Collective Phenomena in Heterogenous Traffic: Power-laws

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Motivation

- Traffic is inherently a driven non-equilibrium system.
  - Non-equilibrium: flux (flow) $\neq 0$.
    - Non-equilibrium stationary state: flux $\neq$ 0 and constant in $time$.
- Exhibits collective phenomena in the presence of a disorder.
  - Self-organization.
  - Power-law coarsening behavior.
  - Order-disorder transition.

Newell's Car-following model

- $\frac{dx_i(t)}{dt} = V(s_i(t - \tau_i)) \quad \forall i, \quad i = 1,2, \dots N$
- $V(s) = \begin{cases} v_f, & s \geq S_c \\ w_b \max\{S_j^{-1}s - 1,0\} & s < S_c \end{cases}$

τ : delay time; v_f: free-flow speed; S_j: jam gap; S_c: critical gap; w_b: back-wave speed

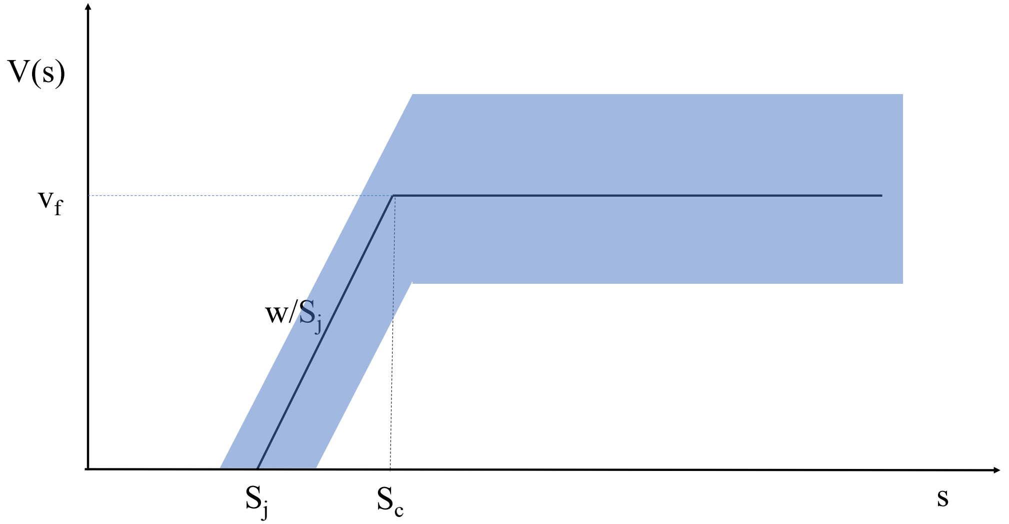


Fig.1: A schematic of Newell's model. Black line is $V(s)$ for a particular choice of v_f, S_j and w_b. Shaded area encompasses all possible $V(s)$.

- v_f, S_j, w_b are drawn from beta distributions:

$$p_A(A) = C(A - A^{min})^{\mu-1}(A^{max} - A)^{\nu-1}I\,(A)$$

$$I(x) = \begin{cases} 1 & if\ x \in [A^{min}, A^{max}] \\ 0 & otherwise \end{cases}$$

$$A = v_f, S_j, w_b$$

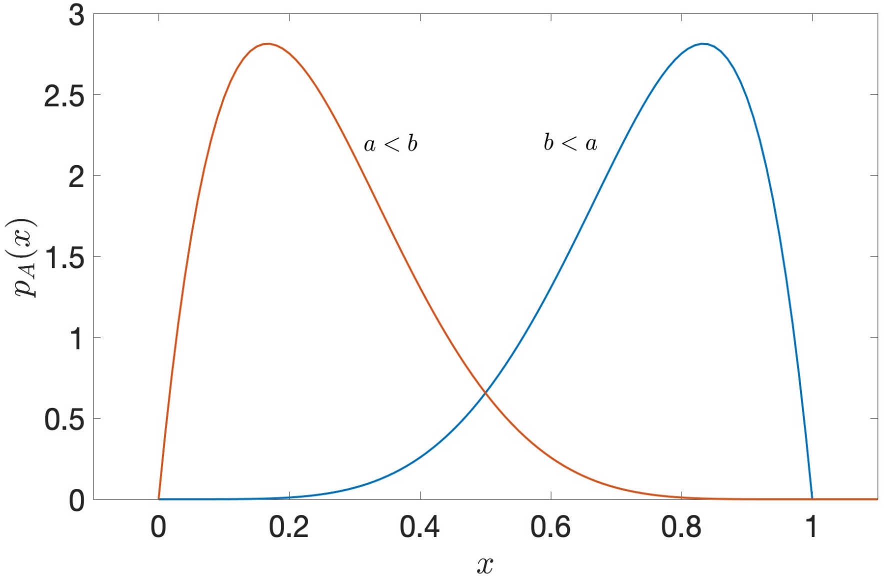


Fig.2: Examples of beta distribution.

$\tau = 0$ case

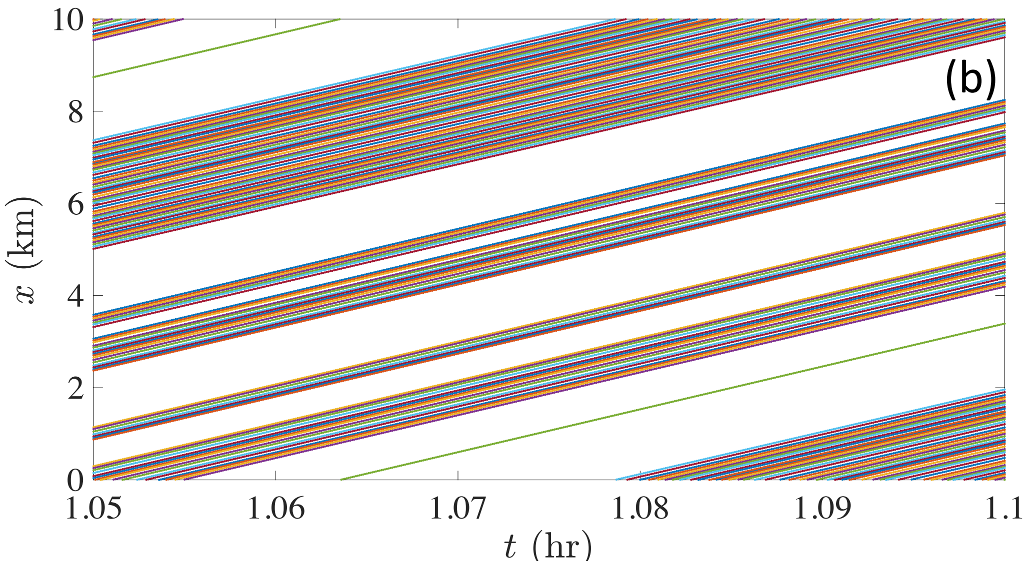
$$\frac{dx_i(t)}{dt} = V\big(s_i(t)\big) \quad \forall i, \qquad i = 1,2,\dots,N$$

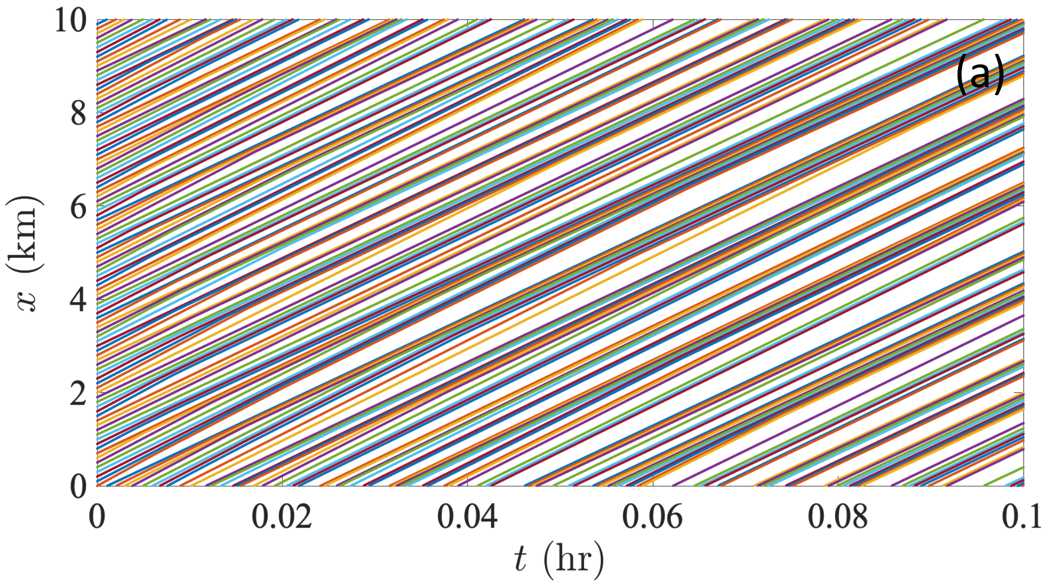
$$V(s) = \begin{cases} v_f, & s \geq S_c \\ w_b \max\{S_j^{-1}s - 1,0\} & s < S_c \end{cases}$$

$v_f \in [90,110]$ km/hr;

$\rho_j \in [110,170]$ veh/km

$w_b \in [10,30]$ km/hr





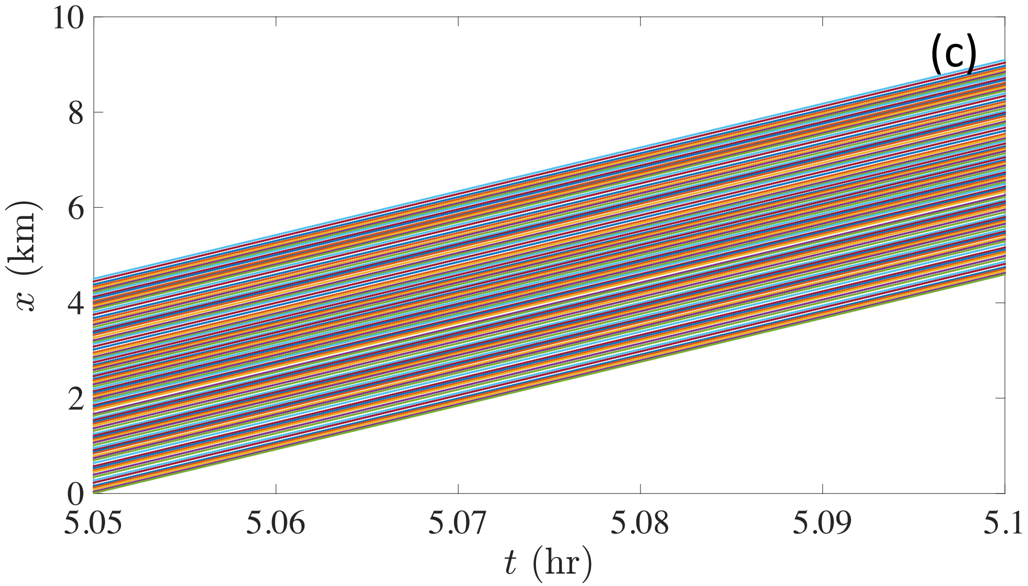


Fig.3 : position-time diagram of vehicles. (a) $0 < t < 0.1hr$, (b) $1hr < t < 1.1hr$, (c) $5hr < t < 5.1hr$

Relaxation to stationary state

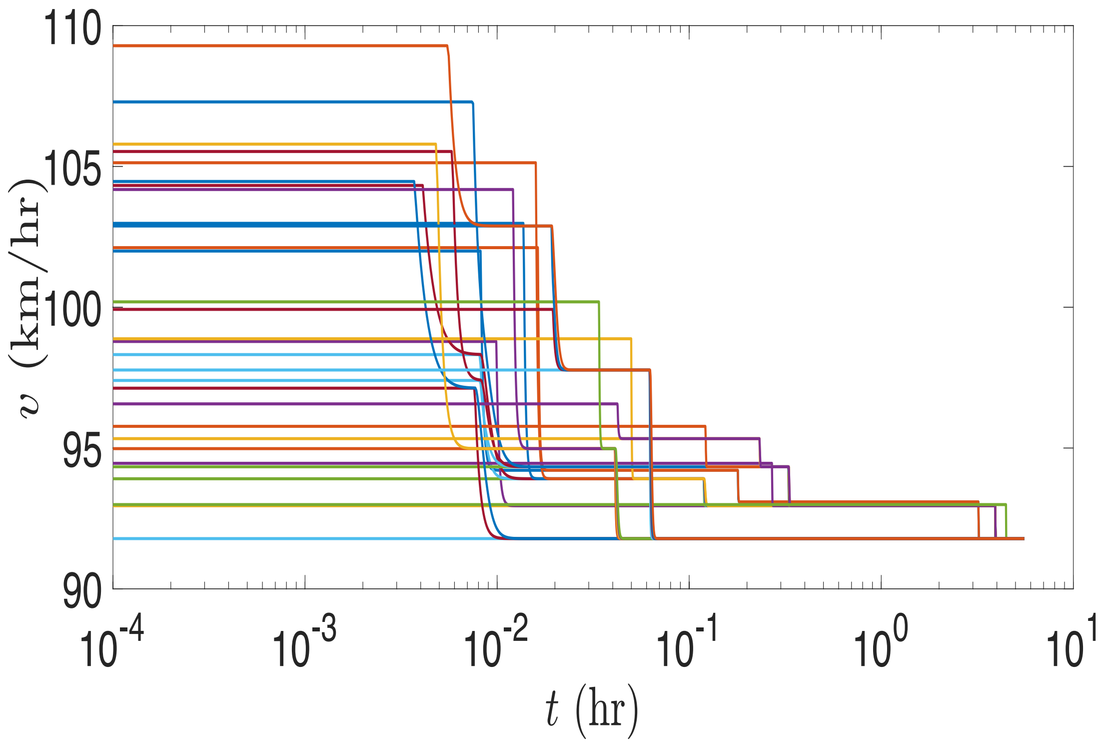


Fig.4: Speeds of vehicles Vs time.

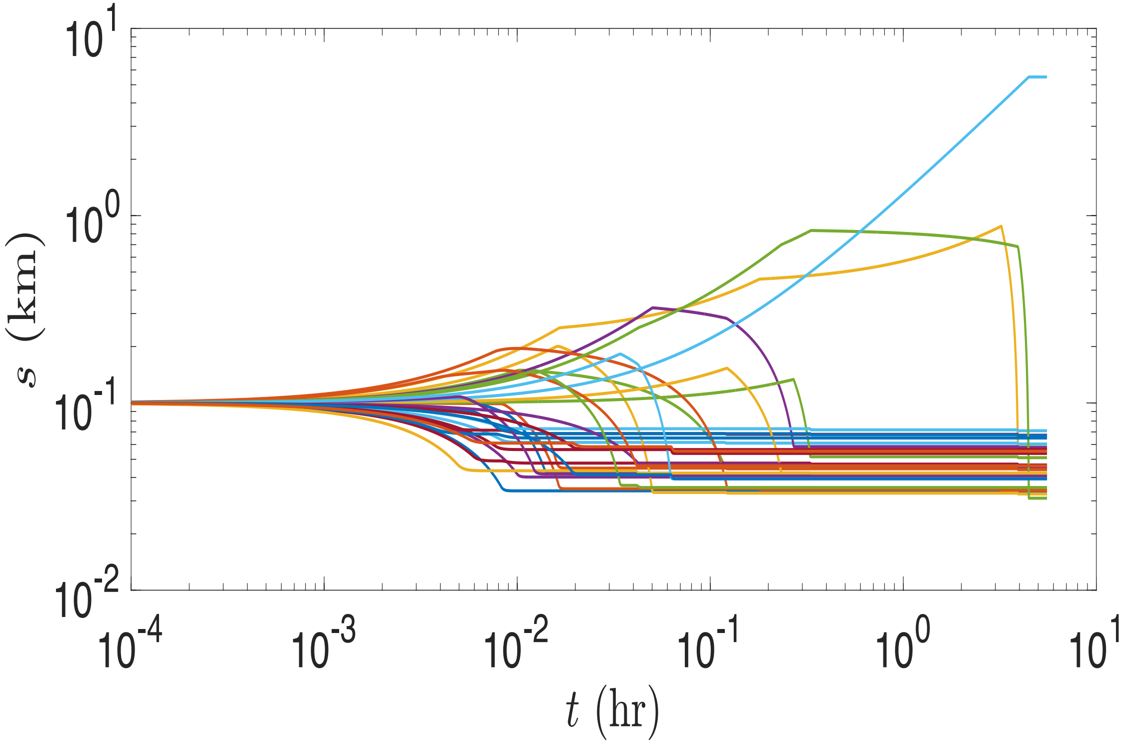


Fig.5: Gaps of vehicles Vs time.

Power-laws in kinetics: Platoon size

- Average platoon size $\langle N_c \rangle \sim t^z$
- Finite size scaling:
  - $t \sim L^{\alpha}$
  - $\langle N_c \rangle \sim L^{\alpha z} \left(\frac{t}{L^{\alpha}}\right)^z$
  - $z = \frac{\mu}{\mu+1}$; $\alpha = 1$
  - $p_{v_f}(v_f) = C\left(v_f - v_f^{min}\right)^{\mu-1}\left(v_f^{max} - v_f\right)^{\nu-1} I\ (A)$

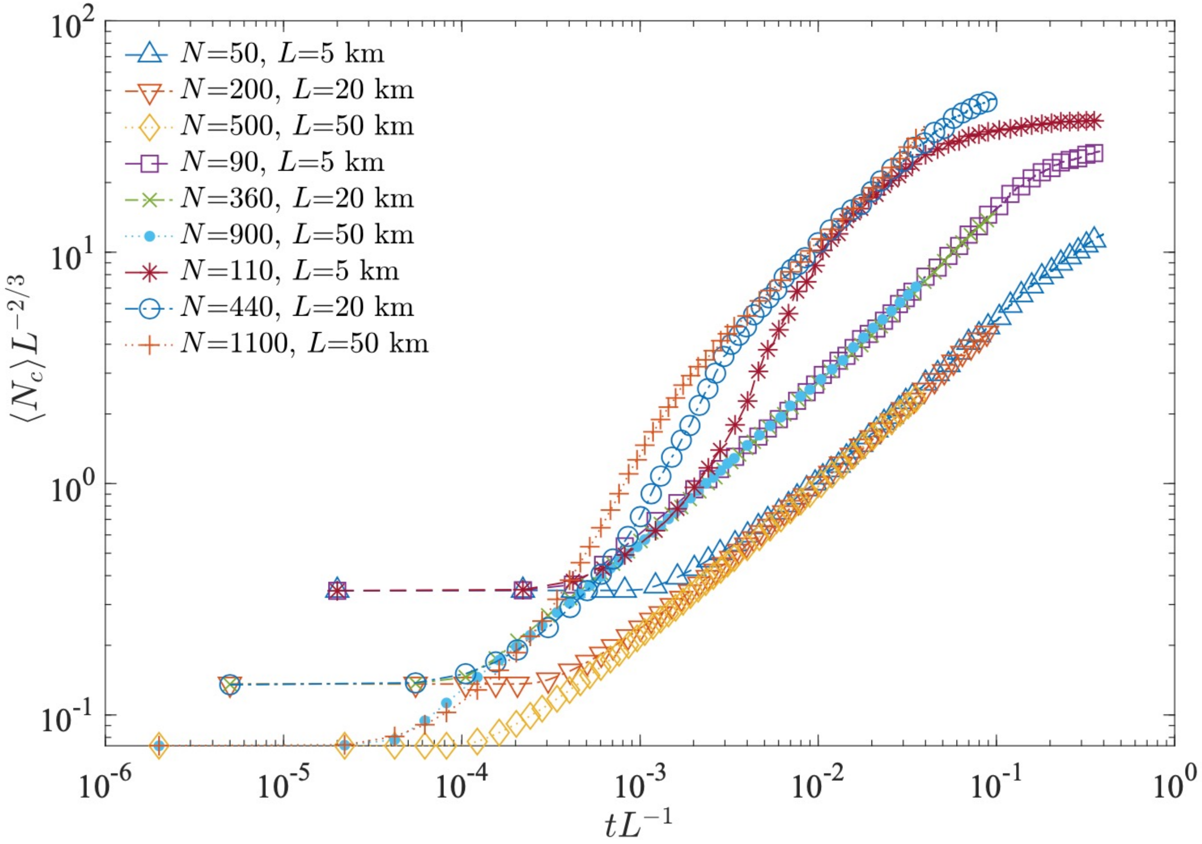


Fig.6: $\langle N_c \rangle$ Vs time t in scaled units for various densities. Scaled curves corresponding to the same density collapse onto single curve indicating $\langle N_c \rangle \sim t^{\frac{2}{3}}$.

Power-laws in kinetics: speed

- $\langle v - v_f^{min} \rangle \sim t^{-z_v}$

- Assuming
  $\langle v - v_f^{min} \rangle \sim L^{-\alpha\, z_v} \left(\frac{t}{L^{\alpha}}\right)^{-Z_v}$,
  we got $\alpha = 1$ and $z_v = \frac{1}{\mu+1}$.

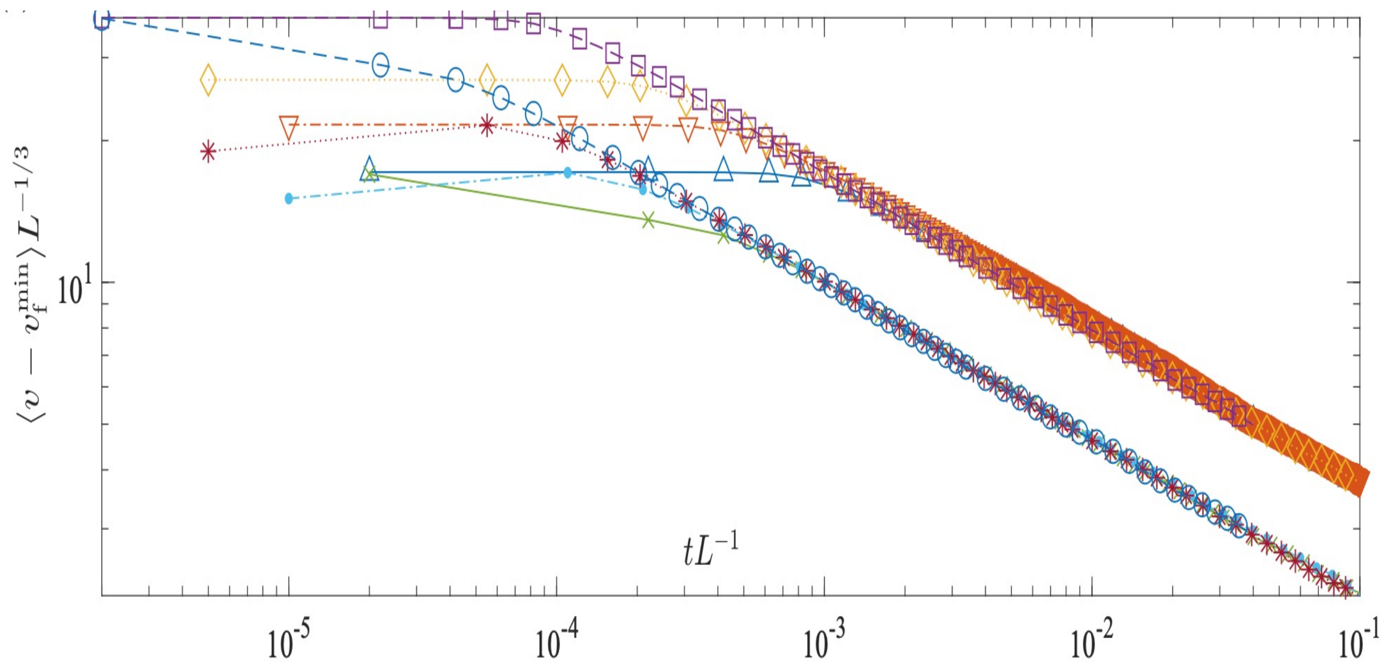


Fig.7: Scaled $\langle v - v_f^{min} \rangle$ Vs t for densities. Scaled curves corresponding to the same density collapse into single curve indicating $\langle v - v_f^{min} \rangle \sim t^{-\frac{1}{3}}$.

Stationary State Gap Distribution

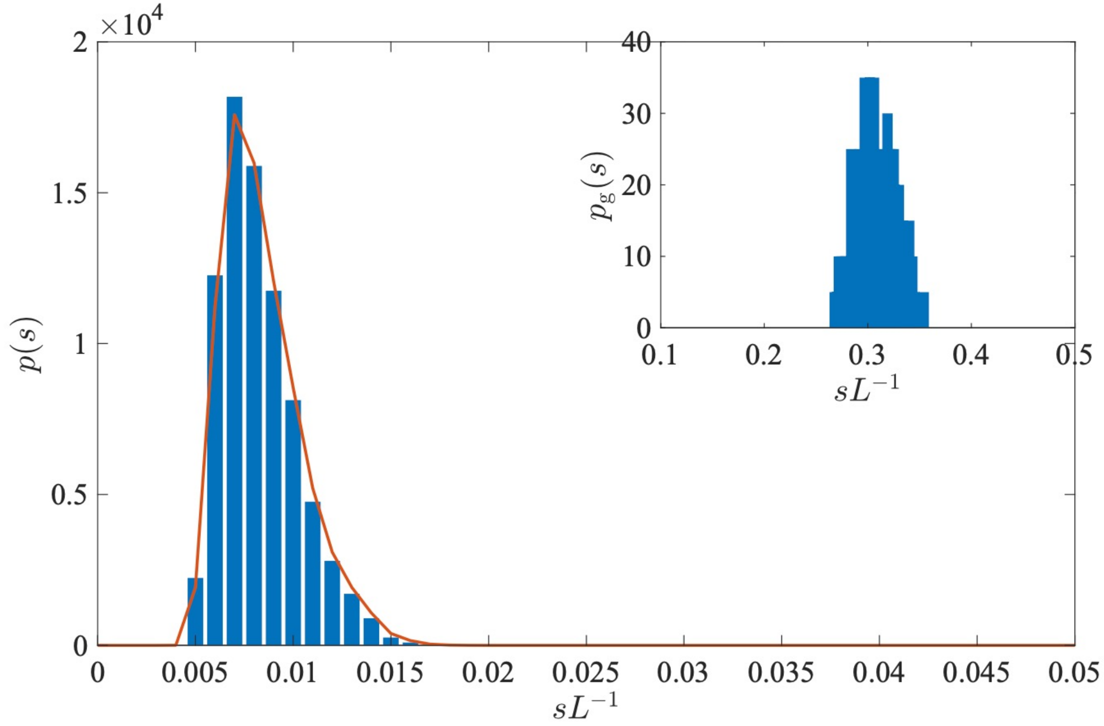


Fig.8: Distribution of gaps in the stationary state $p(s)$ for density $\rho\left(=\frac{N}{L}\right) < \rho_c$. $p(s)$ has two components $p_p(s)$ and $p_g(s)$(inset). Histograms show $p(s)$ determined from the simulation. The solid line is $p_{\{c,l\}}(s)$.

- $p(s) = p_p(s) + p_g(s)$
- $p_p(s) \approx p_{\{c,l\}}(S_c)$ $for\ large\ N$. Where $p_{\{c,l\}}(S_c)$ is the critical gap distribution of the leader.
  - $p_g(s)$ is a Gaussian with mean $\mu \approx L\left(1 - \rho\langle s\rangle_p\right) + \langle s\rangle_p$ and variance $N(\langle s^2\rangle_p - \langle s\rangle_p^2)$ where $\langle s\rangle_p = \int s p_p(s) ds$.
  - The phase transition happens when the head of the platoon meets its tail i.e., μ becomes L independent i.e., $\rho_c = 1/\langle S_p\rangle$.

$\tau \neq 0$ case

$$\frac{dx_i(t)}{dt} = \mathrm{V}\big(s_i(t - \tau_i\,)\big) \quad \forall i, \qquad i = 1,2, \dots, N$$

$$V(s) = \begin{cases} v_f, & s \geq S_c \\ w_b \max\{S_j^{-1} s \; - 1,0\} & s < \; S_c \end{cases}$$

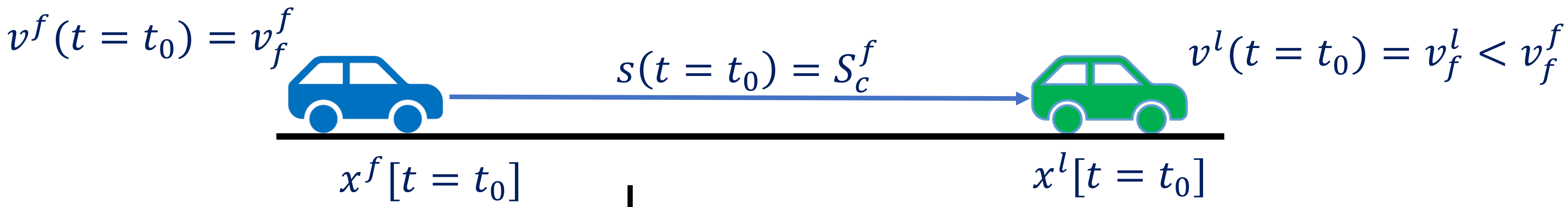
$v_f \in [60,80]$ km/hr;

$\rho_j \in [110,170]$ veh/km

$w_b \in [30,40]$ km/hr

$\tau_i = \frac{s_j}{w_b}$ for each driver.

How does τ affect the dynamics?



1. Position of the leader at any time:

$$x^{\mathrm{l}}[t] = x^{\mathrm{l}}[t_0] + v_{\mathrm{f}}^{\mathrm{l}}(t - t_0).$$

2. Position of the follower at any time:

$$x^{\mathrm{f}}[t] = x^{\mathrm{f}}[t_0] + \int_{t_0}^{t} V[s^{\mathrm{f}}[t' - \tau]]dt'.$$

3. Position of the follower at $t \in [t_0, t_0 + \tau)$

$$x^{\mathrm{f}}[t] = x^{\mathrm{f}}[t_0] + v_{\mathrm{f}}^{\mathrm{f}}(t - t_0) \qquad t \in [t_0, t_0 + \tau)$$

4. Gap of the follower at $t \in [t_0, t_0 + \tau)$

$$\begin{aligned} s^{\mathrm{f}}[t] &= x^{\mathrm{l}}[t] - x^{\mathrm{f}}[t_0] \\ &= S_c^{\mathrm{f}} - v_{\mathrm{f}}^{r}(t - t_0) \qquad t \in [t_0, t_0 + \tau) \end{aligned}$$

5. Gap of the follower at $t \in [t_0 + \tau, t_0 + 2\tau)$

$$\begin{aligned} s^{\mathrm{f}}[t] = S_c^{\mathrm{f}} - v_{\mathrm{f}}^{r}(t - t_0) \\ + \frac{v_{\mathrm{f}}^{r}}{2!}\frac{w_{\mathrm{b}}^{\mathrm{f}}}{S_{\mathrm{j}}}(t - (t_0 + \tau))^2 \qquad t \in [t_0 + \tau, t_0 + 2\tau) \end{aligned}$$

6. Repeating *Ad infinitum*

$$s^{\mathrm{f}}[t] = S_c^{\mathrm{f}} - v_{\mathrm{f}}^{r}(t - t_0) + \sum_{n=1}^{\infty} \frac{(-1)^{n+1} v_{\mathrm{f}}^{r}}{(n+1)!}\left(\frac{w_{\mathrm{b}}^{\mathrm{f}}}{S_{\mathrm{j}}}\right)^n (t - (t_0 + n\tau))^{n+1}\Theta[t - (t_0 + n\tau)]$$

$$\tau = 0$$
$$s^{\mathrm{f}}[t] = S_c^{\mathrm{f}} + v_{\mathrm{f}}^r \frac{S_{\mathrm{j}}}{w_{\mathrm{b}}^{\mathrm{f}}} \left\{ e^{-\frac{w_{\mathrm{b}}^{\mathrm{f}}}{S_{\mathrm{j}}} t} - 1 \right\}$$

$$\tau < \frac{S_j}{w_b^f}; \quad A = \frac{w_b^f}{S_j}; \quad B = v_f^r / A$$

$$s^{\mathrm{f}}[t] \approx S_c^{\mathrm{f}} + B \left\{ \left(1 - tA^2\tau + \frac{1}{2}A^2\tau^2(A^2t^2 - 3tA)\right)e^{-At} - 1 \right\}$$

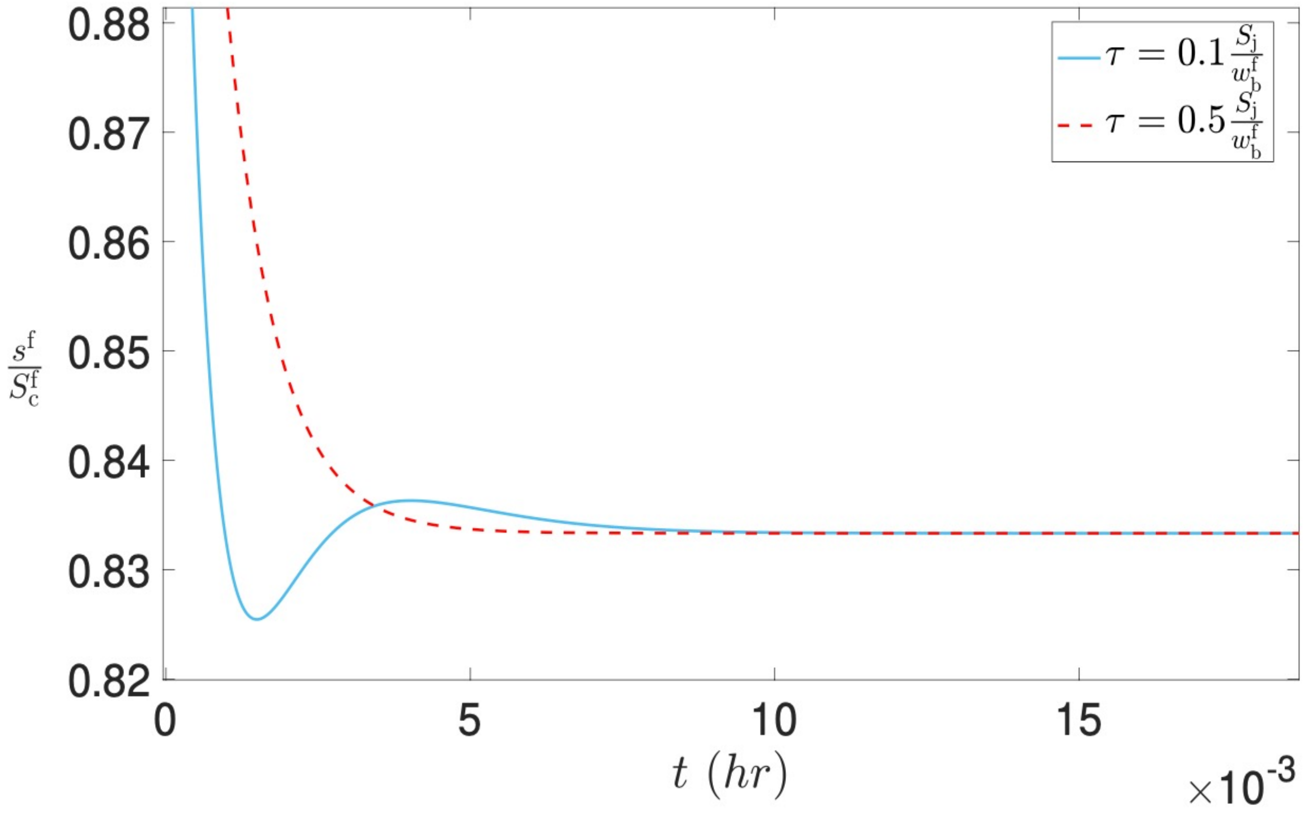


Fig.11: $\frac{s^f}{S_c^f}$ Vs t. For $\tau \ll \frac{S_j}{w_b^f}$, The gap reduces monotonically. As $\tau \sim \frac{S_j}{w_b^f}$, the gap reduces in an oscillatory way.

Formation of Stop-Go Waves

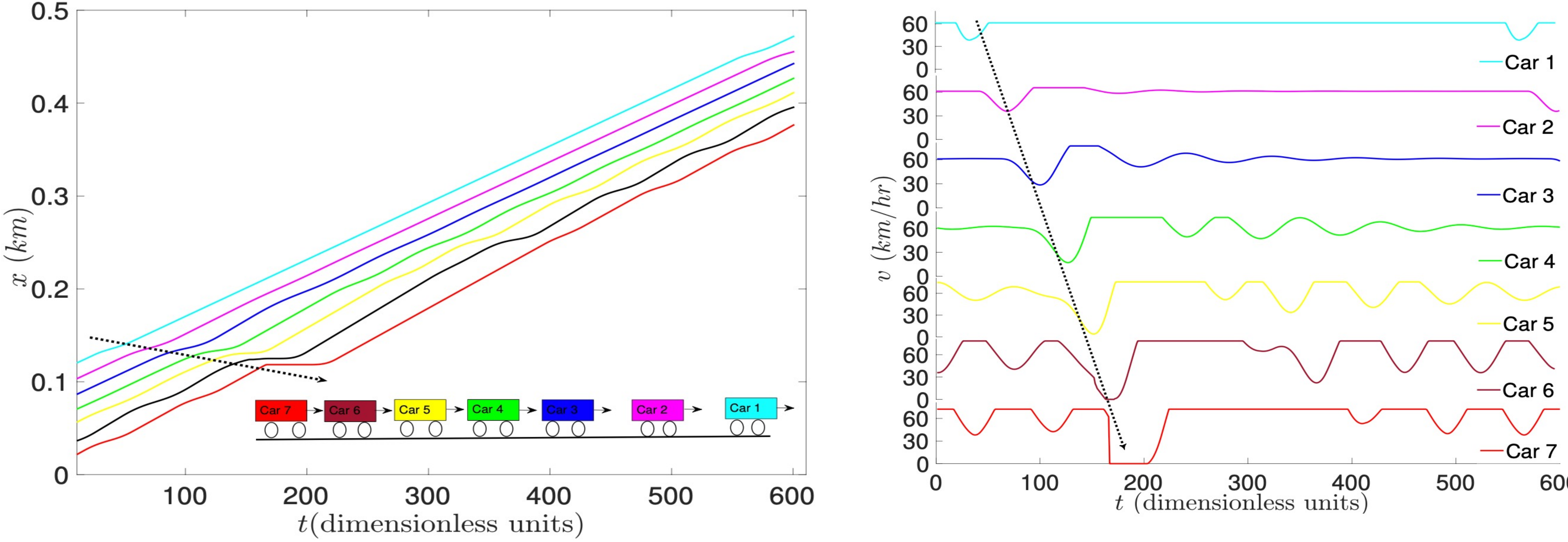


Fig.12: (Left) x Vs t plot and (right) v Vs t plot of seven vehicles. A perturbation in the speed of the leader (car 1) gets amplified as it moves upstream and becomes a stop-go wave at some point.

Self-organization happens in $\tau \neq 0$ case also

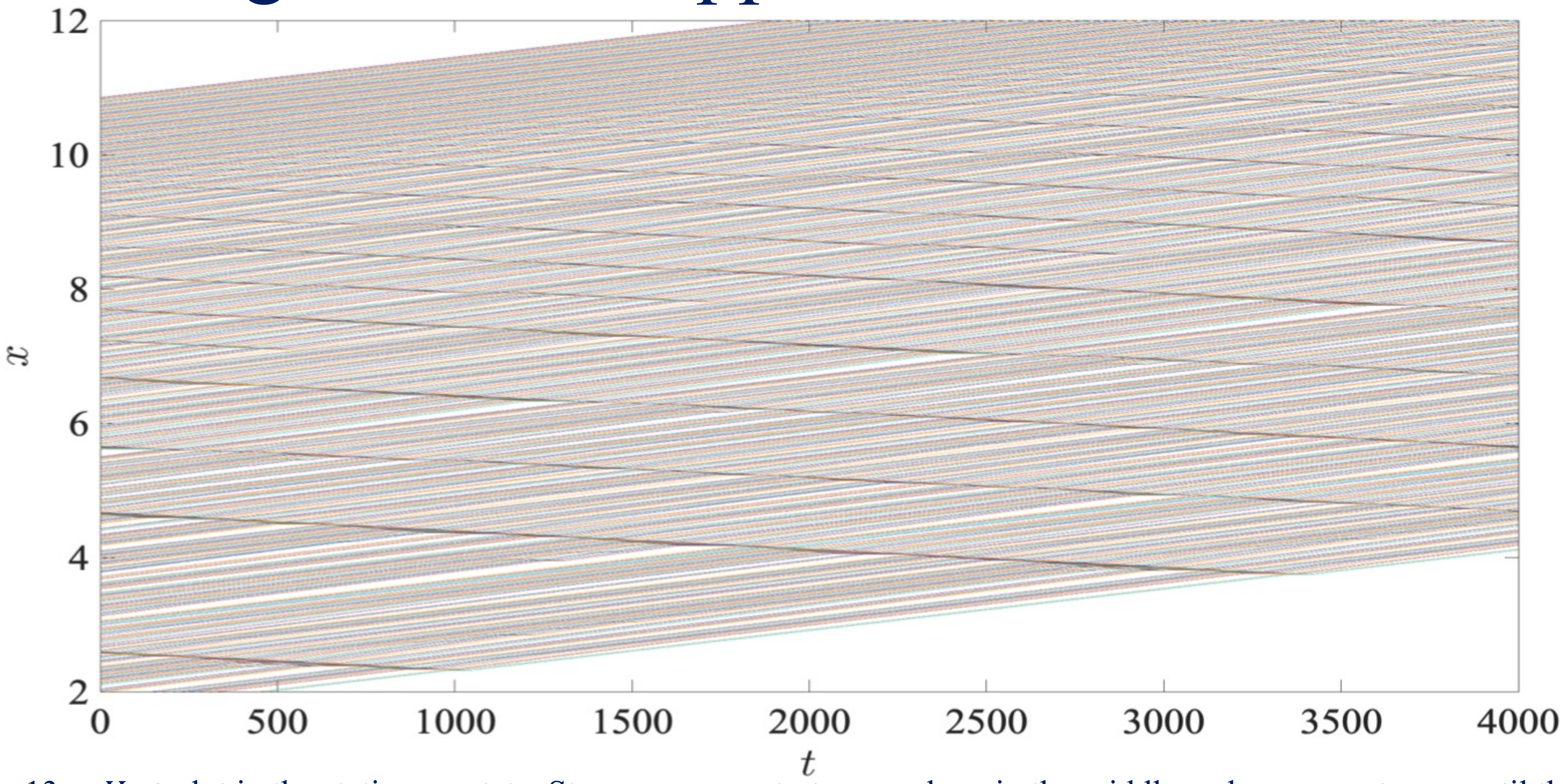


Fig.13: x Vs t plot in the stationary state. Stop-go waves start somewhere in the middle and move upstream until the end of the platoon where they get dissipated.

- The power-laws are same as in the $\tau = 0$ case:

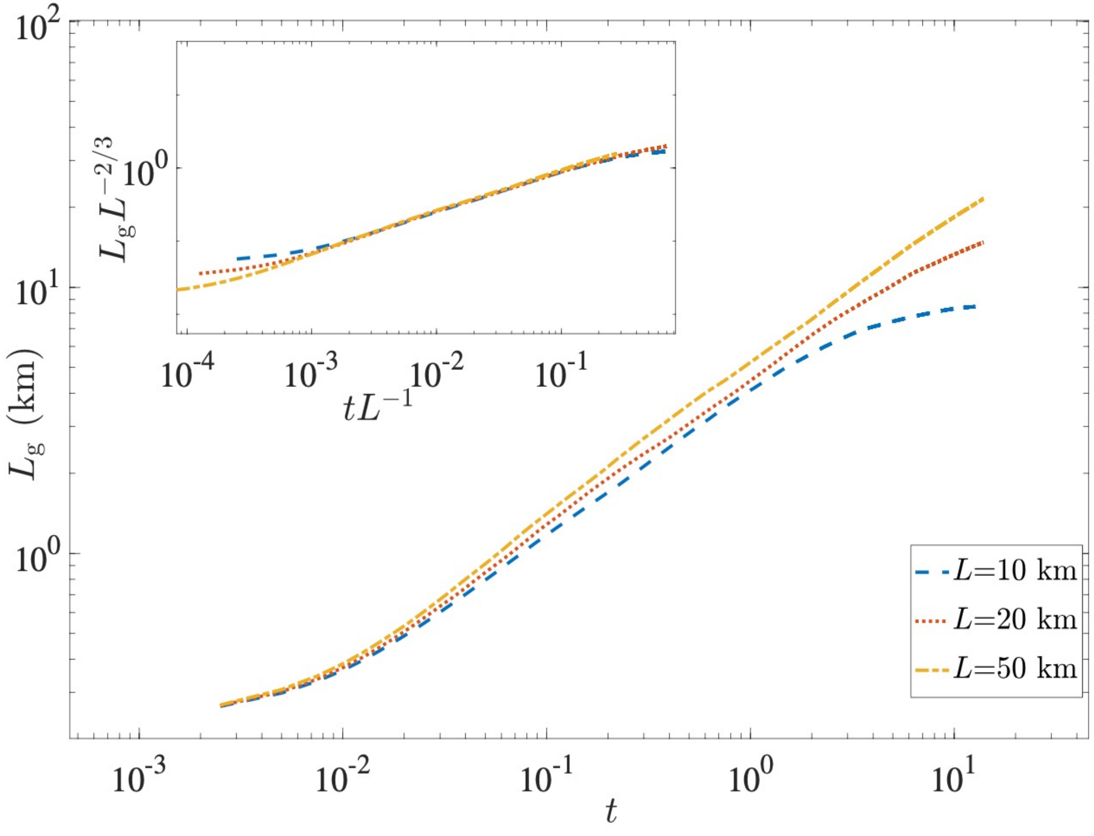


Fig.14. Gap ahead of the largest platoon grows as $\sim t^{\frac{\mu}{\mu+1}}$

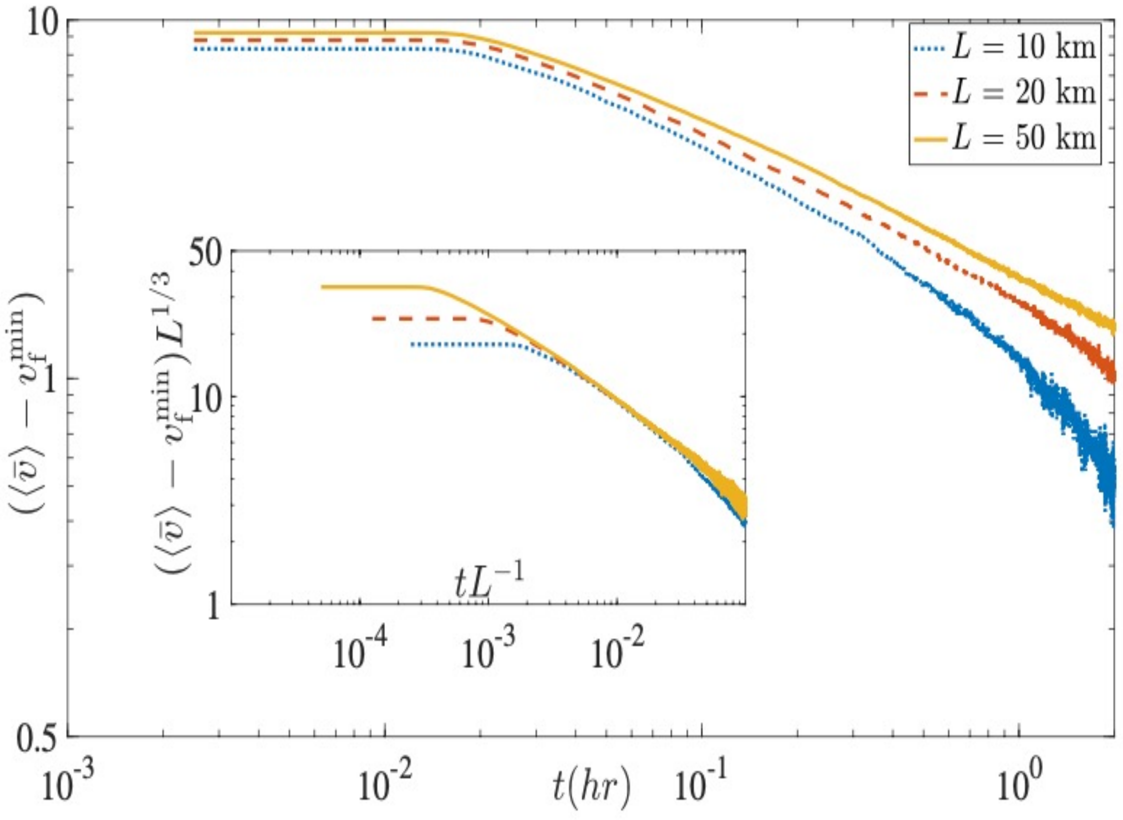


Fig.15. Average speed decays as $\sim t^{-\frac{1}{\mu+1}}$

Stationary state gap distribution

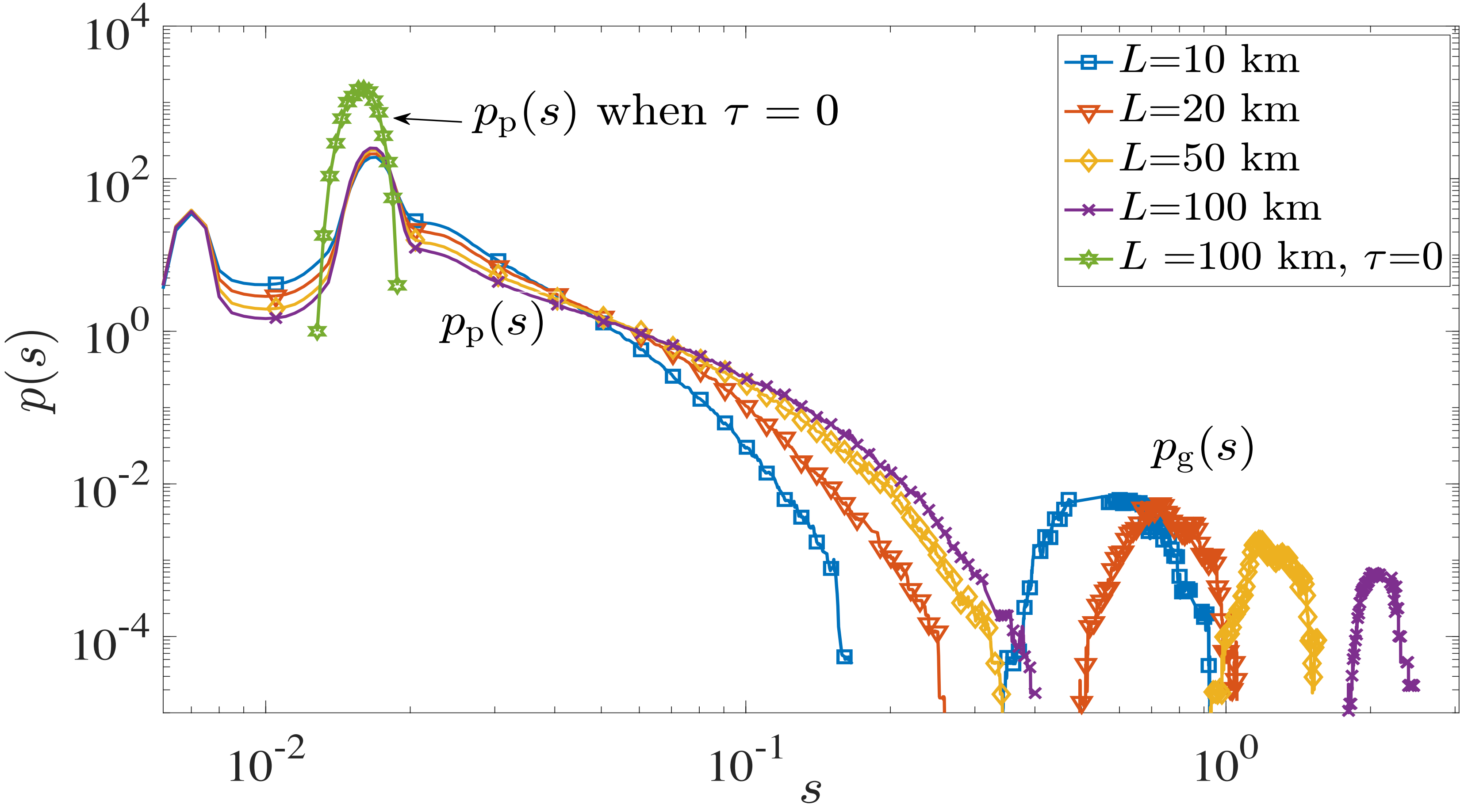


Fig.16: $p(s)$ Vs s for various track lengths after the stationary state is reached. $p_p(s)$ has two peaks; The left most peak is because of stopped vehicles.

Platooning phenomenon when passing is allowed: Ben-Naim Krapivsky model

- Let τ be the mean escape time and let $F_{des}(v)$ be the desired (free-flow) speed distribution.

- Let $f_{des}(v,t)dvdt$ is number of platoons per unit length moving with speed in $(v, v+dv)$ led by a vehicle whose desired speed is v.
- Let $g(v, v_{des}, t)dvdv_{des}dt$ is number of vehicles per unit length moving with speed in $(v, v+dv)$ whose desired speed is in $(v_{des}, v_{des}+dv_{des})$.

- Equations for kinetics: $\frac{\partial f_{des}(v,t)}{\partial t} = \underbrace{\frac{F_{des}(v) - f_{des}(v,t)}{\tau}}_{\text{escape term}} - \underbrace{f_{des}(v,t)\int_0^v dv'(v-v')f_{des}(v',t)}_{\text{collision term}}$

$$
\begin{aligned}
\frac{\partial g(v,v',t)}{\partial t} = -\frac{g(v,v',t)}{\tau} &+ (v-v')f_{des}(v,t)f_{des}(v',t) \\
&+ f_{des}(v',t)\int_{v'}^{v} dv''(v''-v')g(v,v'',t) \\
&- g(v,v',t)\int_{v'}^{v} dv''(v''-v')f_{des}(v'',t)
\end{aligned}
$$

Ben-Naim and Krapivsky model: stationary state

$\tau \to 0$ limit (Passings dominate collisions)

- Stationary state:

$$\frac{F_{des}(v) - f_{des}(v)}{\tau} - f_{des}(v)\int_0^v dv'(v-v')f_{des}(v') = 0$$

- To first order in τ

$$f_{des}(v) = F_{des}(v)(1 - \tau\int_0^v dv'(v-v')F_{des}(v'))$$

- Thus, majority of vehicles move with their desired speeds and there is no platoon formation.

- Alternatively, the above limit occurs if the desired speed v of a vehicle is less than a characteristic speed v^* given by $v^* \sim \tau^{-1/(\mu+2)}$

Large τ limit (Collisions dominate over passings)

- vehicles with $v_{des} < v^*$ still experience free-flow

- vehicles with desired speed $v_{des} > v^*$ experience congestion and form platoons behind those with $v_{des} < v^*$.

- average platoon size: $\Lambda \sim \tau^{\frac{\mu+1}{\mu+2}}$; $F_{des} \sim v^{\mu}$

- This result corroborates with no passing limit case ($\tau \to \infty$)

where $\Lambda \sim t^{\frac{\mu+1}{\mu+2}}$ and $\Lambda \to \infty$ in stationary state

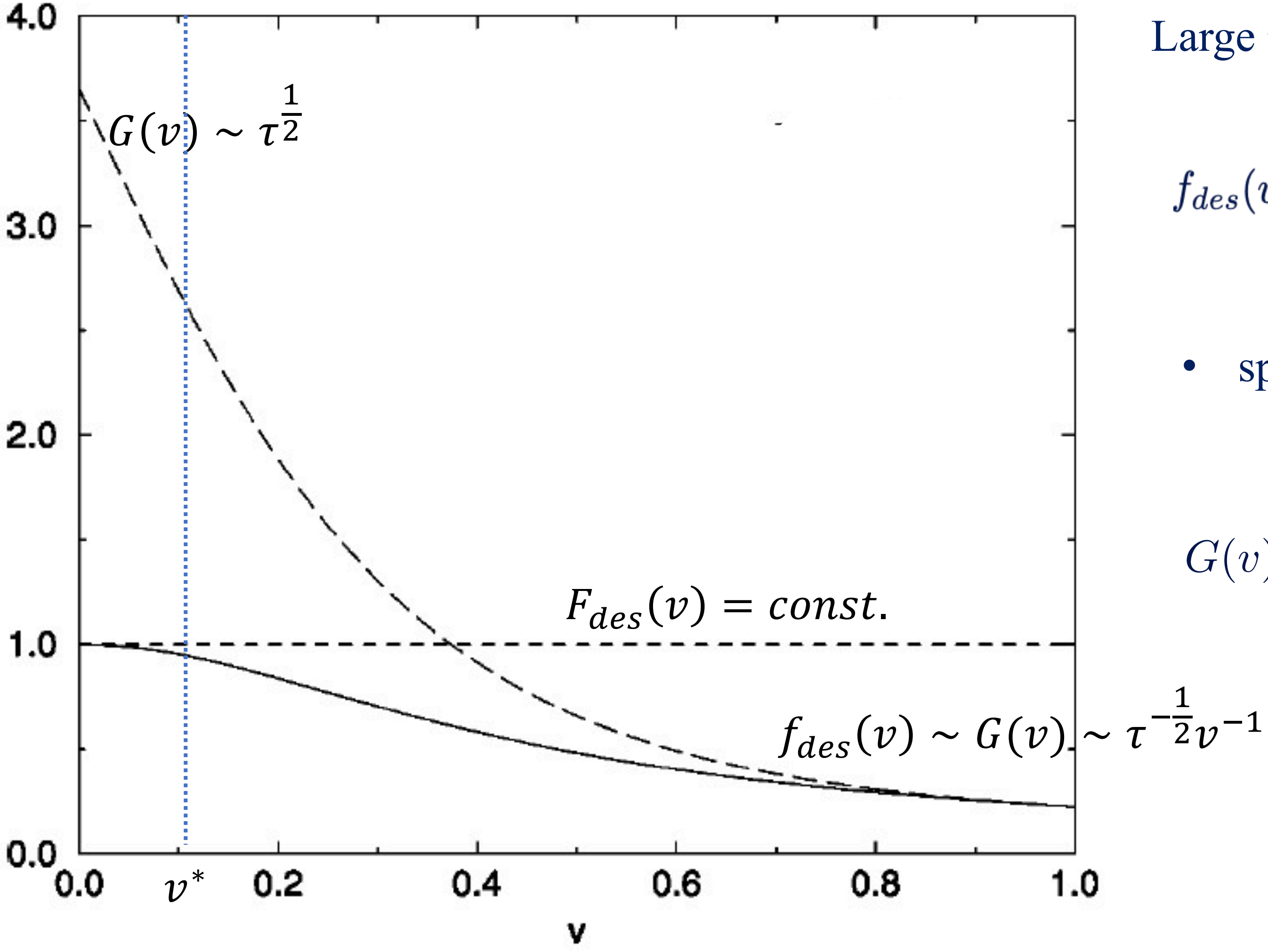


Fig.22: Speed distribution in the stationary state

Large τ limit (Collisions dominate over passings)

$$f_{des}(v) \sim \begin{cases} F_{des}(v) & v << v^* \\ \tau^{-1/2} v^{(\mu-2)/2} & \mu \geq 0, v >> v^* \\ \tau^{-1/(\mu+2)} v^{\mu-1} & \mu < 0, v >> v^* \end{cases}$$

- speed distribution of cars:

$$G(v) \sim \begin{cases} \tau^{\frac{\mu+1}{\mu+2}} v^{\mu} & \mu > -1, v < v^* \\ \tau^{-1/(\mu+2)} v^{\mu-1} & \mu < 0, v >> v^* \\ \tau^{-1/2} v^{\mu/2-1} & \mu \geq 0, v >> v^* \end{cases}$$

Summary: platooning phenomenon

- Single lane traffic and no passing:
  - The system self-organizes and reaches a stationary state in which all the vehicles follow the slowest vehicle forming a single giant platoon.
  - Average platoon size $\sim t^{\frac{\mu+1}{\mu+2}}$ where $P(v_f) \sim (v_f - v_f^{min})^{\mu}$ $\quad v_f \to v_f^{min}$
  - Average speed $\sim t^{-\frac{1}{\mu+1}}$
  - $p_p(s) \sim p_{\{c,l\}}(S_c)$, the critical gap distribution of the slowest vehicle.
  - The gap distribution of vehicles in the platoon $p_p(s)$ has information about the critical density. Essentially, $\rho_c \approx \left\{\int s\, p_p(s) ds\right\}^{-1}$.
  - The self-organization and the coarsening behavior in $\tau \neq 0$ case ($\tau < S_j/w$) are the same as in $\tau = 0$ case.

- Single lane with passing:
  - Vehicles with desired (free-flow) speed more than $v^* \sim \tau^{-1/(\mu+2)}$ experience congestion and form platoons behind vehicles with desired speed $v < v^*$
  - Average size of the platoon depends on the lane changing rate and the expoenent of desired speed distribution in the low speed limit as $\Lambda \sim \tau^{(\mu+1)/(\mu+2)}$.

Thank you