Concentration and quantitative regularity for the Navier-Stokes equations

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Unsteady Navier-Stokes equations

$$\begin{cases} \partial_t U + U \cdot \nabla U - \Delta U + \nabla P = 0\,, & x \in \mathbb{R}^3\,,\ t \in (0,T)\,, \\ \nabla \cdot U = 0\,, & \\ U(\cdot,0) = U_0\,, & x \in \mathbb{R}^3\,. \end{cases}$$

Scaling $\lambda \in (0,\infty)$, $y \in \mathbb{R}^3$, $s \in (0, T/\lambda^2)$,

$$U_{0,\lambda}(y) = \lambda U_0(\lambda y), \quad U_\lambda(y,s) = \lambda U(\lambda y, \lambda^2 s),$$
$$P_\lambda(y,s) = \lambda^2 P(\lambda y, \lambda^2 s).$$

Criticality examples of critical norms

$$\|U\|_{L^5(\mathbb{R}^3\times(0,T))}, \qquad \sup_{t\in(0,T)} \|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}, \qquad \sup_{t\in(0,T)} \sqrt{T-t}\|U(\cdot,t)\|_{L^\infty(\mathbb{R}^3)}.$$

Finite-energy solutions

Energy

$$E(U;t) := \underbrace{\frac{1}{2}\int_{\mathbb{R}^3} |U(\cdot,t)|^2}_{\text{kinetic energy}} + \underbrace{\int_0^t \int_{\mathbb{R}^3} |\nabla U|^2}_{\text{energy dissipation}} \leq \frac{1}{2}\int_{\mathbb{R}^3} |U_0|^2.$$

Leray 1934 Given $U_0 \in L^2_\sigma(\mathbb{R}^3)$, there exists

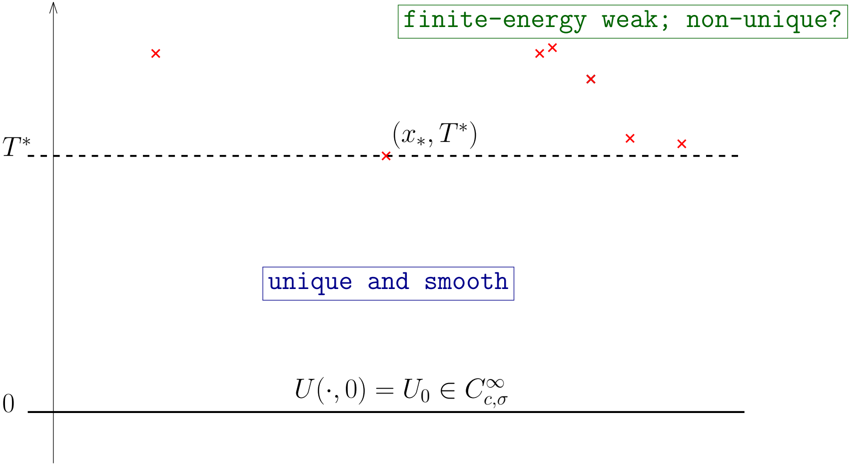
$$U \in L^\infty(0,\infty; L^2_\sigma(\mathbb{R}^3)) \cap L^2(0,\infty; \dot{H}^1(\mathbb{R}^3))$$

a finite-energy global-in-time weak solution, Leray-Hopf solution.

Supercriticality barrier for $U_\lambda := \lambda U(\lambda\cdot, \lambda^2\cdot)$,

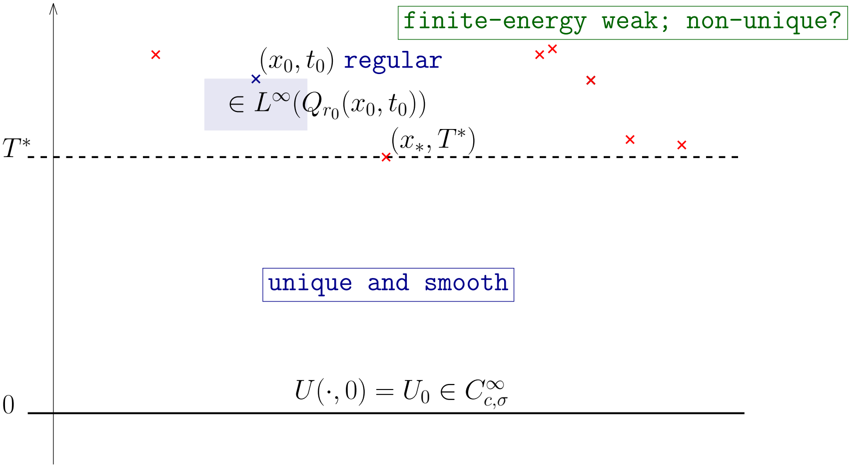
$$E(U_\lambda;\tau) = \frac{1}{\lambda} E(U;\lambda^2\tau), \qquad \text{for} \quad \lambda \in (0,\infty).$$

Blow-up? Non uniqueness?



For $C_c^\infty(\mathbb{R}^3)$ and divergence-free data does the Leray-Hopf solution remain smooth and unique for all time?

Blow-up? Non uniqueness?



Regular vs singular point

A point (x_0, t_0) is a regular point if U is bounded in a parabolic cylinder $Q_{r_0}(x_0, t_0)$ for some $r_0 > 0$, otherwise it is a singular point.

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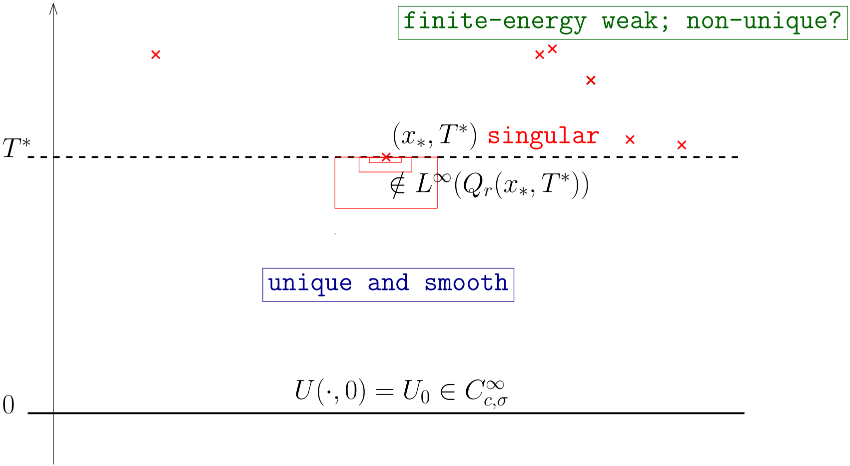

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Scale-critical regime: diffusion vs. convection dominated?

- Heuristics due to Šverák

$$\begin{aligned} |\Delta U| \gg |U \cdot \nabla U| \quad &\Rightarrow \quad \text{smoothness} \\ |\Delta U| \ll |U \cdot \nabla U| \quad &\Rightarrow \quad \text{turbulent behavior?} \end{aligned}$$

- for $|U(x,t)| \leq M|x|^{-1}$,

$$-\Delta U \simeq \frac{M}{|x|^3} \quad \text{vs.} \quad U \cdot \nabla U \simeq \frac{M^2}{|x|^3}.$$

- Open whether Type I condition

$$|U(x,t)| \leq \frac{M}{|x| + \sqrt{-t}} \quad \text{for} \quad M \gg 1$$

implies space-time point $(0,0)$ is regular.

Main results of the talk

New quantitative blow-up rates for critical or slightly supercritical quantities near potential singularities:

Blow-up rate in Type I case

Barker and P. *CMP* (2021)

We get a new localized quantitative blow-up rate for the critical L^3 norm.

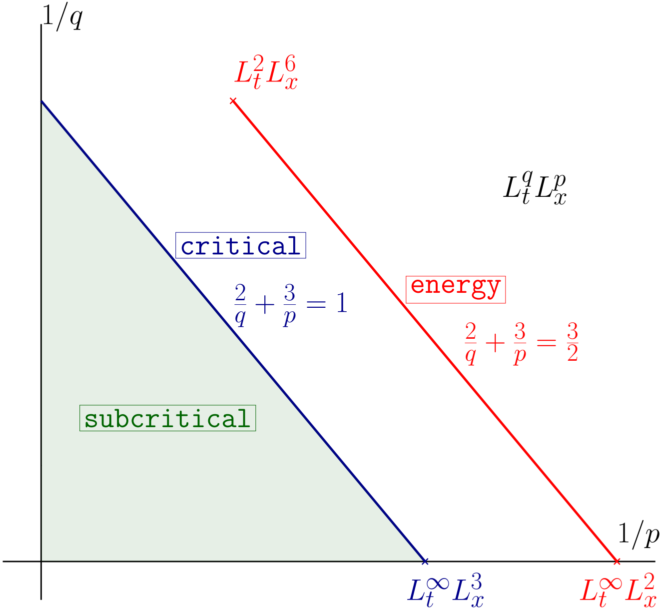
Blow-up of a supercritical Orlicz norm

Barker and P. *JMFM* (2021)

This gives a partial positive answer to a conjecture of Tao (2019).

Qualitative vs. quantitative blow-up of critical norms

Ladyzhenskaya-Prodi-Serrin



Qualitative blow-up in the endpoint $L_t^\infty L_x^3$

Escauriaza, Seregin, Šverák (2003)

Assume that T^* is the first blow-up time of U.
Then

$$\limsup_{t\to T^*} \|U(\cdot,t)\|_{L^3(\mathbb{R}^3)} = \infty.$$

$\boxed{\textbf{Abstract quantitative estimate}}$

There exists F_{univ} such that for all U,

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-\frac{1}{2},0))} \leq F_{univ}\big(\|U\|_{L^\infty(-1,0;L^3(\mathbb{R}^3))}\big).$$

▷ via persistence of singularities from Rusin, Šverák (2011)

Quantitative regularity and blow-up rate in the endpoint $L^\infty_t L^3_x$

Tao (2019)

Quantitative regularity

There exists $c_{univ} > 0$,

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-1/2,0))} \leq \exp\exp\exp\left(\|U\|^{c_{univ}}_{L^\infty(-1,0;L^3(\mathbb{R}^3))}\right).$$

Blow-up rate

Assume T^* is a first blow-up time.
Then, there exists $c'_{univ} > 0$,

$$\limsup_{t\to T^*} \frac{\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{\left(\log\log\log\left(\frac{1}{T^*-t}\right)\right)^{c'_{univ}}} = \infty.$$

Strategy of Tao: frequency bubbles of concentration

Heuristics for regularity Tao (2013)

Let P_N be a projector on frequencies $\sim N$ and apply it to the Navier-Stokes equations

$$
\begin{aligned}
0 &= \partial_t(P_N U) - P_N \Delta U + P_N \mathbb{P} \nabla \cdot (U \otimes U) \\
&\simeq \partial_t(P_N U) - N^2 P_N U + N (P_N U)^2.
\end{aligned}
$$

If $|N^{-1} P_N U| \ll 1$, then $|N(P_N U)^2| \ll |N^2 P_N U|$;
hence diffusion dominates the nonlinearity.

Main goal

If $N^{-1}\|P_N U\|_{L^\infty_{t,x}}$ is not small in terms of $\|U\|_{L^\infty(-1,0;L^3(\mathbb{R}^3))}$, then find an upper bound N_* for N;

$$
N_* \lesssim \exp\exp\exp\left(\|U\|^{c_{univ}}_{L^\infty(-1,0;L^3(\mathbb{R}^3))}\right)
$$

via backward frequency bubbling, transfer of information to the enstrophy, unique continuation, backward uniqueness and summing at final time.

A new strategy for quantitative regularity

Blow-up rate for Type I singularities

Barker, P. (2021)

For M sufficiently large, $\delta \in (0,1)$, if

$$\begin{aligned}&T^* \quad \text{is the first blow-up time of} \quad U,\\ &\text{the space-time point} \quad (0,T^*) \quad \text{is a singularity},\\ &\|U\|_{L^\infty(0,T^*;L^{3,\infty}(\mathbb{R}^3))} \leq M,\end{aligned}$$

then

$$\int_{B_0(R(t))} |U(\cdot,t)|^3 \geq \frac{\log\left((T^*-t)^{-\frac{\delta}{2}}\right)}{\exp\exp(M^{1025})}, \qquad R(t) := O\Big((T^*-t)^{\frac{1-\delta}{2}}\Big)$$

for all $t \in \big(t_*(\delta, M, T^*), T^*\big)$.

Blow-up rate for Type I singularities

Barker, P. (2021)

(Conc)

$$\int_{B_0(R(t))} |U(\cdot,t)|^3 \geq \frac{\log\left((T^*-t)^{-\frac{\delta}{2}}\right)}{\exp\exp(M^{1025})}, \qquad R(t) := O\big((T^*-t)^{\frac{1-\delta}{2}}\big).$$

- In the Type I case, we are able to remove two 'logarithms' for the rate of $\|U(\cdot,t)\|^3_{L^3(\mathbb{R}^3)}$

Tao (2019) $\quad \big(\log\log\log\left((T^*-t)^{-1}\right)\big)^{3c'_{univ}}$,

Barker, P. (2021) $\quad \dfrac{\log\left((T^*-t)^{-\frac{\delta}{2}}\right)}{\exp\exp(M^{1025})}$.

- The rate is optimal for Discretely Backward Self-Similar solutions with sufficient decay

$$U(x,t) = \overline{\lambda}U(\overline{\lambda}x, \overline{\lambda}^2 t), \quad \text{for a fixed} \quad 1 < \overline{\lambda}.$$

Toy model: quantitative regularity under $L^5_{t,x}$ control

Quantitative estimate

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-1/2,0))} \leq G(\|U\|_{L^5(\mathbb{R}^3\times(-1,0))})$$

with G explicit.

Main goal

If the scale-invariant Weissler-Kato type norm

(WK) $$(-t')^{\frac{1}{5}}\|U(\cdot,t')\|_{L^5(\mathbb{R}^3)}$$

is not small, i.e. *concentrates*, then
find a lower bound $-t^*(\|U\|_{L^5_{t,x}})>0$ for $-t'$.

Conclusion

The scale-invariant Weissler-Kato type norm (WK) is uniformly small for $t'\in(-t_*(\|U\|_{L^5_{t,x}}),0)$; hence ε-regularity implies quantitative boundedness.

Toy model: quantitative regularity under $L^5_{t,x}$ control

$\boxed{\textbf{Step 1}}$ backward propagation using mild solution theory, Cannone (1997)

$$\|U(\cdot,t')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon}{(-t')^{\frac{1}{5}}} \Rightarrow \|U(\cdot,t'')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon/2}{(-t'')^{\frac{1}{5}}}, \quad \forall t'' \in (-1, 2t').$$

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Step 2 summation of scales and lower bound for initial concentration

$$\int_{-1}^{2t'} \|U(\cdot,t'')\|^5_{L^5(\mathbb{R}^3)}\, dt'' > \frac{\varepsilon^5}{32} \int_{-1}^{2t'} (-t'')^{-1}\, dt''$$

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$$\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))} \geq \int_{-1}^{2t'} \|U(\cdot,t'')\|^5_{L^5(\mathbb{R}^3)}\,dt'' > \frac{\varepsilon^5}{32}\int_{-1}^{2t'}(-t'')^{-1}\,dt''$$

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$$\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))} \geq \frac{\varepsilon^5}{32}\int_{-1}^{2t'} (-t'')^{-1}\, dt'' = -\frac{\varepsilon^5}{32}\log(-2t')$$

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$$\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))} > -\frac{\varepsilon^5}{32}\log(-2t')$$

Hence:

$$\frac{1}{2} > -t' > \frac{1}{2}\exp\left(-\frac{32\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))}}{\varepsilon^5}\right) =: -t_*.$$

Toy model: quantitative regularity under $L^5_{t,x}$ control

Step 1 backward propagation using mild solution theory, Cannone (1997)

$$\|U(\cdot,t')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon}{(-t')^{\frac{1}{5}}} \Rightarrow \|U(\cdot,t'')\|_{L^5(\mathbb{R}^3)} > \frac{\varepsilon/2}{(-t'')^{\frac{1}{5}}}, \quad \forall t'' \in (-1, 2t').$$

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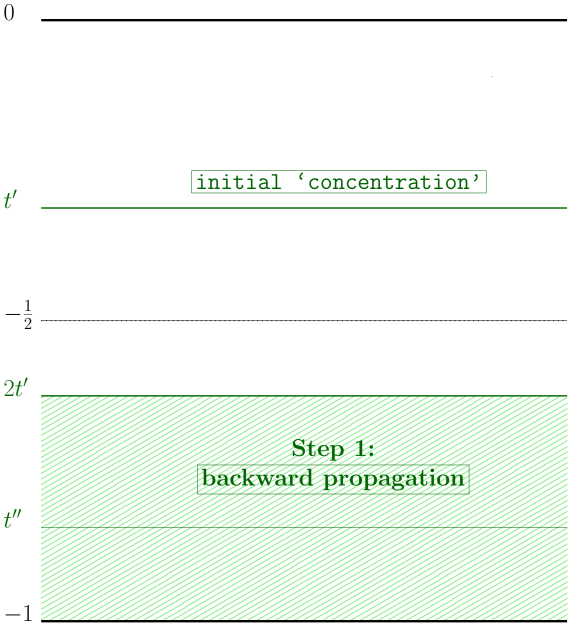
$$\frac{1}{2} > -t' > \frac{1}{2}\exp\left(-\frac{32\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))}}{\varepsilon^5}\right) =: -t_*.$$

Step 3 quantitative boundedness via ε-regularity on $Q_{\sqrt{-t_*/2}}(\xi,0) = B_{\sqrt{-t_*/2}}(\xi)\times(-\frac{t_*}{2},0)$ for $\xi\in\mathbb{R}^3$,

$$\|U\|_{L^\infty(\mathbb{R}^3\times(-\frac{t_*}{2},0))} \lesssim \frac{\varepsilon^{\frac{1}{3}}}{\sqrt{-t_*}} \lesssim \varepsilon^{\frac{1}{3}}\exp\left(\frac{16\|U\|^5_{L^5(\mathbb{R}^3\times(-1,0))}}{\varepsilon^5}\right).$$

Toy model: quantitative regularity under $L^5_{t,x}$ control



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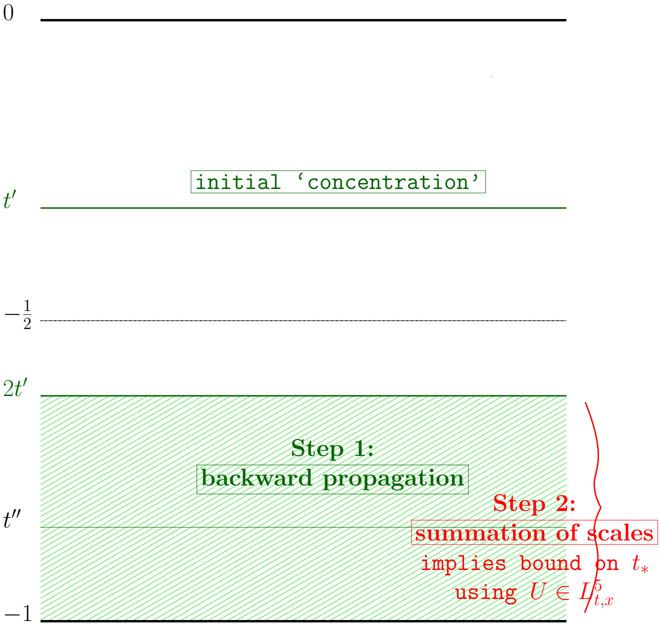

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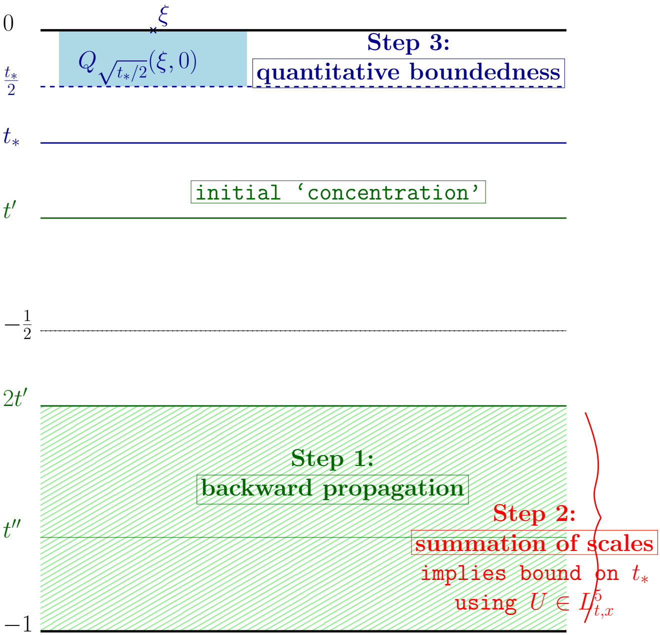


Quantitative regularity in the Type I case

High level strategy same as for toy model $U \in L^5(\mathbb{R}^3 \times (-1, 0))$, with the following main changes:

Quantitative regularity in the Type I case

Quantitative regularity under:	
$\|U\|_{L^5(\mathbb{R}^3\times(-1,0))} < \infty$	$\|U(\cdot,0)\|_{L^3(\mathbb{R}^3)} < \infty$ and Type I condition
initial 'concentration' $\|U(\cdot,t')\|_{L^5} > \varepsilon(-t')^{-\frac{1}{5}}$	
backward propagation via mild solution theory	
summation of scales via integration in time on $(-1,2t')$	

Quantitative regularity in the Type I case

Quantitative regularity under:	
$\|U\|_{L^5(\mathbb{R}^3\times(-1,0))} < \infty$	$\|U(\cdot,0)\|_{L^3(\mathbb{R}^3)} < \infty$ and Type I condition
initial 'concentration' $\|U(\cdot,t')\|_{L^5} > \varepsilon(-t')^{-\frac{1}{5}}$	initial spatial concentration of enstrophy
backward propagation via mild solution theory	backward propagation via local-in-space short-time smoothing
summation of scales via integration in time on $(-1,2t')$	summation of scales at time $t=0$ via quantitative unique continuation and backward uniqueness

Quantitative regularity in the Type I case

Barker, P. (2021)

Let $U \in C^\infty(\mathbb{R}^3 \times (-1,0))$.
For M large enough, assume that U satisfies the Type I bound

$$\|U\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))} \leq M.$$

Then, letting

$$\begin{aligned} t_* &= t_*(M, U(\cdot,0)) \\ &:= -M^{-O(1)} \exp\left(-\exp\exp(M^{1024}) \int_{B_0(\exp(M^{1023}))} |U(\cdot,0)|^3 \right), \end{aligned}$$

we get the quantitative boundedness

$$\|U\|_{L^\infty\left(B_0(M^{O(1)}\sqrt{-t_*})\times(t_*/2,0)\right)} \lesssim \frac{M^{-O(1)}}{\sqrt{-t_*}}.$$

Quantitative regularity in the Type I case

Assumptions

$$\|U\|_{L^\infty(-1,0;L^{3,\infty}(\mathbb{R}^3))} \leq M \quad \text{and} \quad U(\cdot,0) \in L^3(\mathbb{R}^3).$$

Main goal

If the scale-invariant enstrophy

(Enstro) $$\sqrt{-t'} \int_{B_0\left(\sqrt{-t'/S(M)}\right)} |\omega(\cdot,t')|^2$$

with

$$\omega = \nabla \times U \quad \text{vorticity},$$
$$S(M) := O(1)M^{-100} \quad \text{and} \quad 0 < -t' < O(1)M^{-548},$$

is not small, i.e. *concentrates*, then
find a lower bound on $-t^*(M, \|U(\cdot,0)\|_{L^3})$ for $-t'$.

Step 1: backward propagation

Backward propagation of enstrophy concentration

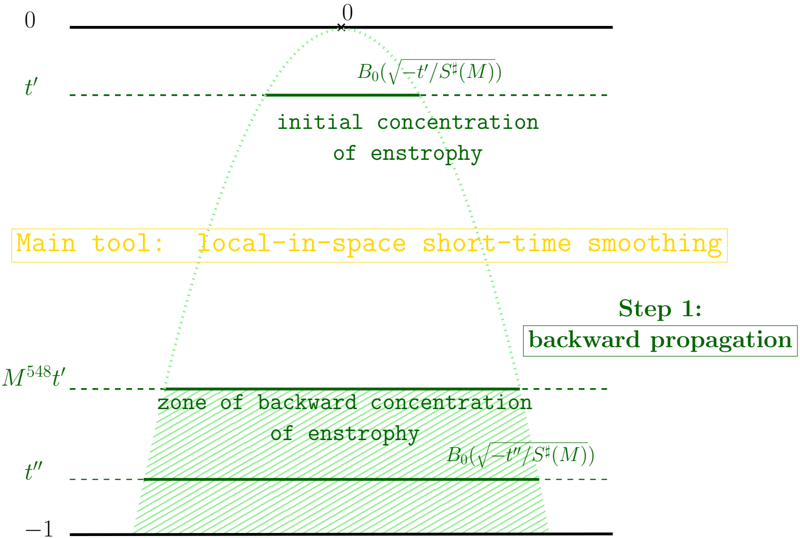
Suppose $t' \in (-1, 0)$ is not too close to -1 and is such that

$$\int\limits_{B_0\left(4\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}}\right)} |\omega(x,t')|^2 dx > \frac{M^2\sqrt{S(M)}}{(-t')^{\frac{1}{2}}}.$$

We show that for all $t'' \in (-1, t')$, such that $-t''$ is well-separated from $-t'$, we have

$$\int\limits_{B_0\left(4\sqrt{S(M)}^{-1}(-t'')^{\frac{1}{2}}\right)} |\omega(x,t'')|^2 dx > \frac{M^2\sqrt{S(M)}}{(-t'')^{\frac{1}{2}}}.$$

Step 1: backward propagation



Key tool: local-in-space short-time smoothing

Barker and P. (2021)

For all $M,\ N>0$ sufficiently large, there exists

$$0<S_*(M,N)=O(1)M^{-30}N^{-70},$$

for all local energy solution (U,P) associated to the initial data

$$\|U_0\|_{L^2_{uloc}(\mathbb{R}^3)}\leq M \quad \text{and} \quad \|U_0\|_{L^2(B_x(1))}\overset{x\to\infty}{\to}0.$$

If in addition

$$\|U_0\|_{L^6(B_0(1))}\leq N,$$

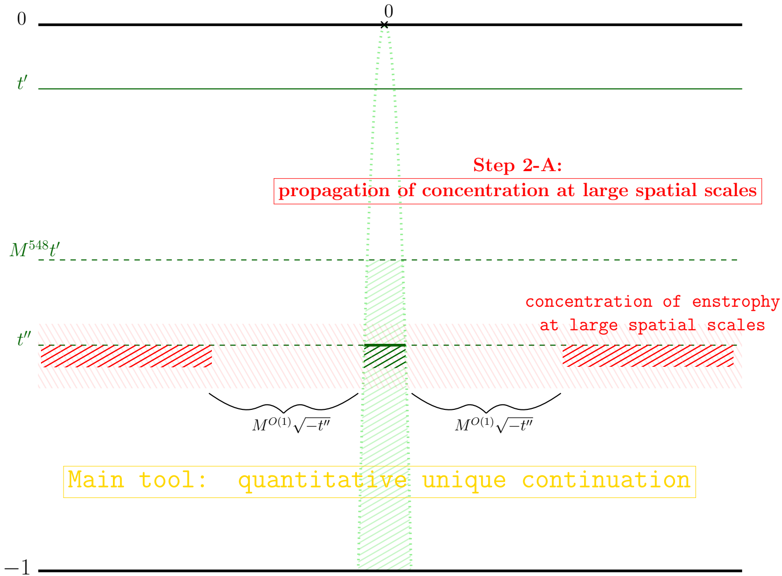
then

$$\|U\|_{L^\infty\left(B_0(\frac{1}{2})\times\left(\frac{3}{4}S_*(M,N),S_*(M,N)\right)\right)}\lesssim M^8N^{19}.$$

Key tool: local-in-space short-time smoothing

- Quantitative version of Jia, Šverák (2014).
- Strategy of the proof:
  $U = a + V$, a mild solution originating from $U_0|_{B_0(1)} \in L^6$, V perturbation; combine
  (1) local energy estimates for the perturbation V
  show local in space smallness of the energy for V
  for times $\sim S_*(M)$
  the difficulty is to handle the nonlocal effects of the pressure,
  (2) an ε-regularity result near a subcritical drift
  proof via a compactness argument similar to Lin (1998) or an iteration similar to Caffarelli, Kohn, Nirenberg (1984).

Step 2: summation of scales



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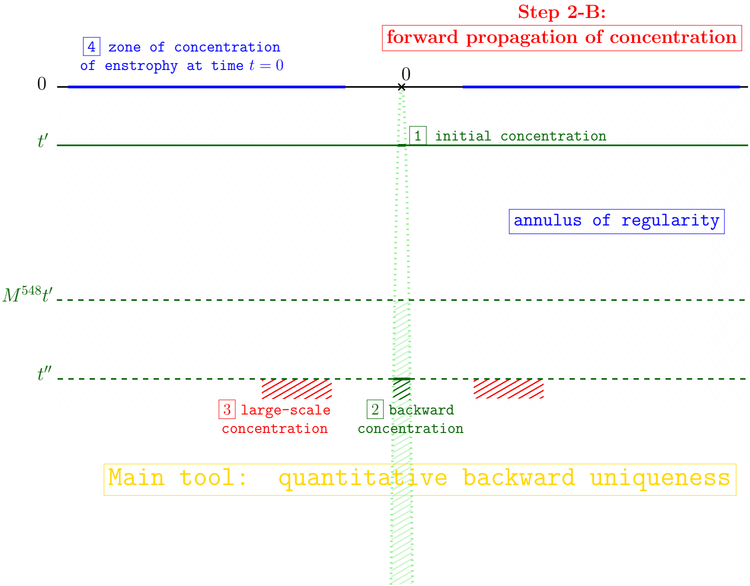

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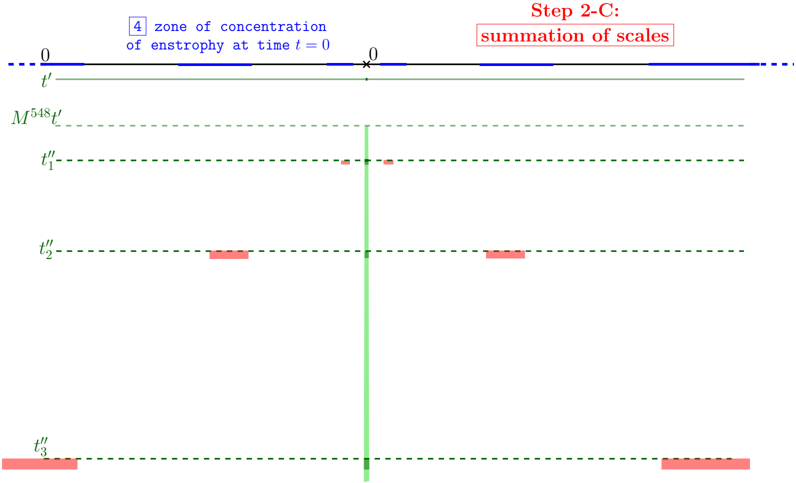
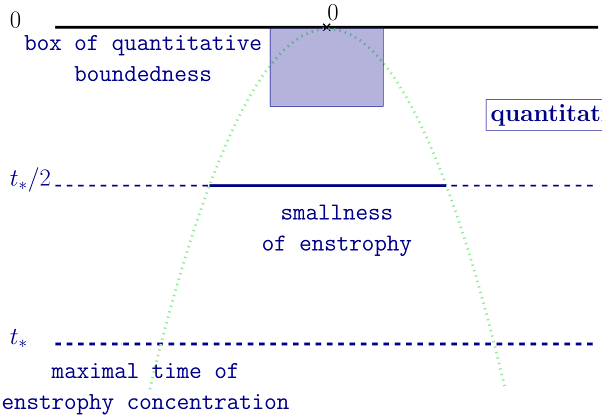


Main tool: summation of well-separated scales implies coercivity on $\|U(\cdot,0)\|_{L^3}$ via Biot-Savart

Step 3: quantitative boundedness



Step 3:
quantitative boundedness

Main tool: local-in-space short-time smoothing

Tao's and Barker-P.'s approach: similarities

Tao (2019)	Barker, P. (2021)
concentration of scale-invariant quantities	
$N^{-1}\lvert P_N U(t',x')\rvert$	$\sqrt{-t'}\int_{B_0(\sqrt{-M^{O(1)}t'})}\lvert\omega(\cdot,t')\rvert^2$
upper bound N_*	lower bound $-t_*$
for concentration	
backward concentration within parabolic domain of dependence	
$x''-x'=O\big(\sqrt{t''-t'}\big)$	$B_0(\sqrt{-M^{O(1)}t''})$

Tao's and Barker-P.'s approach: differences

Tao (2019)	Barker, P. (2021)
quantitative estimates in terms of globally defined quantities	quantitative estimates in terms of locally defined quantities
concentration in Fourier space of globally defined quantities frequency bubbles $N^{-1}\lvert P_N U(t',x')\rvert$	concentration in physical space of locally defined quantities enstrophy $\sqrt{-t'}\int_{B_0\left(\sqrt{-M^{O(1)}t'}\right)} \lvert\omega(\cdot,t')\rvert^2$
frequency analysis	local-in-space short-time smoothing
transfer of scale-invariant information from Fourier space to enstrophy scale-invariant control $\lVert U\rVert_{L^\infty(-1,0;L^3)}<\infty$	concentration is directly on enstrophy no need for a global-in-time scale-invariant bound time-slice regularity criteria

Quantification of Seregin's 2012 qualitative regularity

Barker, P. (2020)

For M sufficiently large, assume that there exists $t_k \to 0$ such that

$$\|U(\cdot, t_k)\|_{L^3(\mathbb{R}^3)} \leq M.$$

We define $M^\flat := \exp(M^6)$. Then for any well-separated subsequence t_j such that

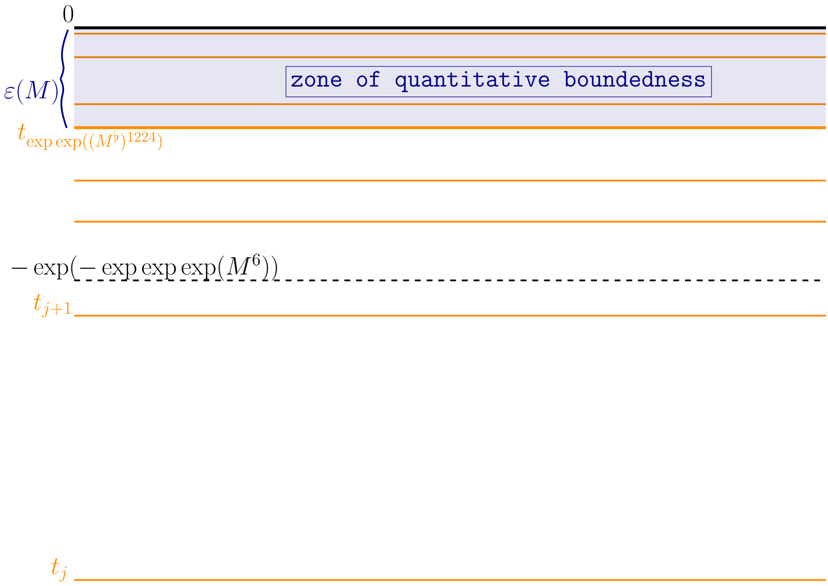
$$\sup_{j \in \mathbb{N}} \frac{-t_{j+1}}{-t_j} \leq \exp(-2(M^\flat)^{1223}),$$

we have the following quantitative regularity estimate:

$$\|U\|_{L^\infty(\mathbb{R}^3 \times (-\varepsilon_M, 0))} \leq C_{univ}(\varepsilon_M)^{-\frac{1}{2}}$$

where
$0 < \varepsilon_M = \varepsilon(M, (t_k)) := -t_{\exp\exp((M^\flat)^{1224})} \lesssim \exp(-\exp\exp(M^\flat))$.

Quantification of Seregin's 2012 qualitative regularity



Step 1: backward propagation in the time slices case

Backward propagation of enstrophy concentration

Fix any $\alpha \geq M^\flat = \exp(M^6)$ and let $t', t'' \in [-1, 0)$ be well-separated. Assume that

$$\|U(\cdot, t')\|_{L^3} \leq M \quad \text{and} \quad \|U(\cdot, t'')\|_{L^3} \leq M.$$

If the vorticity concentrates at time t' in the following way

$$\int\limits_{B_0\left(4\sqrt{S(M)}^{-1}(-t')^{\frac{1}{2}}\right)} |\omega(x,t')|^2\,dx > M^2(-t')^{-\frac{1}{2}}\sqrt{S(M)},$$

then for any $s \in [t'', \frac{t''}{8\alpha^{201}}]$ the vorticity concentrates in the following sense

$$\int\limits_{B_0\left(4(-s)^{\frac{1}{2}}\alpha^{106}\right)} |\omega(x,s)|^2\,dx > \frac{(M+1)^2}{(-s)^{\frac{1}{2}}\alpha^{106}}.$$

Blow-up of slightly supercritical Orlicz norms

‘Good’ quantitative results in the critical regime imply slightly supercritical results

- Nonlinear Wave Equation (slightly supercritical nonlinearity): Tao (2007), Roy (2010), Colombo, Haffter (2019)...
- Surface Quasi-Geostrophic (slightly supercritical diffusion): Dabkowski, Kiselev, Silvestre, Vicol (2014), Coti Zelati, Vicol (2016)...
- Hypodissipative Navier-Stokes (slightly supercritical diffusion): Tao (2010), Barbato, Morandin, Romito (2014), Colombo, De Lellis, Massaccesi (2020), Colombo, Haffter (2021)...

Tao's conjecture

If T^* is a first blow-up time, there exists $c'_{univ} > 0$,

(Tao 2019)
$$\limsup_{t \to T^*} \frac{\|U(\cdot,t)\|_{L^3(\mathbb{R}^3)}}{\left(\log\log\log\left(\frac{1}{T^*-t}\right)\right)^{c'_{univ}}} = \infty.$$

Tao (2019): conjecture

$$\limsup_{t \to T^*} \|U(\cdot,t)\|_{L^3(\log\log\log L)^{-c''_{univ}}(\mathbb{R}^3)} = \infty.$$

A partial positive answer to Tao's conjecture

Barker, P. (2021)

Let $U_0 \in L^2(\mathbb{R}^3) \cap L^4(\mathbb{R}^3)$. Assume that U first blows-up at T^*. Then,

$$\limsup_{t \to T^*} \int_{\mathbb{R}^3} \frac{|U(x,t)|^3}{\left(\log\log\log\left((\log(e^{e^{3e^e}} + |U(x,t)|))^{\frac{1}{3}}\right)\right)^{\theta}} dx = \infty.$$

- Conjecture by Tao: *three* log instead of *four*.
- Log improvement of LPS Chan, Vasseur (2007), Bjorland, Vasseur (2011), Lei, Zhou (2013): regularity under

$$\int_0^T \int_{\mathbb{R}^3} \frac{|U|^5}{\log(1+|U|)} \, dxds < \infty.$$

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Proof via:

(1) estimating the $L^{3-\mu}(\mathbb{R}^3)$ norm for a carefully tuned parameter $\mu > 0$,

(2) and a regularity result in terms of the supercritical norms $L^{3-\mu}$ for μ close to 0.

Mild criticality breaking

Barker, P. (2021)

For M and E sufficiently large, there exists $\delta(M,E) \in (0, \frac{1}{2}]$ such that the following holds.
If

$$\|U_0\|_{L^2(\mathbb{R}^3)}, \quad \|U_0\|_{L^4(\mathbb{R}^3)} \leq M,$$

and

$$\|U\|_{L^\infty(0,\infty;L^{3-\delta(M,E)}(\mathbb{R}^3))} \leq E.$$

Then U is smooth on $\mathbb{R}^3 \times (0,\infty)$.

- ▷ Inspired by Bulut (2020) for NLS.
- ▷ We have an explicit estimate

$$\delta(M,E) \lesssim (\log M)^{-\frac{1}{5}} \exp\exp\exp(-C_{univ} E^{c_{univ}}).$$

- ▷ Non effective regularity criteria; 'mild' criticality breaking.
- ▷ Not enough to rule out Type I blow-ups.

Proof of mild criticality breaking

Assumptions

$\|U_0\|_{L^2(\mathbb{R}^3)}, \quad \|U_0\|_{L^4(\mathbb{R}^3)} \leq M$ and $\|U\|_{L^\infty(0,\infty;L^{3-\delta(M,E)}(\mathbb{R}^3))} \leq E.$

Aim of the game

Show there exists (an explicit) constant $K(M,E)$ such that for $T>0$, $\|U\|_{L^\infty(0,T;L^4(\mathbb{R}^3))} \leq K(M,E).$

Main tools

L^4-energy estimates on small intervals $I_j = (t_j, t_{j+1}) \subset (0,T)$

$$\mathcal{E}_{4,j} \leq \|U(\cdot,t_j)\|^4_{L^4(\mathbb{R}^3)} + C\|U\|_{L^5(\mathbb{R}^3\times I_j)}\mathcal{E}_{4,j}.$$

Quantitative estimate of the subcritical L^5 norm

$$\|U\|_{L^5(\mathbb{R}^3\times(0,T))} \lesssim \big(\log M\big)^{\frac{1}{5}} \exp\exp\exp\Big(C_{univ}\|U\|^{c_{univ}}_{L^\infty(0,T;L^3(\mathbb{R}^3))}\Big)$$

via Tao's quantitative regularity, initial and eventual regularity.

Interpolation $\|U\|_{L^\infty(0,T;L^3(\mathbb{R}^3))} \leq E^{\frac{3-\delta}{3+3\delta}}\big(K(M,E)\big)^{\frac{4\delta}{3+3\delta}}.$

Proof of mild criticality breaking

Strategy

Propagate the subcritical information $U_0 \in L^4$ forward in time.

(1) Split $(0,T)$ into m disjoint intervals $(0,T) = \bigcup_{j=1}^{m} I_j$ such that

$$\|U\|_{L^5(\mathbb{R}^3 \times I_j)} = \varepsilon.$$

(2) Use the quantitative L^5 regularity to estimate m:

$$\varepsilon^5 m \leq \big(\log M\big)^{\frac{1}{5}} \exp\exp\exp\Big(E^{\frac{3-\delta}{3+3\delta}}\big(K(M,E)\big)^{\frac{4\delta}{3+3\delta}}\Big)$$

(3) Carry out L^4-energy estimates on each interval $I_j = (t_j, t_{j+1})$

$$\begin{aligned}\mathcal{E}_{4,j} &\leq \|U(\cdot,t_j)\|^4_{L^4(\mathbb{R}^3)} + C\|U\|_{L^5(\mathbb{R}^3\times I_j)}\mathcal{E}_{4,j} \\ &\leq \|U(\cdot,t_j)\|^4_{L^4(\mathbb{R}^3)} + C\varepsilon\mathcal{E}_{4,j}.\end{aligned}$$

(4) An iteration gives

$$\|U\|^4_{L^4(\mathbb{R}^3\times(0,T))} \lesssim M^4 2^m.$$

Take-away messages

Message A:

Connection between quantitative regularity and concentration

the global **scale-critical** standing assumption		
$U \in L^5_{t,x}$	$U \in L^\infty_t L^{3,\infty}_x$ and $U(\cdot,0) \in L^3$	$U \in L^\infty_t L^3_x$
prevents the following scale-critical quantities		
$(-t)^{\frac{1}{5}} \|U(\cdot,t)\|_{L^5}$	$\sqrt{-t} \int \lvert\omega(\cdot,t)\rvert^2$	$N^{-1}\lvert P_N U(t',x')\rvert$
to concentrate to close to final time		
Hence smallness implies regularity.		

Message B:

Quantification of the critical case implies slight criticality breaking.

Thank you for your attention!