 + \bar{q}_i(t)\mathbb{E}[x(t)]^2 + r_i(t)u_i(t)^2 + \bar{r}_i(t)\mathbb{E}[u_i(t)]^2\right) dt. \end{aligned}$$

The non-cooperative LQ stochastic differential game of mean-field type is

$$\forall i \in \mathcal{I} : \inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}[J_i(u_i, u_{-i})], \text{ s. t. } (3)$$

It can be solved semi-explicitly using an appropriate ansatz (implementation interests).

J. Barreiro-Gomez and H. Tembine *Mean-Field-Type Games for Engineers.*
CRC Press Taylor & Francis Group, ISBN/EAN: 0367566125/9780367566128 pp. 528. November, 2021

Application: Evacuation problems

Objective: Study large-number of decision-makers. Optimal policies for evacuation.

Evacuation Procedure

From initial distribution to the exits (minimize evacuation time)

Traffic Systems

From origin to destination (minimize travel time)

J. M. Lasry and P. L. Lions. *Mean field games.*
Japanese Journal of Mathematics, 2(2007):229-260, 2007

M. Huang, R. P. Malhamé and P. E. Caines. *Large population stochastic dynamic games: closed-loop McKean-Vlasov systems.*
Communications in information and systems, 6(2006):221-251, 2006

A. Bensoussan, J. Frehse and S.C.P. Yam. *Mean Field Games and Mean Field Type Control Theory.*
Springer Briefs in Mathematics, New York, 2013

D. A. Gomes. *Mean field games models–a brief survey.*
Dynamic Games and Applications, 4(2):110-154, 2014

Crowd evacuation problem: Main considerations

Evacuation problems

1. Optimizing time horizon
2. Considering fixed-time horizon and optimizing a running cost

Two main approaches within fixed-time horizon problem

↙ ↘

Local Congestion Terms

Only agents in a neighborhood
Prevent the formation of agglomeration by means of velocity penalty

Global Congestion Terms

All agents in the problem
Crowd aversion (spread of agents)

E. Cristiani and F. S. Priuli and A. Tosin. *Modeling Rationality to Control Self-Organization of Crowds: An Environmental Approach.* **SIAM Journal on Applied Mathematics**, 75(2):605–629, 2015

B. Djehiche, A. Tcheukam and H. Tembine. *A Mean-Field Game of Evacuation in Multilevel Building.* **IEEE Transactions on Automatic Control**, 62(10):5154-5169, 2017

N. Toumi, R. Malhamé and J. Le Ny. *A Tractable Mean Field Game Model for the Analysis of Crowd Evacuation Dynamics.* **Proceedings of the 59th IEEE Control Conference on Decision and Control (CDC)**, DOI: 10.1109/CDC42340.2020.9303802, 2020

J. Barreiro-Gomez, S. E. Choutri and H. Tembine. *Risk-awareness in multi-level building evacuation with smoke:* ***Burj Khalifa*** *case study.* **Automatica**, 129(2021):109625, 2021

A. Aurell and B. Djehiche. *Mean-field type modeling of nonlocal crowd aversion in pedestrian crowd dynamics.* **SIAM Journal on Control and Optimization**, 56(1):434-455, 2018

Evacuation preliminaries

Decision-makers $\mathcal{I} := \{1, \dots, I\}$ pursue to evacuate $\mathbb{D} \subset \mathbb{R}^2$ with exits $\mathbb{S} \subset \mathrm{bd}(\mathbb{D})$.

$$dx_i(t) = u_i(t)dt + \sigma_i(t)dB_i(t), \text{ for all } t \in [0, T], \ x_i(0) \in \mathbb{D},$$

where $x_i(t)$, $u_i(t)$, and $B_i(t)$ are the position, strategy, and Brownian motion for $i \in \mathcal{I}$.

$$\begin{aligned} u(t) &:= (u_1(t), \dots, u_I(t)), \\ u_{-i}(t) &:= (u_1(t), \dots, u_{i-1}(t), u_{i+1}(t), \dots, u_I(t)), \\ x(t) &:= (x_1(t), \dots, x_I(t)). \end{aligned}$$

The decision-makers can identify the crowd within a radius $\varepsilon \in \mathbb{R}_{>0}$

$$\mathcal{B}(x_i(t), \varepsilon) = \{y \in \mathbb{D} \setminus \mathrm{bd}(\mathbb{D}) : d(x_i(t), y) \leq \varepsilon\}.$$

The nearest exit that the decision-maker $i \in \mathcal{I}$ knows (or can identify)

$$\xi_i(t) \in \arg\min_{y \in \mathbb{S}} \ d(x_i(t), y), \ \forall \ i \in \mathcal{I}.$$

J. Barreiro-Gomez *Stochastic differential games for crowd evacuation problems: A paradox.* **Automatica**, 140(2022):110271, 2022

Evacuation preliminaries

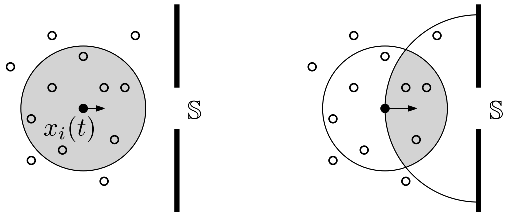

Figure: Example for the local congestion measurement.

$$\mathcal{C}_i^1(x(t),\varepsilon) = \{j \in \mathcal{I} : x_j(t) \in \mathcal{B}(x_i(t),\varepsilon)\},\ \forall\ i \in \mathcal{I},$$
$$\mathcal{C}_i^2(x(t),\varepsilon) = \{j \in \mathcal{I} : x_j(t) \in \mathcal{B}(x_i(t),\varepsilon)\ \wedge\ d(x_j(t),\xi_i) \leq d(x_i(t),\xi_i)\}, \forall\ i \in \mathcal{I}.$$

We will refer to $\mathcal{C}_i(x(t),\varepsilon)$ from now on.

Local congestion term

Let $\gamma_i(x(t),\varepsilon)$ be a local mean-field term for the decision-maker $i \in \mathcal{I}$. Local since $\gamma_i(x(t),\varepsilon)$ only depends on $\mathcal{B}(x_i(t),\varepsilon)$, e.g., $\mathrm{card}(\mathcal{C}_i(x(t),\varepsilon))$.

Global congestion term

Let $\bar{x}(t)$ be a global mean-field term for any decision-maker $i \in \mathcal{I}$. This term is global since $\bar{x}(t) = \frac{1}{\mathrm{card}(\mathcal{I})}\sum_{j\in\mathcal{I}} x_j(t)$ depends on all the decision-makers in the set $\mathcal{I}$.

General evacuation problem

Let us consider the following cost function:

$$G_i(u,\xi^i) = \underbrace{\psi_i(x(T),\bar{x}(T),\xi_i(T))}_{\text{Terminal cost}} + \frac{1}{2}\int_{t_0}^{T} \underbrace{\overbrace{\gamma_i(x(t),\varepsilon)\langle u_i(t),u_i(t)\rangle}^{\text{Kinetic}} + \overbrace{f_i(x(t),\bar{x}(t),\xi_i(t))}^{\text{Potential}}}_{\text{Running cost}} dt.$$

The evacuation problem is given by

$$\inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}[G_i(u,\xi_i(t))],\ \text{s. t.}\ dx_i(t) = u_i(t)dt + \sigma_i(t)dB_i(t),\ t\in[t_0,T],\ x_i(0)\in\mathbb{D}.$$

where $\mathcal{U}_i$ set of measurable feasible control inputs for $i \in \mathcal{I}$.

Best-response strategy

The planning strategy $u_i^*(t)$ to evacuate $\mathbb{D}$ is a best-response strategy if it solves

$$\inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}[G_i(u_i,u_{-i},\xi_i(t)) \mid u_{-i}(\cdot)],$$

subject to SDE. The set of best-response strategies $\mathrm{BR}_i(u_{-i}(t))$, for all $i \in \mathcal{I}$.

Nash equilibria

Strategic profile $u^*(t) \triangleq (u_1^*(t),\ldots,u_I^*(t))$ Nash equilibrium if $u_i^*(t) \in \mathrm{BR}_i(u_{-i}^*(t))$, for all $i \in \mathcal{I}$.

Solution

The optimal cost functional is denoted by $v_i(t, x_i)$ from t up to T as follows:

$$v_i(t, x_i) = \inf_{u_i(\cdot)\in\mathcal{U}_i} \mathbb{E}\left[\psi_i(x_i(T), \bar{x}(T), \xi_i(T)) + \frac{1}{2}\int_t^T \gamma_i(x(s), \varepsilon)\langle u_i(s), u_i(s)\rangle + f_i(x_i(s), \bar{x}(s), \xi_i(s))ds\right].$$

The HJB PDE is given by

$$\frac{\partial}{\partial t} v_i(t, x_i) = -H_i\left(t, x, \frac{\partial}{\partial x_i} v_i, \frac{\partial^2}{\partial x_i^2} v_i\right), \qquad v_i(T, x_i) = \psi_i(x_i(T), \bar{x}(T), \xi_i(T)),$$

where the Hamiltonian is as follows:

$$H_i(t, x, p, S) = \inf_{u_i\in U_i} \mathbb{E}\left[\frac{1}{2}\gamma_i(x, \varepsilon)\langle u_i, u_i\rangle + \langle u_i, p\rangle + \frac{1}{2} f_i(x_i, \bar{x}, \xi_i) + \frac{1}{2}\langle S\sigma_i(t), \sigma_i(t)\rangle\right].$$

Optimizing in u_i, the Hamiltonian becomes

$$H_i(t, x, p, S) = -\frac{1}{2\gamma_i(x(t), \varepsilon)}\langle p, p\rangle + \frac{1}{2} f_i(x_i, \bar{x}, \xi_i) + \frac{1}{2}\langle S\sigma_i(t), \sigma_i(t)\rangle.$$

It follows that, the HJB PDE is re-written

$$\begin{aligned} 0 &= \frac{\partial}{\partial t} v_i(t, x_i) + \frac{1}{2} f_i(x_i(t), \bar{x}(t), \xi_i(t)) \\ &- \frac{1}{2\gamma_i(x(t), \varepsilon)}\left\langle \frac{\partial}{\partial x_i} v_i(t, x_i), \frac{\partial}{\partial x_i} v_i(t, x_i)\right\rangle + \frac{1}{2}\left\langle \frac{\partial^2}{\partial x_i^2} v_i(t, x_i)\sigma_i(t), \sigma_i(t)\right\rangle. \end{aligned}$$

The corresponding optimal cost $G_i(u^*, \xi_i) = v_i(0, x_i(0))$.

Solution

The optimal control input for the decision-maker $i \in \mathcal{I}$ is given by

$$u_i^*(t) = -\frac{1}{\gamma_i(x(t), \varepsilon)}\frac{\partial}{\partial x_i} v_i(t, x_i),$$

where $v_i(t, x_i)$ is the optimal value function from time t up to terminal time T

$$\begin{aligned} \frac{\partial}{\partial t} v_i(t, x_i) &= -\frac{1}{2} f_i(x_i(t), \bar{x}(t), \xi_i) + \frac{1}{2\gamma_i(x(t), \varepsilon)}\left\langle \frac{\partial}{\partial x_i} v_i(t, x_i), \frac{\partial}{\partial x_i} v_i(t, x_i)\right\rangle \\ &- \frac{1}{2}\left\langle \frac{\partial^2}{\partial (x_i)^2} v_i(t, x_i)\sigma_i(t), \sigma_i(t)\right\rangle, \ t \in [0, T), \end{aligned}$$

with terminal boundary condition given by

$$v_i(T, x_i) = \psi_i(x_i(T), \bar{x}(T), \xi_i(T)).$$

The term $\gamma_i(x(t), \varepsilon) = \sum_{j\in\mathcal{I}} \mathbb{1}_{x_j\in\mathcal{B}(x_i(t), \varepsilon)}$ creates some difficulties.

B. Djehiche, A. Tcheukam and H. Tembine. *A Mean-Field Game of Evacuation in Multilevel Building.* IEEE Transactions on Automatic Control, 62(10):5154-5169, 2017

Receding-horizon philosophy

Let $N \in \mathbb{N}$ time intervals in receding-horizon control. Every $\Delta t = T/N$ decision-makers design a new evacuation planning.

$$G_{i|k}(u_k, \xi_{i|k}) = \psi_{i|K}(x(T|k), \bar{x}(T|k), \xi_{i|k}) + \frac{1}{2}\int_{t_0(k)}^{T} \gamma_{i|k}(x(t_0(k)), \varepsilon)\langle u_i(t|k), u_i(t|k)\rangle + f_{i|k}(x(t|k), \bar{x}(t|k), \xi_{i|k})dt,$$

where $k \in [0..N]$, $\xi_{i|k} = \xi_i(t_0(k))$ and $t_0(k) = k\Delta t$.

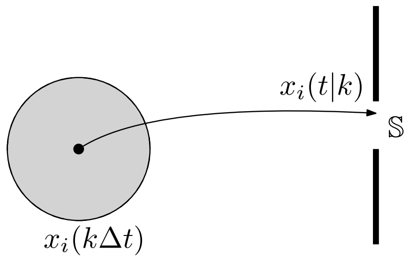

Figure: Simple diagram to explain the receding-horizon control philosophy.

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Automatica, 140(2022):110271, 2022

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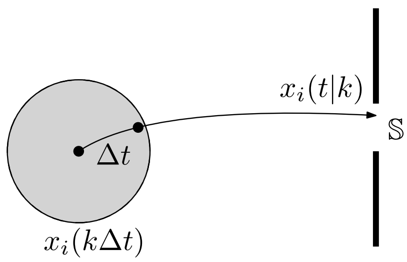

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Receding-horizon philosophy

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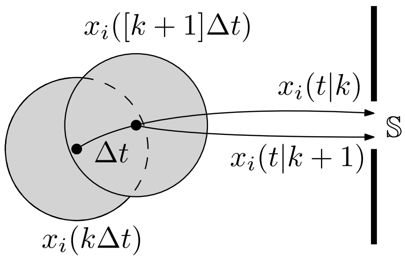

Figure: Simple diagram to explain the receding-horizon control philosophy.

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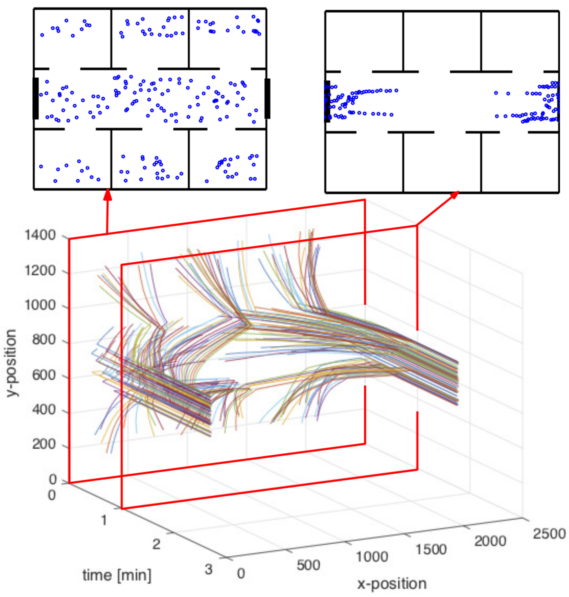
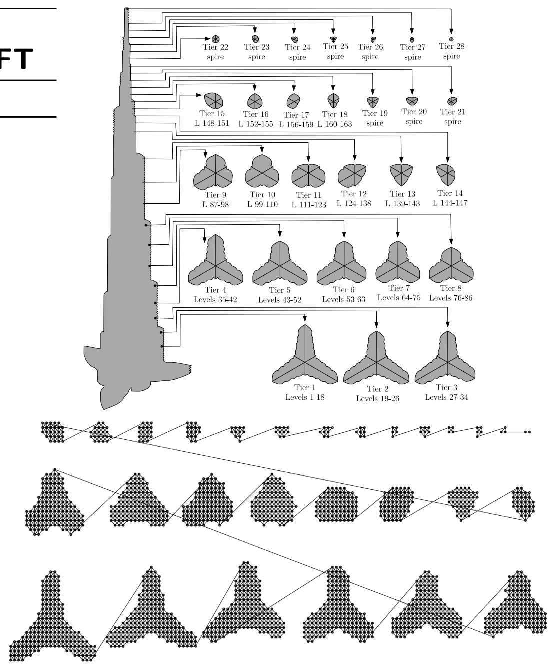
Results: Illustrative example

Total crowd of $\mathrm{card}(\mathcal{I}) = 200$.

Table: Summary of evacuation times for the two scenarios and both evacuation strategies.

	Evacuation Time	
Scenario	**Local MFT**	**Local/Global MFT**
Multiple exits	2[min]	1.5[min]





Neural network differential games of mean-field type

Some important considerations. Decision-makers:

- do not have knowledge about the model
- only have access to data of the system behavior
- can interact with the system to "learn" from it

This leads to the need ot AI techniques

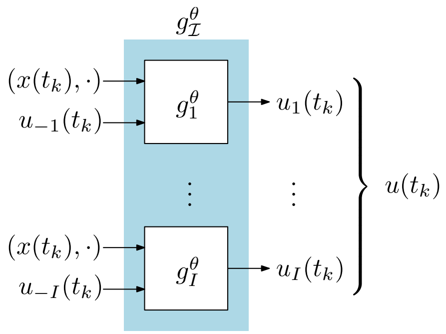


Figure: Computation architecture.

Layer functions, input dim d_1, output dim d_2, and activation func $h_j : \mathbb{R} \to \mathbb{R}$, $j \in \{0, \dots, n\}$, by

$$\mathbb{L}^{h_j}_{d_1,d_2} = \left\{ \phi : \mathbb{R}^{d_1} \to \mathbb{R}^{d_2} | \exists b \in \mathbb{R}^{d_2}, \exists W \in \mathbb{R}^{d_2 \times d_1}, \forall i \in \{1, \dots, d_2\}, \phi(z)_i = h_j\left(b_i + \sum_{k=1}^{d_1} W_{ik} z_k \right) \right\}$$

where $z := (z_1, \dots, z_{d_1})$ input vector.

J. Barreiro-Gomez, S. E. Choutri, and B. Djehiche *Stability of Stochastic Mean-Field-Type Games Via Non-Cooperative Neural Networks Adversarial Training*. **Preprint**, 2022

R. Carmona and M. Laurière *Convergence analysis of machine learning algorithms for the numerical solution of mean field control and games: II-the finite horizon case*. **Annals of Applied Probability**, (to appear)

R. Carmona and M. Laurière *Convergence Analysis of Machine Learning Algorithms for the Numerical Solution of Mean Field Control and Games I: The Ergodic Case*. **SIAM Journal on Numerical Analysis**, 59(3):1455-1485, 2021

Neural network differential games of mean-field type

Set of NNs with n hidden layers and one output

$$\mathcal{U}_i^{\mathbf{NN}} = \left\{ g_i^{\theta} : \mathbb{R}^{d_0} \to \mathbb{R}^{d_{n+1}} | \forall j \in \{0 \dots, n_i\}, \exists \phi_i^j \in \mathbb{L}^{h_j}_{d_j,d_{j+1}},\ g_i^{\theta} = \phi_i^{n_i - 1} \circ \phi_i^{n_i} \circ \dots \circ \phi_i^0 \right\},$$

for all $i \in \mathcal{I}$. Parameters to be trained $\theta_i := \left\{ W_i^{(0)}, b_i^{(0)}, W_i^{(1)}, b_i^{(1)}, \dots, W_i^{(n-1)}, b_i^{(n-1)} \right\}$, and

$$g_{\mathcal{I}}^{\theta} := (g_1^{\theta}, \dots, g_I^{\theta}) \in \mathcal{U}_1^{\mathbf{NN}} \times \dots \times \mathcal{U}_I^{\mathbf{NN}}.$$

For a finite $T > 0$ and $N_T \in \mathbb{N}^+$, let $\Delta t = T/N_T$ and $t_k = k\Delta t,\ \ k \in \{0, \dots, N_T - 1\}$.

$$J_i^N(u_i, u_{-i}) = \frac{1}{N} \sum_{j=1}^{N} \left(\psi_i(x^j(t_{N_T}), \mu_x(t_{N_T})) + \sum_{k=0}^{N_T - 1} \ell_i(t_k, x^j(t_k), \mu_x(t_k), u_i^j(t_k), \mu_{u_i}(t_k))\Delta t \right),$$

where,

$$\mu_x(t_k) = \frac{1}{N} \sum_{j=1}^{N} x^j(t_k), \qquad \mu_{u_i}(t_k) = \frac{1}{N} \sum_{j=1}^{N} u_i^j(t_k), \text{ for all } i \in \mathcal{I}$$

and $\mu_u(t_k) := (\mu_{u_1}(t_k), \dots, \mu_{u_I}(t_k))$.

J. Barreiro-Gomez, S. E. Choutri, and B. Djehiche *Stability of Stochastic Mean-Field-Type Games Via Non-Cooperative Neural Networks Adversarial Training*. **Preprint**, 2022

Discretized Stochastic Differential Game Problem

The non-cooperative problem:

$$\forall i \in \mathcal{I}: \inf_{u_i(\cdot)\in\mathcal{U}_i} J_i^N(u_i, u_{-i}), \text{ s. t. } x^j(t_{k+1}) = x^j(t_k) + f(t_k, x^j(t_k), \mu(t_k), u^j(t_k), \mu_u(t_k)))\Delta t + \sigma B_k^j,$$

$x_0^i \sim \mu_0,\ B_k^j \sim \mathcal{N}(0, \Delta t),\ k \in \{0, \ldots, N_T - 1\},\ \ j \in \{1, \ldots, N\},$ and $x_0^j \sim \mu_0$.

Neural network as best-response functions:

$$L_i^N(\theta) = \sum_{j=1}^{N} \sum_{k=0}^{N_T-1} \left\| g_i^\theta(\cdot, u_{-i}^{j*}(t_k)) - u_i^{j*}(t_k) \right\|, \text{ for all } i \in \mathcal{I},$$

Minimize $L_i^N(\theta)$ by suitable NNs $g_{\mathcal{I}}^{\bar{\theta}}(z) \in \mathcal{U}_1^{\text{NN}} \times \cdots \times \mathcal{U}_I^{\text{NN}}$

$$\forall\ i \in \mathcal{I}: \quad \inf_{\theta_i} L_i^N(\theta_i), \text{ s. t. : compatible architecture } g_{\mathcal{I}}^\theta, \text{ and } u_i^{j*},\ j = 1, \ldots, N, \text{ given.}$$

The optimal weight and bias parameters are obtained as

$$\theta_i^* \in \arg\min_{\theta_i} L_i^N(\theta_i).$$

J. Barreiro-Gomez, S. E. Choutri, and B. Djehiche *Stability of Stochastic Mean-Field-Type Games Via Non-Cooperative Neural Networks Adversarial Training*. **Preprint**, 2022

Toy LQ Example

Let us consider the following two-decision-maker $\mathcal{I} = \{1, 2\}$ discrete-time dynamics:

$$x(t_{k+1}) = b_1 x(t_k) + \bar{b}_1 \mathbb{E}[x(t_k)] + b_{21} u_1(t_k) + \bar{b}_{21} \mathbb{E}[u_1(t_k)] + b_{22} u_2(t_k) + \bar{b}_{22} \mathbb{E}[u_2(t_k)] + \sigma B_k,$$

and the cost functional for the $i-$th decision-maker:

$$J_i(u_i, u_{-i}) = q_i(t_{N_T})(x(t_{N_T}) - \mathbb{E}[x(t_{N_T})])^2 + \bar{q}_i(t_{N_T})\mathbb{E}[x(t_{N_T})]^2$$
$$+ \sum_{k=0}^{N_T-1} \Big(q_i(t_k)(x(t_k) - \mathbb{E}[x(t_k)])^2 + \bar{q}_i(t_k)\mathbb{E}[x(t_k)]^2 + r_i(t_k)(u_i(t_k) - \mathbb{E}[u_i(t_k)])^2 + \bar{r}_i(t_k)\mathbb{E}[u_i(t_k)]^2 \Big)$$

Parameters: $b_1 = 1,\ \bar{b}_1 = 0.5,\ b_{21} = 1,\ \bar{b}_{21} = 1.5,\ b_{22} = 2,\ \bar{b}_{22} = 2.5,\ \sigma = 1,\ q_1 = 5,\ \bar{q}_1 = 5,$ $q_2 = 10,\ \bar{q}_2 = 10,\ r_1 = 1,\ \bar{r}_1 = 1,\ r_2 = 2,\ \bar{r}_2 = 2,$ for all $k = 0, \ldots, N_T$.

Table: Considered neural network architectures

	Non-cooperative computation	
	Decision-maker 1: g_1^θ	**Decision-maker 2:** g_2^θ
Layers	3	5
Neurons per layer	$\{3, 10, 2\}$	$\{3, 10, 10, 10, 2\}$
Total number of neurons	14	34
Activation functions	$\{\text{lin}, \tanh, \text{lin}\}$	$\{\text{lin}, \tanh, \tanh, \tanh, \text{lin}\}$

Stability: Closed-loop using the NN

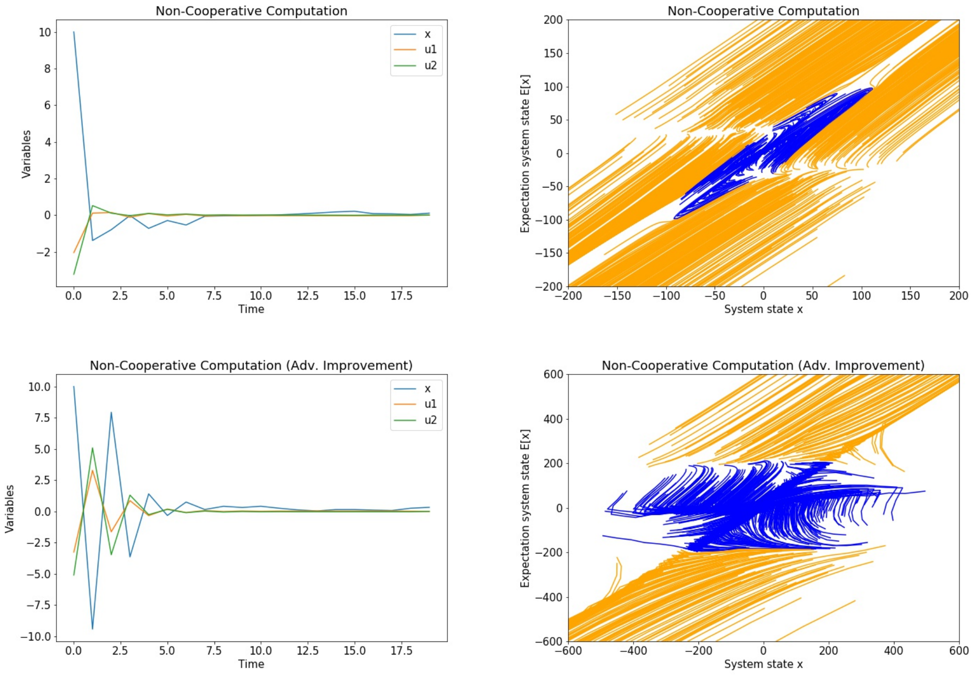


Figure: Closed-loop behavior using the trained neural network, and stability characterization.

Other SITE Applications

- Water distribution network

S. E. Choutri and B. Djehiche. *Stochastic Maximum Principle for Mean-Field Control with Almost Sure Pure State Constraints.* On Going Research, **2022**

J. Barreiro-Gomez and H. Tembine. *Mean-Field-Type Model Predictive Control: An Application to Water Distribution Networks.* IEEE Access, 7, 135332-135339, **2019**

- COVID-19 propagation

H. Tembine. *COVID-19: data-driven mean-field-type game perspective.* Games, 11(4):51, **2020**

Z. El Oula Frihi, J. Barreiro-Gomez, S. E. Choutri and H. Tembine. *Toolbox to Simulate and Mitigate COVID-19 Propagation.* SoftwareX, 100673, **2021**

- Price dynamics for smart grids

Z. E. O. Frihi, S. E. Choutri, J. Barreiro-Gomez, H. Tembine. *Hierarchical Mean-Field Type Control of Price Dynamics for Electricity in Smart Grid.* Journal of Systems Science and Complexity, 35(1):1-17, 2022

- Leadership design

Z. E. O. Frihi, J. Barreiro-Gomez, S. E. Choutri, H. Tembine. *Hierarchical structures and leadership design in mean-field-type games with polynomial cost.* Games, 11(3):30, 2020

- Opinion Dynamics

J. Barreiro-Gomez *et al. Distributed data-driven UAV formation control via evolutionary games: Experimental results.* Journal of the Franklin Institute, 358 (10), 5334-5352, 2021

Thank you very much for your attention

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