Long Time Behavior and Local Regularity for solutions to the Navier-Stokes equations

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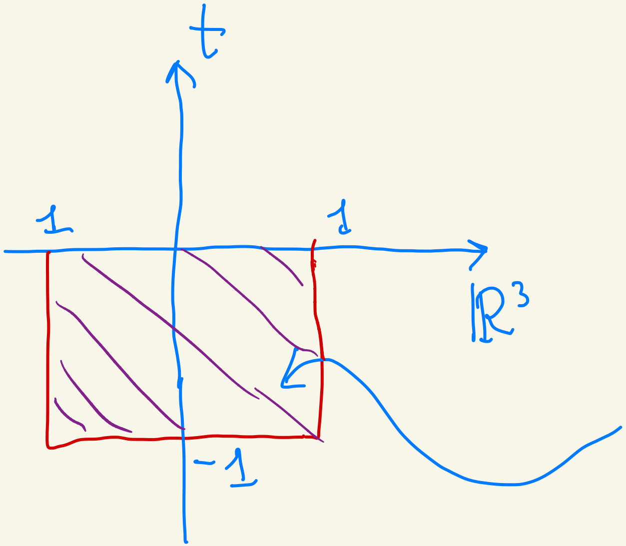
Local Regularity Setting



$$Q = B \times \,]-1, 0[$$

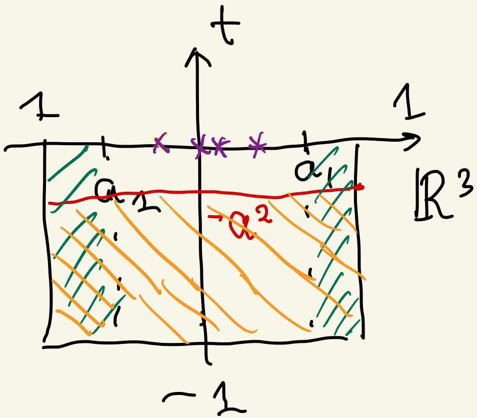
$$\begin{cases} \partial_t w + w \cdot \nabla w - \Delta w = -\nabla r \\ \operatorname{div} w = 0 \end{cases} \qquad \text{in } Q$$

First time singularity scenario

1. Energy solution: $w \in L_{2,\infty}(Q) = L_\infty(-1,0;L_2(B))$,

$\nabla w \in L_2(Q)$, $\quad r \in L_{3/2}(Q)$

2. Mimic the first time singularity:



$u \in L_\infty(\{a<|x|<1\} \times]-0,1[)$

$u \in L_\infty(B \times]-1,-a^2[)$

for any $0 < a < 1$

3. Type I singularity

for example

$$|w(x,t)| \leq \frac{C_d}{|x| + \sqrt{-t}}$$

Potential singularity is $z = (x,t) = (0,0)$

Question: Is $z=0$ is a regular point of w? In other words, is it true $w \in L_\infty(Q)$?

The answer is Yes if c_d is sufficiently small

Question 2. How small?

Our way to attack the problem is via duality.

Assume that $z=0$ is a singular point of w, then $w \notin L_\infty(Q(\delta))$, $\forall\ 0<\delta<1$

① + ② $\Longrightarrow$ $\exists$ non-trivial mild bounded ancient solution $\bar{u} : \mathbb{R}^3 \times]-\infty, 0[$ such that

(i) $\bar{u} \in L_\infty(\mathbb{R}^3 \times]-\infty, 0[)$ $(|\bar{u}| \leq 1)$

(ii) $\exists\ \bar{p} \in L_\infty(-\infty, 0; BMO)$

(iii) $\left.\begin{array}{l} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} - \Delta \bar{u} = -\nabla \bar{p} \\ \operatorname{div} \bar{u} = 0 \end{array}\right\}$ $\mathbb{R}^3 \times]-\infty, 0[$

(iv) $|\bar{u}(0)| = 1.$!!!

Properties of mild bounded ancient solutions:

$\bar{u}$ is of class C^∞, all derivatives of $\bar{u}$ and $\nabla \bar{p}$ are bounded.

Decay ?

3 $\Rightarrow$ $$|\bar{u}(x,t)| \leq \frac{C_d}{|x| + \sqrt{-t}}$$

$\forall\ x \in \mathbb{R}^3, \quad \forall\ t \leq 0$

It is more convenient to work with

$$u(x,t) = \overline{u}(x,-t) \quad \text{for } t > 0.$$
$$p(x,t) = \overline{p}(x,-t)$$

$$(i)\quad \left.\begin{aligned} -\partial_t u + u\cdot\nabla u - \Delta u &= -\nabla p \\ \operatorname{div} u &= 0 \end{aligned}\right\} \text{ in } Q_+ = \mathbb{R}^3 \times]0,\infty[$$

$$(ii)\quad |u(x,t)| \le 1, \qquad p \in L_\infty(0,\infty; BMO)$$

$$(iii)\quad |u(x,t)| \le \frac{c_*}{|x| + \sqrt{t}}$$

$$(iv)\quad |u(0)| = 1.$$

Question: $(i) + (ii) + (iii) \Longrightarrow u \equiv 0$ in Q_+ ?

Duality approach

I. $\partial_t v - u\cdot\nabla v - \Delta v - \nabla q = -\operatorname{div} F$

$(\text{or } \partial_t v - u\cdot\nabla v \mp v\cdot\nabla u - \nabla q = -\operatorname{div} F),$

$$\operatorname{div} v = 0 \quad \text{in } Q_+ = \mathbb{R}^3\times]0,\infty[$$

$$v\Big|_{t=0} = 0$$

$F \in (F_{ij}) \in C_0^\infty(\mathbb{R}^3), \qquad F = -F^T$

$$\int_{Q_+} u \cdot \operatorname{div} F \, dx = - \lim_{T \to \infty} \int_{\mathbb{R}^3} u(x,T) \cdot v(x,T) \, dx$$

$$|u(x,t)| \leq \frac{C_d}{|x| + \sqrt{t}} \quad \Rightarrow$$

$$\left| \int_{\mathbb{R}^3} u(x,t) \cdot v(x,t) \, dx \right| \leq \frac{C_d}{\sqrt{t}} \| v(\cdot,t) \|_{L_1(\mathbb{R}^3)}$$

$$\longrightarrow 0 \quad ?$$

Theorem. $\forall\, m = 0, 1, 2, \ldots$

$$\| v(\cdot, t) \|_1 \le c(m, C_d, F) \frac{\sqrt{t}^{\,3/2}}{\ln^m (t+e)}$$

and

$$\| v(\cdot, t) \|_2 \le c(m, C_d, F) \frac{1}{\ln^m (t+e)}$$

$\forall\; t \ge 0$

II. $\partial_t v - u\cdot\nabla v - \Delta v - \nabla q = 0$

$$\left(\text{or } \partial_t v - u\cdot\nabla v \mp v\cdot\nabla u - \Delta v - \nabla q = 0\right)$$

$$\operatorname{div} v = 0 \qquad \text{in } Q_+$$

$$v\Big|_{t=0} = v_0 \in C^\infty_{0,0}(\mathbb{R}^3)$$

$$C^\infty_{0,0}(\mathbb{R}^3) = \{ v \in C^\infty_0(\mathbb{R}^3) : \operatorname{div} v = 0 \}$$

$$\int_{\mathbb{R}^3} u(x,t)\cdot v(x,t)\,dx = \int_{\mathbb{R}^3} u(x,0)\cdot v_0(x)\,dx \qquad \forall\, t \geq 0$$

$$? \Big\downarrow \qquad \text{as } t \to \infty$$

$$0$$

yes

$$\Longrightarrow \quad u(x,0) = 0 \quad \left(\text{but} \quad \|u(0)\| = 1 \right)$$

Theorem. $\forall\, m = 0, 1, 2, \ldots\ldots$

$$\| v(\cdot, t) \|_1 \leq c(m, C_d, v_0) \frac{\sqrt{t}^{\,3/2}}{\ln^m (t+e)}$$

and

$$\| v(\cdot, t) \|_2 \leq c(m, C_d, v_0) \frac{1}{\ln^m (t+e)}$$

$\forall\, t \geq 1$

So, we need to study the decay of $\|v(\cdot, t)\|_{L_p}$ if $t \gg 1$ and $p = 1$ or $p = 2$

What would we have for heat equation: $\partial_t v - \Delta v = 0$, $v\big|_{t=0} = u_0$

$$\Longrightarrow \quad \|v(\cdot, t)\|_{L_p} \le \|u_0\|_{L_p} \quad \text{if } p \ge 1.$$

$$\|$$

$$\|u_0\|_p$$

Why it is problem to estimate $\|v(\cdot,t)\|_1$? (Heuristic only)

Heat equation:

$$\partial_t v - \Delta v = 0 \qquad \times \quad \frac{v}{|v|}$$

$$\Rightarrow \quad \partial_t \int_{\mathbb{R}^3} |v| \, dx \leq 0$$

Stokes equations:

$$\partial_t v - \Delta v = u \cdot \nabla v + \nabla q$$

$$\Delta q = \operatorname{div}(u \cdot \nabla v)$$

If $u(\cdot,t) \cdot \nabla(\cdot,t) \in \mathcal{H}^1$ (Hardy space)

$$\Rightarrow \nabla q(\cdot,t) \in \mathcal{H}^1$$

$$\Rightarrow \quad \partial_t \int_{\mathbb{R}^3} |v| \, dx \leq c \, \| u(\cdot,t) \cdot \nabla v(\cdot,t) \|_1$$

However,

$$\|u\cdot\nabla v\|_{\mathcal{L}^1} \le c\left(\|u\|_2\,\|\nabla v\|_2 + \|\nabla u\|_2\,\|v\|_2\right)$$

$$\sup_{0\le t<\infty} \|u(\cdot,t)\|_2 \;-\; ?$$

Perturbation method (unfortunately)

Later on

$$\partial_t v - \underbrace{u \cdot \nabla v} \qquad - \Delta v - \nabla q = 0$$

This equation can be rewritten

$$\partial_t v - \operatorname{div}(v \otimes u) - \Delta v - \nabla q = 0$$

with

$$\Delta q = - \operatorname{div} \operatorname{div}(v \otimes u)$$

energy estimate

$$\|v(\cdot,t)\|^2_{L_2} + 2\int_0^t \int_{\mathbb{R}^3} |\nabla v|^2 \, dx\,dt'$$

$$= \|u_0\|^2_{L_2}$$

So, we have perturbed heat equation

$$\partial_t v - \Delta v = \operatorname{div} F$$

where

$$F = \mathscr{X}(u,v) + q\,\mathbb{I}$$

and

$$\mathscr{X}(u,v) = v \otimes u$$

and

$$v\Big|_{t=0} = u_0 \quad \text{in} \quad \mathbb{R}^3$$

Weak point (perturbation)

$$\partial_t v + b \cdot \nabla v - \Delta v = 0 \text{ with } \operatorname{div} b = 0, \quad v|_{t=0} = u_0$$

$$\Longrightarrow \quad \int_{\mathbb{R}^3} |v(x,t)|\,dx = \int_{\mathbb{R}^3} |u_0(x,t)|\,dx$$

However, if we use representation formula

$$v(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)\,u_0(y)\,dy + \int_0^t \int_{\mathbb{R}^3} \nabla \Gamma(x-y,t-\tau) \cdot b(y,\tau)\,v(y,\tau)\,d\tau$$

Positive results are coming from smallness of b !!!

$$v_i(x,t) = v_i^1(x,t) + v_i^2(x,t)$$

where

$$v_i^1(x,t) = \int_{\mathbb{R}^3} \Gamma(x-y,t)\, u_0(y,t)\, dy \iff \begin{aligned} &\partial_t v^1 - \Delta v^1 = 0 \\ &v^1\big|_{t=0} = u_0 \end{aligned}$$

and

$$v_i^2(x,t) = \int_0 \int_{\mathbb{R}^3} K_{ijl}(x-y,t-\tau)\, \mathcal{X}_{lj}(y,\tau)\, dy\, d\tau$$

$$\mathcal{X} = v \otimes u$$

$$K_{ijl}(x,t) = \frac{\partial}{\partial x_j} K_{il}(x,t), \quad \leftarrow \quad (\text{Ozeen})$$

where

$$K_{il} = \Phi_{,il} - \delta_{il} \Delta \Phi$$

and

$$\Delta \Phi = \Gamma$$

Elementary estimates

$$|K(x,t)| \le \frac{c_1}{(|x|^2 + t)^2}$$

and

$$\Downarrow$$

$$\int_{\mathbb{R}^3} |K(x,t)|\, dx\, d\tau \le \frac{c_*}{\sqrt{t}}$$

Recall

Theorem. $\forall\, m = 0, 1, 2, \ldots\ldots$

$$\| v(\cdot, t) \|_1 \leq c(m, C_d, v_0) \frac{\sqrt{t}^{\,3/2}}{\ln^m(t+e)}$$

and

$$\| v(\cdot, t) \|_2 \leq c(m, C_d, v_0) \frac{1}{\ln^m(t+e)}$$

$\forall\, t \geq \underline{1}$

Remark

Weak Leray-Hopf solution

$$\left.\begin{aligned} \partial_t v + v\cdot\nabla v - \Delta v &= -\nabla q \\ \operatorname{div} v &= 0 \end{aligned}\right\} \quad \text{in } \mathbb{R}^3 \times]0,\infty[$$

$$v\big|_{t=0} = u_0 \in C^\infty_{0,0}(\mathbb{R}^3)$$

$$\|v(\cdot,t)\|_{L_2} \leq \frac{c}{\sqrt{t}} \qquad t \geq 1$$

(Solv ?)

Proof of Theorem A

1° Estimate of $\|v(\cdot,t)\|_{L_1}$ via $\|v(\cdot,t)\|_{L_2}$

Boundedness of $\|v(\cdot,t)\|_{L_2} \Longrightarrow \|v(\cdot,t)\|_{L_1} \le c(\sqrt{t})^{3/2}$

2° Modification of Schonbek and Wiegner techique $\Longrightarrow$

$$\|v(\cdot,t)\|_{L_2} \le \frac{c}{\ln t} \qquad t \gg 1$$

3°. Iteration

Little bit about Step 1.

(i) $\frac{6}{5} < p \le \frac{3}{2}$

$$A_p(t) := \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{-\frac{5p-6}{2p}} c_d \|v^2(\cdot,s)\|_2^d \, s$$

$\|\cdot\|_p$ via $\|\cdot\|_2$

$$\|v^2(\cdot,t)\|_p \le C(p)\left(c(c_d, v_0, p)\, f^{\frac{3(1-p)}{p}}(t) + A_p(t)\right)$$

$$f(t) = \max\{1, \sqrt{t}\}$$

$$0 < \varepsilon < \frac{3}{10}$$

(ii)

$$\|v^2(\cdot,t)\|_1 \le C(\varepsilon, C_d) \int_0^t \frac{ds}{\sqrt{t-s}} \sqrt{s}^{\frac{-3+10\varepsilon}{6+5\varepsilon}} \left(c(C_d, v_0, \varepsilon) \sqrt{s}^{-\frac{3(1+5\varepsilon)}{6+5\varepsilon}} + A_{-\frac{6}{5}+\varepsilon}(s)\, ds \right)$$

$\|v\|_1$ vice $\|v\|_{\frac{6}{5}+\varepsilon}$

$$\Rightarrow \quad \|v^2(\cdot,t)\|_1 \le c \sqrt{t}^{\frac{3}{2}}$$

Energy estimate

Comments to Step 2

$$\partial_t y + 2 \quad \|\nabla v(\cdot,\tau)\|^2_{L_2} \, d\tau = 0$$

where $y = \|v(\cdot,t)\|^2_{L_2}$

Fourier Transform

$$\partial_t y = -\int_{\mathbb{R}^3} |\xi|^2 |\hat{v}(\xi,t)|^2 \, d\xi = -\int_{|\xi|\le g(t)} \dots - \int_{|\xi|>g(t)} \dots$$

$$\le -g^2(t) \int_{\mathbb{R}^3} |\hat{v}(\xi,t)|^2 dy + \int_{|\xi|\le g(t)} (g^2(t) - |\xi|^2) |\hat{v}(\xi,t)|^2 d\xi + \cancel{\int_{|\xi|>g(t)} (g^2(t) - |\xi|^2) |\hat{v}(\xi,t)|^2 d\xi}$$

From the equation

$$\partial_t \hat{v} + |\xi|^2 \hat{v} = -\hat{H},$$

where $H = -\operatorname{div}(v \otimes u) - \nabla q = -\operatorname{div}(v \otimes u + q\mathbb{I})$

$$\Downarrow$$

$$\hat{v}(\xi, t) = -\int_0^t e^{-|\xi|^2(t-\tau)} \hat{H}(\xi, \tau)\, d\tau + e^{-|\xi|^2 t} \hat{v}_0(\xi)$$

$$\Rightarrow |\hat{H}(\xi, \tau)| \leq c\, |\xi|\, \| v(\cdot, \tau) \otimes u(\cdot, \tau) \|_1$$

$$\partial_t y + g^2(t)\, y \le c\, g^5(t)\left[g(t)\sqrt{t}\int_0^t a^2(s)\, ds + c(v_0)\right] =: K(s)$$

$$a(s) = \| \, |u(\cdot, s)| \; |v(\cdot, s)| \, \|_1$$

$$y(t) \le c \int_0^t e^{-\int_s^t g^2(\tau)\, d\tau} K(s)\, ds + y(0)\, e^{-\int_0^t g^2(\tau)\, d\tau}$$

$$g^2(t) = \frac{h'(t)}{h(t)}, \quad h(t) \simeq \ln^k(t+s) \overset{m=0}{\Longrightarrow} \|v^2(\cdot, t)\|_2 \le \frac{c}{\ln(t+e)}$$

$$k > 2m+2$$

Upper estimate for c_d
(1st one)

$$\partial_t v - u\cdot\nabla v - \nabla v - \nabla q = -\operatorname{div} F, \quad \operatorname{div} v = 0$$

$$v\big|_{t=0} = 0$$

Theorem. Assume that

$$4 c_* c_\delta \le 1,$$

$$\boxed{\int_{\mathbb{R}^3} |K(x,y)|\,dy \le \frac{c_*}{\sqrt{t}}}$$

then

$$\int_{\mathbb{R}^3} u(x,T)\cdot v(x,T)\,dx \longrightarrow 0 \qquad T \longrightarrow \infty$$

$$f(t) \leq c_* \int_0^t \frac{1}{\sqrt{t-s}} \left(\frac{c_d}{\sqrt{s}} f(s) + \|F(\cdot, s)\|_1 \right) ds, \tag{5.1}$$

where $f(t) := \|v(\cdot, t)\|_1$. Since F is compactly supported, (5.1) can be reduced to the following form:

$$f(t) \leq A + c_* c_d \int_0^t \frac{1}{\sqrt{t-s}\sqrt{s}} f(s) ds.$$

Now, fix an arbitrary $T > 0$. Then, for any $t \in]0, T]$, we have

$$f(t) \leq A + 4 c_* c_d M(T),$$

where $M(T) = \sup_{0<t\leq T} f(t)$. Hence,

$$M(T) \leq A + 4 c_* c_d M(T)$$

for any $T > 0$. Finally, we see that

$$\|v(\cdot, t)\|_1 \leq c = \frac{A}{1 - 4 c_* c_d}$$

for all $t > 0$. Therefore,

$$\left| \int_{\mathbb{R}^3} u(x,t) \cdot v(x,t) dx \right| \leq \frac{c}{\sqrt{t}} \to 0$$

as $t \to \infty$. $\square$

Second estimate

$$K : \mathcal{L}_2 \to J_2$$

$$\mathcal{L}_2 = \{ \mathcal{F} = (\mathcal{F}_{ij}) \in L_2(\mathbb{R}^3) : \quad \operatorname{div} \operatorname{div} \mathcal{F} = 0 \}$$

$$J_2 = \{ v \in L_2(\mathbb{R}^3) : \quad \operatorname{div} v = 0 \}$$

$$\Rightarrow \quad A_F = K\mathcal{F} \iff \operatorname{rot} A_{\mathcal{F}} = - \operatorname{div} \mathcal{F}$$

$$\Downarrow$$

$$- \Delta A_{\mathcal{F}} = - \operatorname{rot} \operatorname{div} \mathcal{F}$$

$$\mathbb{M} : L_2(\mathbb{R}^3; \mathbb{M}^{3\times 3}) \longrightarrow L_2(\mathbb{R}^3)$$

$$q_r = \mathbb{M}\,\mathbb{F} \iff \Delta q_r = -\operatorname{div}\operatorname{div}\mathbb{F}$$

Theorem C If $c_d \leq \dfrac{\sqrt{3}}{2\|K\|(1+\sqrt{3}\|\mathbb{M}\|)}$

$$\Longrightarrow \lim_{T\to\infty} \int_{\mathbb{R}^3} \sigma(x,T)\cdot u(x,T)\,dx = 0.$$

The last theorem (Sohr's idea)

Translation from semigroup language

PROOF OF THEOREM 1.4 Assume that F is skew symmetric and therefore satisfies condition (1.3).

Equation (1.1) can be written as follows:

$$\partial_t v - \Delta v = \operatorname{div} F_0, \tag{5.2}$$

where

$$F_0 = v \otimes u + \nabla q \mathbb{I} - F.$$

We know from previous results that

$$F_0 \in L_{2,\infty}(Q_+), \qquad \operatorname{div} F_0 \in L_2(Q_+). \tag{5.3}$$

Since $\operatorname{div}\operatorname{div} F_0 = 0$, we can apply the elliptic theory and conclude that there exists a divergence free field $A(\cdot,t)$ such that

$$\operatorname{rot} A(\cdot,t) = \operatorname{div} F_0(\cdot,t) \tag{5.4}$$

in $\mathbb{R}^3$ and the following estimate holds

$$\|A(\cdot,t)\|_2 \leq \|K\| \|F_0(\cdot,t)\|_2 \tag{5.5}$$

for all $t \in]0,\infty[$.

Taking into account the definition of the operator M, one can go further and derive from (5.5)

$$\|A(\cdot,t)\|_2 \leq \|K\|(\|v(\cdot,t)\otimes u(\cdot,t)\|_2 + \|q_{v\otimes u}(\cdot,t)\mathbb{I}\|_2 + \|F(\cdot,t)\|_2) \leq$$

$$\leq \|K\|(1+\sqrt{3}\|M\|)\|v(\cdot,t)\otimes u(\cdot,t)\|_2 + h(t),$$

where $h(t) = \|K\|\|F(\cdot,t)\|_2$ and thus

$$\|A(\cdot,t)\|_2 \leq \|K\|(1+\sqrt{3}\|M\|)\frac{c_d}{\sqrt{t}}\|v(\cdot,t)\|_2 + h(t) \qquad (5.6)$$

With the above A, let us consider the Cauchy problem

$$\partial_t B - \Delta B = A \tag{5.7}$$

$$B(\cdot, 0) = 0. \tag{5.8}$$

Problem (5.7), (5.8) has a unique solution defined for all positive t and $B \in W_2^{2,1}(Q_T)$ for all $T > 0$. Since $A(\cdot, t)$ is divergence free, so is $B(\cdot, t)$. Now, let $w = \operatorname{rot} B$. Then we can see that w is a solution to equation (5.2) and sinse it vanishes at $t = 0$, we can state that $w = v$.

$\Rightarrow$ $v = \operatorname{rot} B$

Now, let us analyse the Cauchy problem for B. It is easy to see that B satisfies the energy identity

$$\frac{1}{2}\partial_t \|B(\cdot,t)\|_2^2 + \|\nabla B(\cdot,t)\|_2^2 = \int\limits_{\mathbb{R}^3} A(x,t)\cdot B(x,t)dx. \tag{5.9}$$

Taking into account the simple identity

$$\|v(\cdot,t)\|_2 = \|\nabla B(\cdot,t)\|_2,$$

one can derive from (5.6) the following estimate

$$\frac{1}{2}\partial_t \|B(\cdot,t)\|_2^2 + \|v(\cdot,t)\|_2^2 \leq \|K\|(1+\sqrt{3}\|M\|)\frac{c_d}{\sqrt{t}}\|v(\cdot,t)\|_2 \|B(\cdot,t)\|_2 +$$

$$+h(t)\|B(\cdot,t)\|_2.$$

Applying the Young inequality, we find

$$\frac{1}{2}\partial_t \|B(\cdot,t)\|_2^2 \leq \|K\|^2(1+\sqrt{3}\|M\|)^2\frac{c_d^2}{4t}\|B(\cdot,t)\|_2^2 + \frac{1}{2}h(t))(\|B(\cdot,t)\|_2^2+1)$$

Let us introduce the important constant

$$l = \|K\|^2(1+\sqrt{3}\|M\|)^2\frac{c_d^2}{2}.$$

Then the previous inequality leads to

$$\|B(\cdot,t)\|_2^2 \le t^l \int\limits_0^t \frac{h(\tau)}{\tau^l} \exp\Big(-\int_\tau^t h(s)ds\Big)d\tau.$$

$h(s) = \|K\| \, \|F(\cdot,t)\|_2$

Taking into account that F is compactly supported in Q_+, we have

$$\|B(\cdot,t)\|_2^2 \leq c_F t^l.$$

From here, it is easy to derive the following:

(Energy Inequality (5.9) and $\|v(\cdot,t)\|_2 =$ $= \|\nabla B(\cdot,t)\|_2$)

$$\int\limits_o^t \|v(\cdot,s)\|_2^2 ds \leq c_F t^l. \tag{5.10}$$

We denote all the constant depending of F and its support by c_F.

Having estimate (5.10) in mind, let us go back to equation (5.2) multiplying it by tv and integrating result over $\mathbb{R}^3\times]0,t[$, as a result, we find the following differential inequality

$$\frac{1}{2}t\|v(\cdot,t)\|_2^2 + \int\limits_0^t \|\nabla v(\cdot,s)\|_2^2 ds = \frac{1}{2}\|v(\cdot,t)\|_2^2 + \int\limits_0^t \int\limits_{\mathbb{R}^3} sF(x,s)\cdot v(x,s)ds \leq$$

$$\leq c_F(\int\limits_0^t \|v(\cdot,s)\|_2^2 ds + 1).$$

The latter, together with boundedness of $\|v(\cdot,t)\|_2$, implies the bound

$$\|v(\cdot,t)\|_2^2 \leq c_F(t+1)^{l-1},$$

which, in turn, allows to improve the decay of $\|v(\cdot,t)\|_1$. To this end, we are going back to (2.4) and (2.5). Indeed, by the assumption of the theorem $l < 3/4$,

$$A_p(t) \leq c \int_0^t \frac{1}{\sqrt{t-s}} s^{-\frac{5p-6}{4p}} (s+1)^{l-1} ds \leq c \int_0^t \frac{1}{\sqrt{t-s}} s^{-\frac{5p-6}{4p}+l-1} ds \leq$$

$$\leq ct^{\frac{6-3p}{4p}+l-1}.$$

Letting $p = 6/5 + \varepsilon$, for sufficiently small positive ε, we find

$$\|v(\cdot,t)\|_1 \leq c(\sqrt{t})^{\frac{3}{2}+2(l-1)}.$$

This shows

$$\left| \int_{\mathbb{R}^3} v(\cdot,t) \cdot u(\cdot,t) dx \right| \leq c(\sqrt{t})^{\frac{1}{2}+2(l-1)} \to 0$$

as $t \to \infty$ provided $l < \frac{3}{4}$. □