Mean Field Limits for Singular Flows

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SITE online conference: Long Time Behavior and Singularity Formation in PDEs

The discrete coupled ODE system

Consider

$$H_N(x_1, \ldots, x_N) = \frac{1}{2} \sum_{1 \le i \ne j \le N} \mathrm{g}(x_i - x_j), \qquad x_i \in \mathbb{R}^d$$

Model case:

$$\begin{cases} \mathrm{g}(x) = \dfrac{1}{|x|^s} & 0 \le s < d \quad \textbf{Riesz case} \\ \mathrm{g}(x) = -\log|x| & s = 0 \quad \textbf{log case} \end{cases}$$

Evolution equation

$$\dot{x}_i = -\frac{1}{N} \left(\nabla_i H_N(x_1, \ldots, x_N) \right) \qquad \textbf{gradient flow}$$

$$\dot{x}_i = -\frac{1}{N} \mathbb{J} \nabla_i H_N(x_1, \ldots, x_N) \qquad \textbf{conservative flow} \quad (\mathbb{J}^T = -\mathbb{J})$$

$$\ddot{x}_i = -\frac{1}{N} \nabla_i H_N(x_1, \ldots, x_N) \qquad \textbf{Newton's law}$$

possibly with added noise $\sqrt{\theta} dW_i^t$, N independent Brownian motions, θ=temperature

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Questions

For a general system

$$\dot{x}_i = \frac{1}{N}\sum_{j\neq i} K(x_i - x_j) + \sqrt{\theta} dW_i^t$$

- What is the limit of the **empirical measure**? Is there μ^t such that for each t

$$\frac{1}{N}\sum_{i=1}^{N} \delta_{x_i^t} \rightharpoonup \mu^t \tag{1}$$

- if $f_N^0(x_1, \ldots, x_N)$ is the probability density of position of the system at time 0, what is the limit behavior of f_N^t?
- **propagation of chaos** (Boltzmann, Kac, Dobrushin): if $f_N^0(x_1, \ldots, x_N) \simeq \mu^0(x_1) \ldots \mu^0(x_N)$ is it true that

$$f_N^t(x_1, \ldots, x_N) \simeq \mu^t(x_1) \ldots \mu^t(x_N)?$$

in the sense of convergence of the k-point marginal $f_{N,k}$.
- In fact (1) is equivalent to propagation of chaos. Check : $\varphi \in C_c^\infty$

$$\int (f_{N,1} - \mu^t)\varphi = \int f_N(x_1, \ldots, x_N) \left(\frac{1}{N}\sum_{i=1}^{N} \varphi(x_i) - \int \mu\varphi \right) dx_1 \ldots dx_N$$

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Formal limit

Use

$$\partial_t \delta_{x(t)} + \mathrm{div}\,\left(\dot{x}\delta_{x(t)}\right) = 0$$

or Liouville equation + BBGKY hierarchy

$$\partial_t f_N + \sum_{i=1}^{N} \nabla_{x_i}\left(f_N \frac{1}{N}\sum_{i=1}^{N} K(x_i - x_j)\right) = 0$$

We *formally* expect $\mu_N^t := \frac{1}{N}\sum_{i=1}^N \delta_{x_i^t} \rightharpoonup \mu^t$ where μ^t solves the mean-field equation

$$\boxed{\partial_t \mu = \mathrm{div}\,((K*\mu)\mu) + \frac{1}{2}\theta\Delta\mu}$$

or in the second order case the Vlasov / McKean-Vlasov equation

$$\boxed{\partial_t \rho + v\cdot\nabla_x \rho + (K*\mu)\cdot\nabla_v \rho + \frac{1}{2}\theta\Delta\rho = 0 \qquad \mu = \int_{\mathbb{R}^d} \rho(x,v)dv}$$

How to prove convergence?

- Classical method ([Sznitman]...): compare true **trajectories** of points to trajectories following characteristics for the limit equation, show they remain close. OK for K Lipschitz.
  If K not regular, try to control the minimal distance between points in order to control $K(x_i - x_j)$ [Hauray-Jabin]
- Find a good metric, typically Wasserstein W_1 such that

$$\partial_t W_1(\mu_1(t), \mu_2(t)) \leq C W_1(\mu_1(t), \mu_2(t))$$

  for two solutions of the mean-field evolution. Apply to μ_N^t and μ^t. [Braun-Hepp, Dobrushin, Neunzert-Wick]
- Use a **relative entropy method**: show a Gronwall relation for

$$0 \leq \mathcal{H}_N(f_N | \rho^{\otimes N}) := \frac{1}{N} \int f_N \log \frac{f_N}{\rho^{\otimes N}} dx_1 \dots dx_N.$$

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Specialization to the Coulomb or Riesz interaction

Limiting equations

$$\boxed{\partial_t \mu = \mathrm{div}\,(\mathbb{M}\nabla(\mathrm{g} * \mu)\mu)} \qquad (MF)$$

with $\langle \mathbb{M}\xi, \xi\rangle \geq 0$

or Vlasov-Poisson type eq

$$\boxed{\partial_t \rho + v \cdot \nabla_x \rho + (\nabla \mathrm{g} * \mu) \cdot \nabla_v \rho = 0 \qquad \mu = \int_{\mathbb{R}^d} \rho(x, v)\, dv}$$

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Previous results

- $d = 2$ log, **point vortex system** $\rightarrow$ 2D incompressible Euler in vorticity form [Goodman-Hou-Lowengrub '90, Schochet '96] with noise $\rightarrow$ 2D Navier-Stokes [Osada '87, Fournier-Hauray-Mischler '14]
- [Hauray' 09, Carrillo-Choi-Hauray '14] ($s < d - 2$) stability in Wasserstein W_∞
- [Carrillo-Ferreira-Precioso '12, Berman-Onnheim '15] ($d = 1$) Wasserstein gradient flow, use **convexity** of the interaction in 1D
- [Duerinckx '15] ($d \leq 2$, $s < 1$) **modulated energy method**

for convergence to Vlasov-Poisson/Riesz

- [Hauray-Jabin '15, Jabin-Wang '17] $s < d - 2$, **relative entropy method**. Coulomb interaction (or more singular) remains open.
- [Boers-Pickl '16, Lazarovici '16, Lazarovici-Pickl '17] with N-dependent cut-off of the interaction kernel

The modulated energy method

<[S '16, Ginzburg-Landau vortex dynamics]
Idea: use **Riesz-based metric**:

$$\|\mu - \nu\|^2 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{g}(x-y)\,d(\mu-\nu)(x)\,d(\mu-\nu)(y).$$

Observe weak-strong uniqueness property of the solutions to (MF) for $\|\cdot\|$:

$$\|\mu_1^t - \mu_2^t\|^2 \le e^{Ct}\|\mu_1^0 - \mu_2^0\|^2 \qquad C = C(\|\nabla^2(\mathrm{g} * \mu_2)\|_{L^\infty})$$

In the discrete case, let X_N denote $(x_1, \dots, x_N)$ and take for modulated energy,

$$F_N(X_N, \mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} \mathrm{g}(x-y)\,d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)(x)\,d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu\Big)(y)$$

where $\triangle$ denotes the diagonal in $\mathbb{R}^d \times \mathbb{R}^d$, and μ^t solves (MF).
Analogy with "relative entropy" and "modulated entropy" methods [Dafermos '79] [DiPerna '79] [Yau '91] [Brenier '00]....

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Theorem (S '18 ($d-2 \le s < d$), Nguyen-Rosenzweig-S '21)

Assume (MF) admits a solution $\mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d))$ *with*

$$\begin{cases} \nabla^2 g * \mu^t \in L^\infty([0,T], L^\infty(\mathbb{R}^d)) \text{ in the Coulomb case} \\ \text{additional regularity conditions on } \mu^t \text{ in the other cases} \end{cases}$$

There exist constants C_1, C_2 *depending on the norms of* μ^t *and* $\beta > 0$ *depending on* d, s, σ*, s.t.* $\forall t \in [0,T]$

$$|F_N(X_N^t, \mu^t)| \le \left(|F_N(X_N^0, \mu^0)| + C_1 N^{-\beta}\right) e^{C_2 t}.$$

In particular, if $\mu_N^0 \rightharpoonup \mu^0$ *and is such that*

$$(*) \qquad \lim_{N\to\infty} F_N(X_N^0, \mu^0) = 0,$$

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- well-prepared assumption $(*)$ implied by

$$\lim \frac{1}{N^2} H_N(X_N^0) = \iint \mathrm{g}(x-y)\,d\mu^0(x)\,d\mu^0(y).$$

- regularity assumption on μ^t allow for "patches" i.e. measures which are only L^∞, as in vortex patch solutions to Euler's eq [Chemin, Serfati]
- Self-similar solutions of patch type are attractors in the Coulomb case ([S-Vazquez]). For general s, self-similar *Barenblatt solutions* of the form

$$t^{-\frac{d}{2+s}}\left(a - bx^2 t^{-\frac{2}{2+s}}\right)_+^{\frac{s-d+2}{2}}$$

- [Rosenzweig '20] improved the result in the Coulomb case: $\mu^0 \in L^\infty$ suffices (for short times if $d \geq 3$).
- limiting equation called fractional porous medium equation
- required propagation of regularity ok ([Lin-Zhang, Masmoudi-Zhang, Xiao-Zhou, Caffarelli-Vazquez, Caffarelli-Soria-Vazquez, Ambrosio-S, S-Vazquez, Choi-Jeong, Constantin et al.]
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$$t^{-\frac{d}{2+s}}(a - bx^2 t^{-\frac{2}{2+s}})_+^{\frac{s-d+2}{2}}$$
- [Rosenzweig '20] improved the result in the Coulomb case: $\mu^0 \in L^\infty$ suffices (for short times if $d \geq 3$).
- limiting equation called fractional porous medium equation
- required propagation of regularity ok ([Lin-Zhang, Masmoudi-Zhang, Xiao-Zhou, Caffarelli-Vazquez, Caffarelli-Soria-Vazquez, Ambrosio-S, S-Vazquez, Choi-Jeong, Constantin et al.]
- can include the case of multiplicative noise

The case with noise

[Bresch-Jabin-Wang '19] incorporate the modulated energy into their relative entropy method: use a **modulated free energy**

$$\mathcal{F}_N^\theta(f_N, \rho) = \theta \mathcal{H}_N(f_N | \rho^{\otimes N}) + \int f_N(X_N) F_N(X_N, \rho) dx_1 \dots dx_N$$

Theorem

Assume f_N^0 *is an initial density and* f_N^t *solves*

$$\partial_t f_N^t = \sum_{i=1}^N \mathrm{div}_i \left(\frac{1}{N} \sum_{i \neq j} \nabla_i \mathcal{H}_N(x_i - x_j) f_N^t(X_N) \right) + \frac{1}{2}\theta \sum_{i=1}^N \Delta_i f_N^t.$$

Then for μ^t *a solution to*

$$\boxed{\partial_t \mu = \mathrm{div}\ (\nabla(\mathrm{g} * \mu)\mu) + \frac{1}{2}\theta\Delta\mu}$$

with μ^0 *regular enough (Lipschitz), we have*

$$\mathcal{F}_N^\theta(f_N^t, \mu^t) \leq \left(\mathcal{F}_N^\theta(f_N^0, \mu^0) + C_1 N^{-\beta}\right) e^{C_2 t}$$

and for all t, $f_N^t \rightharpoonup (\mu^t)^{\otimes N}$.

They extend the method to the case with moderate attraction (e.g. Patlak-Keller-Segel).
Global-in-time convergence [Guillin, Le Bris, Monmarché '21]
Dissipative case only

Global-in-time convergence via the modulated energy method

Theorem (Rosenzweig-S '21)

Assume $d \geq 3$, $0 < s < d-2$, *first order evolution with additive noise. Assume* μ^t *is a global in time bounded solution to the limiting equation*

$$\partial_t \mu = -\mathrm{div}\,(\mu \mathbb{M} \nabla \mathrm{g} * \mu) + \frac{1}{2}\theta \Delta \mu.$$

Then

$$\mathbb{E}\left(|F_N(X_N^t, \mu^t)|\right) \leq C|F_N(X_N^0, \mu^0)| + N^{-\beta}.$$

In the logarithmic case, we get instead a t^σ control.

Convergence to Vlasov-Poisson in the monokinetic case

Here $d-2 \le s < d$. Let $Z_N = ((x_1, v_1), \ldots, (x_N, v_N))$ where $v_i = \dot{x}_i$.
Monokinetic version of (VP) (**pressureless Euler-Poisson**):

$$\rho^t(x,v) = \mu^t(x)\delta_{v=u^t(x)}$$

$$\boxed{\partial_t \mu + \mathrm{div}\,(\mu u) = 0 \qquad \partial_t u + u \cdot \nabla u = -\nabla \mathrm{g} * \mu} \qquad (PEP)$$

Use modulated energy

$$E_N(Z_N, (\mu, u)) := \frac{1}{N}\sum_{i=1}^{N} |u(x_i) - v_i|^2 + F_N(X_N, \mu)$$

Theorem (Duerinckx-S '18)

Assume Z_N^t solves Newton's law with initial data Z_N^0. Assume (μ, u) is a sufficiently regular solution to (PEP) on $[0, T]$. Then

$$E_N(Z_N^t, (\mu^t, u^t)) \le \left(E_N(Z_N^0, (\mu^0, u^0)) + N^{-\beta}\right) e^{C_2 t}$$

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Coerciveness of the modulated energy

$$F_N(X_N,\mu) := \iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} \mathrm{g}(x-y)\,d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)(x)\,d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu\Big)(y)$$

- Formally $\dot H^{\frac{s-d}{2}}$ norm, but positivity not clear.
- Use suitable **truncations** obtained by replacing δ_{x_i} by $\delta_{x_i}^{(\eta)}$
- almost **monotonicity** with respect to η provided by the superharmonicity of g near 0 (if not, use dimension extension)
- $$F_N(X_N,\mu) \geq \left\| \frac{1}{N}\sum_{i=1}^N \delta_{x_i}^{(\eta)} - \mu \right\|^2_{\dot H^{\frac{s-d}{2}}} - Error_{N,\eta}$$

The commutator estimate / main functional inequality

When computing $\frac{d}{dt}F_N(X_N^t, \mu^t)$ we find we need to control

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} (\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y)d(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu)^{\otimes 2}(x,y)$$

with $\psi = \mathbb{M}\nabla \mathrm{g} * \mu^t$.

The commutator estimate / main functional inequality

Proposition (S, NRS)

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle} (\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y)d(\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu)^{\otimes 2}(x,y)$$
$$\le C\|D\psi\|_{L^\infty}(F_N(X_N,\mu)+N^{-\beta}),$$

Why commutator ? Let $f=\frac{1}{N}\sum_{i=1}^N \delta_{x_i}-\mu$

$$\int \psi\cdot\nabla(\mathrm{g}*f)-\mathrm{g}*(\nabla\cdot(\psi f))=\langle f,\left[\psi,\frac{\nabla}{(-\Delta)^{\frac{d-s}{2}}}\right]f\rangle_{L^2}$$

Topic of *singular integrals / Christ-Journé operators*, [Hadzic, Seeger, Smart, Street] (all $\mathrm{div}\,\psi=0$).

Estimate used to treat the *quantum* Coulomb mean-field limit [Golse-Paul, Rosenzweig], *quasi-neutral* limits [Iacobelli-Han Kwan, Rosenzweig]

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Proof in the Coulomb case

Set $h^f = \mathrm{g} * f$. In the Coulomb case

$$-\Delta h^f = c_d f$$

We have by IBP

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{g}(x-y) df(x) df(y) = \int_{\mathbb{R}^d} h^f df = -\frac{1}{c_d} \int_{\mathbb{R}^d} h^f \Delta h^f = \frac{1}{c_d} \int_{\mathbb{R}^d} |\nabla h^f|^2.$$

Stress-energy tensor

$$[\nabla h^f]_{ij} = 2\partial_i h^f \partial_j h^f - |\nabla h^f|^2 \delta_{ij}.$$

For regular f,

$$\mathrm{div}\,[\nabla h^f] = 2\Delta h^f \nabla h^f = -\frac{2}{c_d} f \nabla h^f.$$

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$$\begin{aligned}\iint_{\mathbb{R}^d\times\mathbb{R}^d} &(\psi(x)-\psi(y))\cdot\nabla g(x-y)df(x)df(y)\\ &= 2\int_{\mathbb{R}^d}\psi(x)\cdot\nabla h^f(x)df(x) = -c_d\int_{\mathbb{R}^d}\psi\cdot\mathrm{div}\ [\nabla h^f]\\ &= c_d\int_{\mathbb{R}^d} D\psi\cdot[\nabla h^f] \le c_d\|D\psi\|_{L^\infty}\iint g(x-y)df(x)df(y)\end{aligned}$$

Then needs to be **renormalized**.

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$$\begin{aligned}
&\iint_{\mathbb{R}^d \times \mathbb{R}^d} (\psi(x) - \psi(y)) \cdot \nabla \mathrm{g}(x-y) df(x) df(y) \\
&\quad = 2 \int_{\mathbb{R}^d} \psi(x) \cdot \nabla h^f(x) df(x) = -c_d \int_{\mathbb{R}^d} \psi \cdot \mathrm{div}\ [\nabla h^f] \\
&\quad = c_d \int_{\mathbb{R}^d} D\psi \cdot [\nabla h^f] \le c_d \|D\psi\|_{L^\infty} \iint \mathrm{g}(x-y) df(x) df(y)
\end{aligned}$$

Then needs to be **renormalized**.

Proof in the non-Coulomb case

[Nguyen-Rosenzweig-S]: dispense with stress-tensor structure and from paraproduct commutator estimates. Purely space-based.
Rewrite

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d\setminus\triangle}(\psi(x)-\psi(y))\cdot\nabla \mathrm{g}(x-y)df(x)df(y)$$

and $f=\mathrm{div}\ \nabla\Delta^{-1}f=\mathrm{div}\ f_1$

$$\iint k(x,y)\,\mathrm{div}\ f_1(x)\,\mathrm{div}\ f_1(x)$$

Case $s>d-2$. After integrations by part + symmetrizations (and paying attention to the diagonal...)

$$\leq C\|\nabla\psi\|_{L^\infty}\left(\iint\frac{|f_1(x)-f_1(y)|^2}{|x-y|^{s+2}}dx\,dy\right)=C\|\nabla\psi\|_{L^\infty}\|f\|^2_{\dot H^{\frac{s-d}{2}}}$$

Case $s=d-2$. Finish with Christ-Journé theorem on Calderón commutators
Case $s<d-2$. Iterated integrations by parts + Riesz transform estimates.
Then needs to be **renormalized** via truncation procedure.

Global in time result

Step 1. Prove that μ^t **decays** fast enough in long time thanks to the dissipation.

Proposition

Let $1 \leq p \leq q \leq \infty$.

$$\|\mu^t\|_{L^q} \leq C_{p,q} \left(\frac{2\pi\theta t}{1/p - 1/q} \right)^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{q})} \|\mu^0\|_{L^p}$$

Inspired by [Carlen-Loss] (case of divergence-free vector field).

Step 2. **Optimized** version of the functional inequality: use that $\psi^t = \mathbb{M}\nabla \mathrm{g} * \mu^t$ and optimize over η

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d \setminus \triangle} (\psi^t(x) - \psi^t(y)) \cdot \nabla \mathrm{g}(x-y) d\Big(\frac{1}{N}\sum_{i=1}^N \delta_{x_i} - \mu^t\Big)^{\otimes 2}(x,y)$$

$$\leq C \|\mu^t\|_{L^\infty}^{\frac{s+d}{2}} \left(F_N(X_N^t, \mu^t) + N^{-1+\frac{s}{d}}\right),$$

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$$\|\mu^t\|_{L^q} \le C_{p,q} \left(\frac{2\pi\theta t}{1/p - 1/q} \right)^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{q})} \|\mu^0\|_{L^p}$$

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$$\le C \|\mu^t\|_{L^\infty}^{\frac{s+d}{2}} (F_N(X_N^t, \mu^t) + N^{-1+\frac{s}{d}}),$$

THANK YOU FOR YOUR ATTENTION!