Gravitational Collapse for Newtonian stars

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- Balance between pressure and gravity in a static star;
- As gas burns, balance shifts;

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Classical model of a star: sphere of gas under Newtonian gravity.

- Balance between pressure and gravity in a static star;
- As gas burns, balance shifts;
- Possible collapse? Supernova?

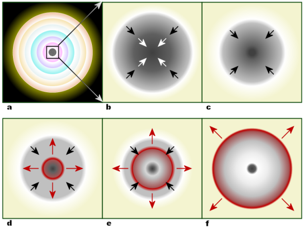

Figure: Image credit: R.J. Hall

Euler-Poisson equations

The Euler-Poisson equations of gas dynamics with Newtonian gravity:

$$\begin{cases} \partial_t \rho + \mathrm{div}_{\mathbf{x}}(\rho \mathbf{u}) = 0, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\ \rho\big(\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u}\big) + \nabla_{\mathbf{x}} p(\rho) = -\rho \nabla \Phi, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3, \\ \Delta \Phi = 4\pi\rho, & (t, \mathbf{x}) \in \mathbb{R} \times \mathbb{R}^3. \end{cases} \tag{1}$$

ρ is density, $\mathbf{u}$ is velocity, p is pressure, Φ is gravitational potential.
We assume the equation of state

$$p = p(\rho) = \rho^{\gamma}, \quad \gamma \in \big(1, \frac{4}{3}\big).$$

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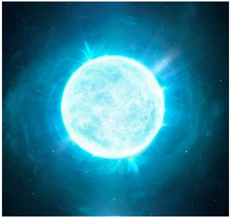

Example adiabatic exponents
$\gamma = \frac{5}{3}$ – monatomic gas, used for fully convective star cores (e.g. red giants);
$\gamma = \frac{4}{3}$ – high mass white dwarf stars, main-sequence stars (e.g. the Sun).
In general, as γ decreases, density is increasingly weighted towards centre.

Collapse

Collapse is the formation of a *singularity* at the origin, i.e.

$$\rho(t,0) \to \infty \quad \text{as} \quad t \to 0- \, .$$

- For $\gamma > \frac{4}{3}$, no finite mass and energy collapse possible.
- For $\gamma = \frac{4}{3}$, Goldreich–Weber collapse - unsuitable model for outer core.

Supernova expansion

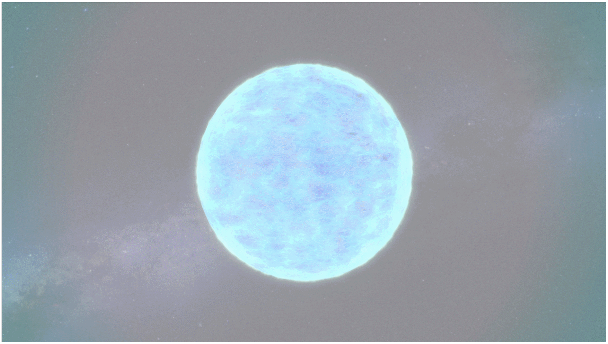

Figure: Image credit: NASA

Self-similar singularity formation

Self-similarity and singularities interact in a wide range of problems.

- Stellar collapse;
- Formation/expansion of shock waves;
- Shock reflection;
- Droplet pinch-off;
- Bacterial growth;
- Geometric wave equations;
- Yang–Mills;
- ...

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Key Features:

- Non-linearity;
- Intertwining of spatial and time scales;
- Good initial data leads to badly behaved solutions!

Scaling and Self-similarity

Scaling

Let $\rho = \rho(t, r)$, $\mathbf{u} = u(t, r)\frac{\mathbf{x}}{|\mathbf{x}|}$, $r = |\mathbf{x}|$, solve Euler-Poisson, $\lambda > 0$. Then

$$\rho_\lambda(t, r) = \lambda^{-\frac{2}{2-\gamma}}\rho\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big), \quad u_\lambda(t, r) = \lambda^{-\frac{\gamma-1}{2-\gamma}}u\Big(\frac{t}{\lambda^{\frac{1}{2-\gamma}}}, \frac{r}{\lambda}\Big)$$

is also a solution. (NB: This is a *unique* scaling!)

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Self-similarity

We define a *self-similar* variable

$$y = \frac{r}{(-t)^{2-\gamma}},$$

and search for

$$\rho(t,r) = (-t)^{-2}\tilde{\rho}(y), \quad u(t,r) = (-t)^{1-\gamma}\tilde{u}(y).$$

Mass and energy criticality

Natural notions of mass and energy for Euler-Poisson:

$$M[\rho] = \int_0^\infty \rho\, r^2 \mathrm{d}r, \quad E[\rho, u] = \int_0^\infty \left(\frac{1}{2}\rho u^2 + \frac{\rho^\gamma}{\gamma - 1} + \frac{1}{2}\rho\Phi\right) r^2 \mathrm{d}r,$$

where Φ solves $\Delta\Phi = 4\pi\rho$ is the gravitational potential.
Under scaling,

$$M[\rho_\lambda] = \lambda^{\frac{4-3\gamma}{2-\gamma}} M[\rho], \quad E[\rho_\lambda, u_\lambda] = \lambda^{\frac{6-5\gamma}{2-\gamma}} E[\rho, u].$$

Thus $\gamma = \frac{4}{3}$ is *mass-critical*, $\gamma = \frac{6}{5}$ is *energy-critical*.

ODE system

Defining a convenient variable $\omega(y) = \tilde{u}(y)/y + 2 - \gamma$, self-similar Euler-Poisson becomes

$$\begin{aligned} \tilde{\rho}' &= \frac{y\tilde{\rho}h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2}, \\ \omega' &= \frac{4 - 3\gamma - 3\omega}{y} - \frac{y\omega h(\tilde{\rho},\omega)}{\gamma\tilde{\rho}^{\gamma-1} - y^2\omega^2}, \end{aligned} \tag{2}$$

where $h(\tilde{\rho},\omega)$ is a quadratic function.

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Definition (Sonic point)

Let $(\tilde{\rho}(\cdot),\omega(\cdot))$ be a C^1-solution to the self-similar Euler-Poisson system on the interval $(0,\infty)$. A point $y_* \in (0,\infty)$ such that

$$\gamma\tilde{\rho}^{\gamma-1}(y_*) - y_*^2\omega^2(y_*) = 0$$

is called a *sonic point*.

Theorem

Initial/boundary conditions

For a regular solution, we require

$$\tilde{\rho}(0) > 0, \quad \omega(0) = \frac{4 - 3\gamma}{3},$$

$$\tilde{\rho}(y) \sim y^{-\frac{2}{2-\gamma}} \text{ as } y \to \infty, \quad \lim_{y\to\infty} \omega(y) = 2 - \gamma.$$

NB: this forces the existence of a sonic point!

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NB: this forces the existence of a sonic point!

Theorem (Guo–Hadzic–Jang–S. '21)

For each $\gamma \in (1, \frac{4}{3})$, there exists a global, real-analytic solution $(\tilde{\rho}, \omega)$ of self-similar Euler-Poisson with a single sonic point y_* such that:

$$\tilde{\rho}(y) > 0 \textit{ for all } y \in [0,\infty), \quad -\frac{2}{3}y < u(y) < 0 \textit{ for all } y \in (0,\infty).$$

In addition, both ρ and ω are strictly monotone:

$$\tilde{\rho}'(y) < 0 \textit{ for all } y \in (0,\infty), \quad \omega'(y) > 0 \textit{ for all } y \in (0,\infty).$$

Connection to previous literature

Classical and numerical work

- Taylor, Von Neumann, Sedov, Güderley '40s: study implosion and explosion for Euler equations;
- Larson–Penston '69: numerical solution for $\gamma = 1$;
- Hunter '77: family of numerical solutions for $\gamma = 1$;
- Yahil '83: numerical solutions for $\gamma \in [\frac{6}{5}, \frac{4}{3})$;
- Maeda–Harada '01: numerical evidence towards mode stability of Larson–Penston.

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Recent works

- Merle–Raphaël–Rodnianski–Szeftel '19: existence of a imploding self-similar solutions for Euler;
- Guo–Hadzic–Jang '20: construction of LP solution.

Overview of key difficulties

Regularity

Expect stability tied to regularity (MRRS '19). Requires smoothness through sonic point.

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Methods need to be adapted to specific non-linearities (no general recipe for solving such problems).

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Non-autonomous system

Non-autonomous forces evolving phase portrait. No fixed phase portrait analysis for invariant regions.

Reference solutions and sonic point

Two explicit solutions

Far-field solution (ρ_f, ω_f) and Friedman solution (ρ_F, ω_F):

$$(\rho_f(y), \omega_f(y)) = (k_\gamma y^{-\frac{2}{2-\gamma}}, 2-\gamma), \qquad (\rho_F(y), \omega_F(y)) = (\frac{1}{6\pi}, \frac{4}{3} - \gamma).$$

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- Sonic points at $y_f(\gamma) < y_F(\gamma)$.
- Far-field satisfies asymptotic boundary condition as $y \to \infty$.
- Friedman satisfies boundary condition at origin.

Idea: Use $\omega_f = 2 - \gamma$, $\omega_F = \frac{4}{3} - \gamma$ as barriers.

Sonic point

Taylor expansion

For a candidate sonic point $y_* > 0$, try to solve with Taylor expansion:

$$\rho(y; y_*) = \sum_{j=0}^{\infty} \rho_j (y - y_*)^j, \quad \omega(y; y_*) = \sum_{j=0}^{\infty} \omega_j (y - y_*)^j.$$

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Zero order

Recall EP in form

$$\rho' = \frac{y\rho h(\rho,\omega)}{G(y;\rho,\omega)}, \quad \omega' = \frac{4 - 3\gamma - 3\omega}{y} - \frac{y\omega h(\rho,\omega)}{G(y;\rho,\omega)}$$

with $G(y;\rho,\omega) = \gamma\rho^{\gamma-1} - y^2\omega^2$.
Require $h(\rho_0,\omega_0) = G(y_*;\rho_0,\omega_0) = 0$.
Smooth one-to-one mapping between $\omega_0 \in [\frac{4}{3} - \gamma, 2 - \gamma]$ and $y_* \in [y_f(\gamma), y_F(\gamma)]$.

First order coefficients I

In case $\gamma = 1$, apparently two branches of (ρ_1, ω_1): LP and Hunter

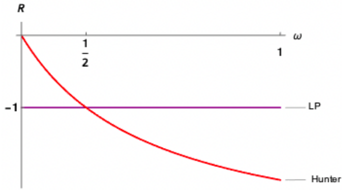
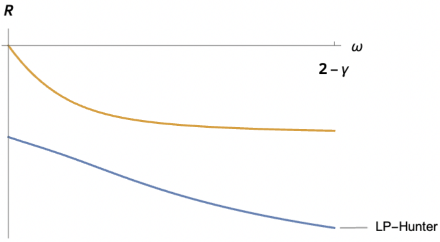


Figure: Plot of $R = \frac{\rho_1 y_*}{\rho_0}$ as function of ω_0 for $\gamma = 1$.

First order coefficients II

For $\gamma \in (1, \frac{4}{3})$, branch separation:



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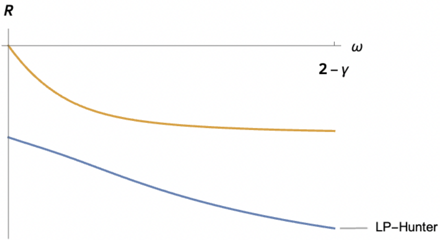

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Theorem

For all $\gamma \in (1, \frac{4}{3})$, there exists $\nu > 0$ such that for all $y_* \in [y_f(\gamma), y_F(\gamma)]$, there exists an analytic solution to self-similar Euler-Poisson on $(y_* - \nu, y_* + \nu)$ with a single sonic point at y_*.

Solving to the right

Lemma

For each $\gamma \in (1, \frac{4}{3})$, *each* $y_* \in [y_f(\gamma), y_F(\gamma)]$, *the local solution* $(\rho(\cdot; y_*), \omega(\cdot; y_*))$ *obtained by Taylor expansion extends globally to the right on* $[y_*, \infty)$, *remains supersonic, and satisfies the asymptotic boundary conditions.*

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Key ideas

- Use structure of $h(\rho, \omega)$ and $G(y; \rho, \omega)$ to derive dynamical invariances to the right.
- Show ω remains trapped between $\frac{4}{3} - \gamma$ and $2 - \gamma$.
- Extend dynamical invariance to show flow remains supersonic.
- Asymptotics follow easily from structure of flow.

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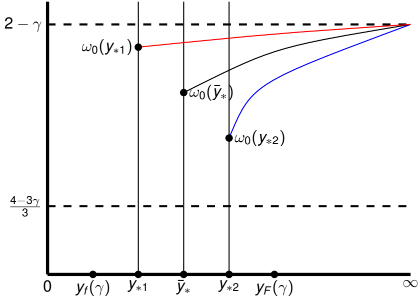

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Aim

Find $\bar{y}_*$ such that local solution $(\rho(\cdot;\bar{y}_*),\omega(\cdot;\bar{y}_*))$ extends smoothly to $y=0$.
Aim for solution with

$$\frac{4}{3}-\gamma \leq \omega(y;\bar{y}_*) < 2-\gamma, \qquad \lim_{y\to 0}\omega(y;\bar{y}_*) = \frac{4}{3}-\gamma.$$

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Idea

Look for critical $\bar{y}_*$ as infimum of

$$Y = \left\{ y_* \in (y_f, y_F) \,\middle|\, \exists y \text{ such that } \omega(y;\tilde{y}_*) = \frac{4-3\gamma}{3} \ \forall\, \tilde{y}_* \in [y_*, y_F) \right\}.$$

Key idea: Prove monotonicity for both $\rho(\cdot;y_*)$ and $\omega(\cdot;y_*)$ as long as $y_* \in Y$ and $\omega(\cdot;y_*) \ge \frac{4}{3}-\gamma$.

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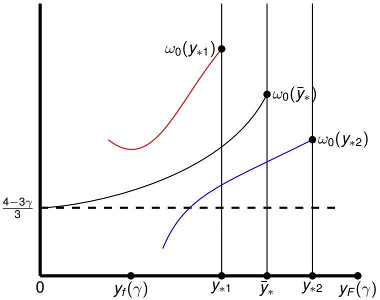

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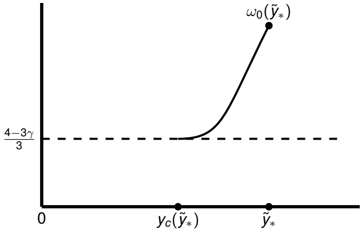
Monotonicity Argument

For $y_* \in Y$, define first touching time

$$y_c(y_*) = \inf\{y \leq y_* \,|\, \omega(\tilde{y}; y_*) > \frac{4-3\gamma}{3} \;\forall \tilde{y} \in (y, y_*)\}.$$

Suppose $\exists$ maximal $\tilde{y}_* \in Y$ such that $\exists\, y_0 \in [y_c(\tilde{y}_*), \tilde{y}_*]$ with $\omega'(y_0; \tilde{y}_*) = 0$.



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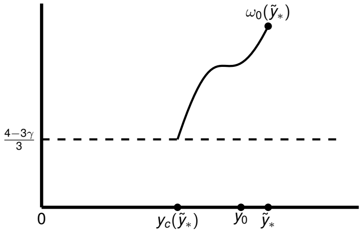
- $h < 0$ on $[y_c, \tilde{y}_*]$, hence $\omega'(y_c) > 0$, so that $y_0 \in (y_c, \tilde{y}_*)$;



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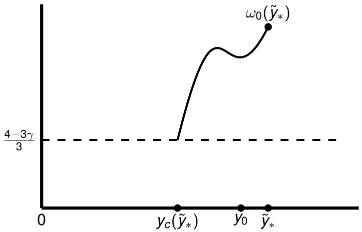
- $\exists\ \delta_1 > 0$ such that $\omega'(y; \tilde{y}_*) < 0$ on $(y_0 - \delta_1, y_0)$;



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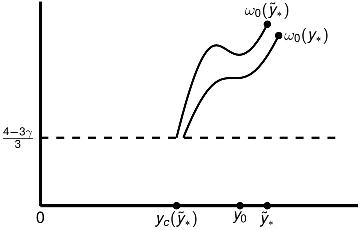
Suppose $\exists$ maximal $\tilde{y}_* \in Y$ such that $\exists\, y_0 \in [y_c(\tilde{y}_*), \tilde{y}_*]$ with $\omega'(y_0; \tilde{y}_*) = 0$.

- $\omega'(y; y_*)$ is uniformly continuous with respect to (y, y_*) on a neighbourhood of $(y_0, \tilde{y}_*)$.



Future Programme

Linear Stability

- Appropriate self-similar coordinates;
- Non-self-adjoint problem (complex eigenvalues);
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Future directions

- Non-linear stability;
- Einstein-Euler (relativistic self-similar fluid implosion) and its stability (cf. Guo–Hadžić–Jang ’21).
- Continuation and expansion?

Thank you!