Existence of non-trivial stationary solutions to the 2D Euler equation

Jaemin Park

University of Barcelona

European Research Council
Established by the European Commission

European Commission

Project: STG 852741 CAPA

Joint work with Javier Gómez-Serrano, Jia Shi and Yao Yao

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Long Time Behavior and Singularity Formation in PDEs: Part V

June 01, 2022

2D Euler Equation

- 2D incompressible Euler equation in vorticity form:

$$\begin{cases} \omega_t + u \cdot \nabla \omega = 0 & \text{for } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2 \\ u = \nabla^\perp \Delta^{-1} \omega, & \nabla^\perp = (-\partial_{x_2}, \partial_{x_1}). \\ \omega(0,\cdot) = \omega_0(\cdot) \end{cases}$$

- Biot-Savart Law: $u = \nabla^\perp (\omega * \mathcal{N}) =: \nabla^\perp \Psi$, where $\mathcal{N}(x) := \frac{1}{2\pi} \log |x|$.
- Vortex Patch : If $\omega_0 = 1_D$ for some bounded domain D, then

$$\omega(t,x) = 1_{D_t}, \text{ where } D_t = X_t(D),$$

where $X_t : \mathbb{R}^2 \mapsto \mathbb{R}^2$ is the flow map generated by u.
- More generally, one can consider $\omega = \sum_i \omega_i 1_{D_i}$

$\omega = \Sigma_{i=1}^n \omega_i 1_{D_i}$

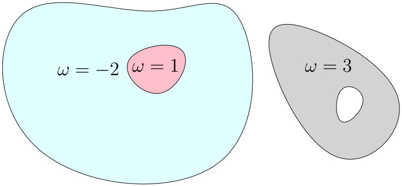


Figure: Linear combination of patches

- Stationary solutions:

$$\omega(t, x) = \omega_0(x)$$

- Trivial stationary solutions: Any radial vorticity is a stationary solution.
- If $\omega = 1_D$ is stationary, then $u \cdot \vec{n} = 0$ on ∂D.

Questions

Under what condition, must a stationary compactly supported vorticity be radially symmetric?

- If $\omega = 1_D$ and D is simply connected, then D has to be a disk (moving plane method) Fraenkel ('00).
- A stationary solution without a stagnation point must be shear flow Hamel, Nadirashvili ('17, '19)

Theorem (Gómez-Serrano, P, Shi, Yao '19)

Let $\omega = \sum_i \omega_i 1_{D_i}$ where $\omega_i \geq 0$ be a stationary solution to 2D Euler equation. Then ω is radially symmetric (up to a translation).

Questions

Allowing ω to change its sign, can we construct a non-radial solution?

- Trivial/ non-trivial stationary vorticity:
  - In $\mathbb{T}^2$, there are "many" stationary weak solutions Choffrut, Székelyhidi ('14)
  - In $\mathbb{T}^2$, there are non trivial stationary solutions near Kolmogorov and Poiseuille (Coti-Zelati, Elgindi, Widmayer ('20))
  - Flexibility results in various domains Constantin, Drivas, Ginsberg ('21)
  - In $\mathbb{R}^2$, smooth non-radial stationary ω without compact support. Musso, Pacard, Wei ('12)
  - In $\mathbb{R}^2$, there exists a non-trivial stationary solutions with compact support of velocity u, where u is continuous but not C^1. David Ruiz (forthcoming)

Theorem ((Infinite kinetic energy) Gómez-Serrano, P, Shi '21)

There exists a non-radial stationary patch solution to the 2D Euler equation with infinite kinetic energy $\int_{\mathbb{R}^2} |\vec{u}|^2 dx = \infty$.

Theorem ((finite Kinetic energy) Gómez-Serrano, P, Shi '21)

There exists a non-radial stationary patch solution to the 2D Euler equation with finite kinetic energy $\int_{\mathbb{R}^2} |\vec{u}|^2 dx < \infty$ *and compactly supported velocity field.*

- Locally radial solutions:

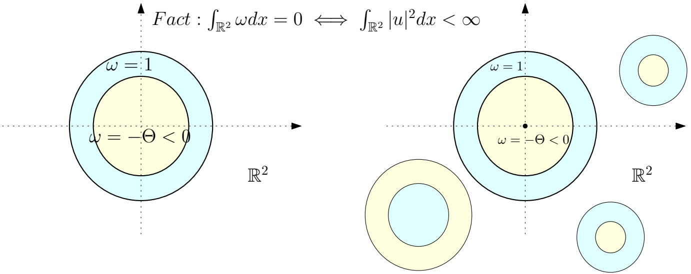


Figure: Locally radial stationary solution

Finite energy solutions

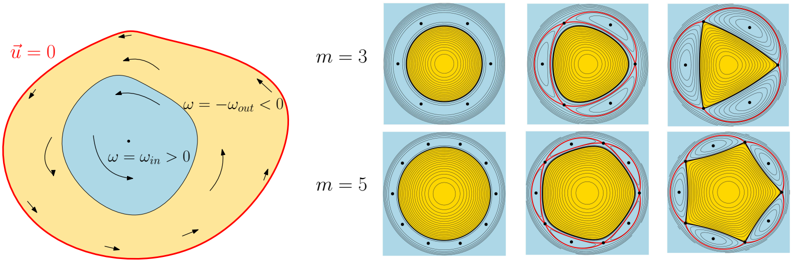
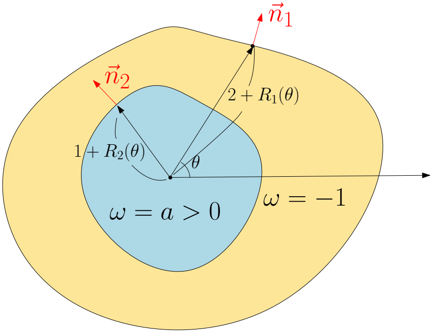


Figure: Stationary vorticity with compactly supported velocity and rotating patches

- The velocity vanishes on the outer boundary.
- A finite energy solution can be defined in any bounded domain.
- Stagnation point? Regularity might break down (Rotating patch solutions, Stokes conjecture (water waves)). In our solutions, the boundary is analytic.

Problem setting



- (a, R) determines the flow: vorticity, velocity, stream functions.. etc. $\omega = \omega(a, R)$, $\vec{u} = \vec{u}(a, R)$, $\Psi = \Psi(a, R)$.
- Goal: Choose $a > 0$ and $R_1, R_2 \in H^k(\mathbb{T})$ so that (denoting $R = (R_1, R_2)$)

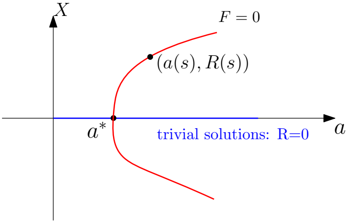
$$F(a, R) = \begin{pmatrix} \text{normal velocity on the outer boundary} \\ \text{normal velocity on the inner boundary} \end{pmatrix} = \begin{pmatrix} u \cdot \vec{n}_1 \\ u \cdot \vec{n}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Radial vorticity is stationary: $F(a, 0) = 0$.

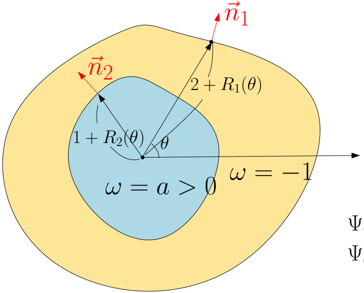
Crandall-Rabinowitz theorem

- Crandall–Rabinowitz theorem: Consider $F = F(a, R) : \mathbb{R} \times X \mapsto Y$ for a Hilbert spaces X, Y. If
  (1) $F(a, 0) = 0$ for all $a \in \mathbb{R}$,
  (2) For some $a^* \in \mathbb{R}$, $Ker(D_R F(a^*, 0))$ and $Im(D_R F(a^*, 0))^{\perp}$ have one-dimensional.
  (3) (Transversality) $\partial_a D_R F(a^*, 0)[h] \notin Im(D_R F(a^*, 0))$, for $h \in Ker(D_R F(a^*, 0))$.

  Then, there exists a curve $s \mapsto (a(s), R(s))$ such that $F(a(s), R(s)) = 0$ and $R(s) \neq 0$.



- Key ingredient of CR (Lyapunov-Schmidt): Under (2), $\exists\ \phi : \mathbb{R}^2 \mapsto \mathbb{R}$ s.t.
  - $\phi(0, 0) = 0$
  - If $\phi(x, y) = 0$ for some $(x, y) \neq 0$, then there exists a nontrivial solution to $F(a, r) = 0$.
  - Transversality is just a necessary condition for $\nabla\phi \neq 0$.



$\Psi_r :=$ radial derivative of Ψ
$\Psi_\theta :=$ angular derivative of Ψ

- Recall that our functional is $F : \mathbb{R} \times H^k(\mathbb{T}) \mapsto H^{k-1}(\mathbb{T})$,

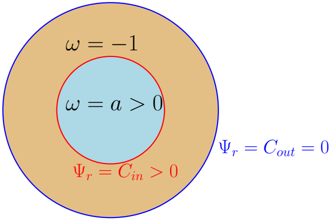
$$F(a, R) = \text{ normal velocity on each boundary} = u \cdot \vec{n} = \begin{pmatrix} \Psi_r|_{out}\partial_\theta R_1 + \Psi_\theta|_{out} \\ \Psi_r|_{in}\partial_\theta R_2 + \Psi_\theta|_{in} \end{pmatrix}.$$

- Ex) $\Psi(r, \theta)$ (on each boundary) $\implies \partial_\theta \left(\Psi(2 + R_1(\theta), \theta)\right) = 0$
- At the linear level, (linearization at the vorticity $\omega(a, R)$),

$$D_R F(a, R)[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}] = \begin{pmatrix} \Psi_r|_{out}\partial_\theta h_1 \\ \Psi_r|_{in}\partial_\theta h_2 \end{pmatrix} + K(a, R)[h]$$
$$= \Psi_r(a, R)\partial_\theta h + K(a, R)[h].$$

for some linear operator $K(a, R) : H^k \mapsto H^k$.

Finite energy case: $\int_{\mathbb{R}^2} |u|^2 dx < \infty$



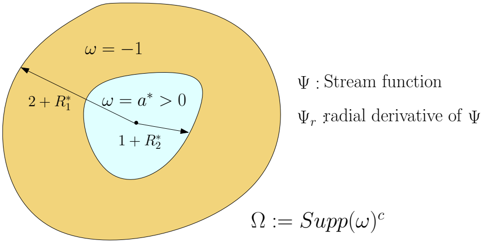
- If $a > 0$ is chosen s.t. $\int_{\mathbb{R}^2} \omega dx = 0$, then $\Psi_r = \begin{cases} C_{in} > 0 & \text{on inner bdry,} \\ C_{out} = 0 & \text{on outer bdry.} \end{cases}$
- Recall $D_R F(a,0)[h] = \Psi_r \partial_\theta h + K(a,0)[h]$ for some $K(a,0) : H^k \mapsto H^k$.
- For instance (H:Hilbert transform):

$$D_R F(a,0)[\begin{pmatrix} h_1 \\ h_2 \end{pmatrix}] = \begin{pmatrix} Hh_1 \\ C_{in}\partial_\theta h_2 + Hh_2 \end{pmatrix} = \begin{pmatrix} -i\,\mathrm{sgn}(n) & 0 \\ 0 & i(C_{in} n - \mathrm{sgn}(n)) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \end{pmatrix}.$$

- Choose $C_{in} = C_{in}(a)$ so that $Ker(D_R F(a,0)), Im(D_R F(a,0))^\perp$ are one-dimensional as a map between $H^k \times H^k \mapsto H^k \times H^{k-1}$.
  while $F : H^k \times H^k \mapsto H^{k-1} \times H^{k-1}$ (regularity mismatch).
- $Im(D_R F(a,0))^\perp$ cannot be finite dimensional.

Compactly supported velocity

- If (a^*, R^*) defines a stationary solution with finite kinetic energy, then the velocity vanishes on the exterior domain.



- $\Psi(a^*, R^*)$, solves

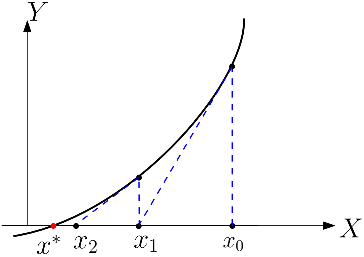
$$\begin{cases} \Delta\Psi = 0, & \text{in } \Omega, \\ \Psi = const. & \text{on } \partial\Omega. \end{cases}$$

- The maximum principle: max/min of Ψ is on $\partial\Omega$ or at "infinity" ex) $\log|x|$.
- $\int_{\mathbb{R}^2} \Delta\Psi dx = 0$ prevents Ψ from having its min/max at "infinity".
- $\Psi = Const.$ in Ω.

- Two issues:
  - CR is not directly applicable (Lyapunov-Schmidt reduction cannot work) .
  - Regularity mismatch at nonlinear/linear level.
- One dimensionality still holds but in different regularity space...
- The difficulty can be resolved using: 1) Proof of CR theorem without the Lyapunov-Schmidt reduction 2) Nash-Moser scheme.

Newton's method

- Goal: Given $G : X \mapsto Y$ for Hilbert spaces X, Y, find $x^* \in X$ such that $G(x^*) = 0$.



- $x_{n+1} := x_n - DG(x_n)^{-1}[G(x_n)]$.
- $|G(x_{n+1})| \lesssim |DG(x_n)^{-1}|^2 |G(x_n)|^2$.
- Requirement: 1) Good initial guess: $|G(x_0)| \leq \epsilon$. 2) Invertibility of DG: $|DG^{-1}| = O(1)$. Then, $|G(x_n)| \sim \epsilon^{2^n} \mapsto 0$.
- The condition 2) can be relaxed using "approximate inverse": If $T(x) : Y \mapsto X$ such that $|DG(x_n) \circ T(x_n) - I| = O(|G(x_n)|)$,, then we still have

$$|G(x_{n+1})| \lesssim |T(x_n)|^2 |G(x_n)|^2. \implies |G(x_n)| \sim \epsilon^{2^n} \to 0.$$

Newton's method framework: Removing parameter

- Recall our functional:

$$F(a, R) = \text{normal velocity on each boundary.}$$

- $Ker(D_R F(\tilde{a}, 0)), Im(D_R F(\tilde{a}, 0))^{\perp}$ are one-dimensional as a map between $H^k \times H^k \mapsto H^k \times H^{k-1}$, while $F : H^k \times H^k \mapsto H^{k-1} \times H^{k-1}$.
- For $\epsilon > 0$, let $G_\epsilon(R) := F(\tilde{a} + P[R], \epsilon v + (I - P)[R])$.

$$\text{where } 0 \neq v \in Ker(D_R F(\tilde{a}, 0)), \quad P : H^k \mapsto Ker D_R F(\tilde{a}, 0).$$

- Goal: For small $\epsilon > 0$, find R s.t. $G_\epsilon(R) = 0$.
- $G_\epsilon(0) = F(\tilde{a}, \epsilon v) = O(\epsilon^2)$. Good initial guess!. Need to check the "invertibility" of DG_ϵ.
- Invertibility: $DG_\epsilon(R)[h] = \underbrace{A(R)}_{\text{isomorphism}} [h] + \Psi_r(R)|_{\text{out}} \partial_\theta h$.
- Dirichlet-Neumann type estimate: $|\Psi_r|_{out}(R)| = O(|G_\epsilon|(R))$.
- $A(R)^{-1}$ plays a role of an approximate inverse.

Thank You for Your Attention!