On the fully nonlinear 2D Peskin (immersed boundary) problem

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Yoichiro Mori (University of Pennsylvania).

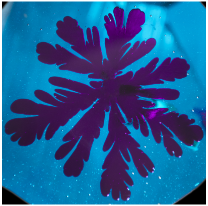
arXiv:2009.03360 and arXiv:2112.00692

NYU ABU DHABI,
Long Time Behavior and Singularity Formation in PDEs: Part V,
Wednesday, June 1, 2022.

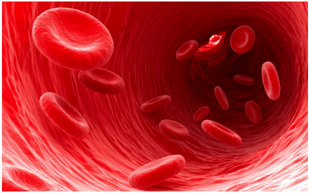
Dynamical mathematical models with free boundaries

(a) Kelvin-Helmholtz

(b) Water waves

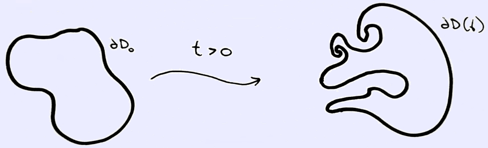


(c) Hele-Shaw cell



(d) Red blood cells

Main Interests and Questions



- Local-in-time existence and uniqueness
- Regularity of the interface (smoothing effects)
- Global-in-time existence and uniqueness
- Large time behavior
- Non-small initial data (nonlinear effects)
- Critical regularity
- Viscosity (nonlocal effects)

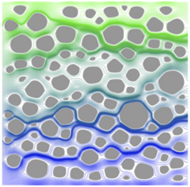
example: Incompressible Porous Medium

Equations (Darcy law)

$$
\begin{cases}
\partial_t \rho(x,t) + u(x,t)\cdot\nabla\rho(x,t) = 0, \quad x\in\mathbb{R}^d, \quad t\geq 0 \\[2ex]
\nabla\cdot u(x,t) = 0, \\[2ex]
\frac{\mu}{\kappa}u(x,t) = -\nabla p(x,t) - \rho(x,t) g e_d, \quad (d=2,3),
\end{cases}
$$

where

- u, p, ρ, μ are the velocity, pressure, density, and viscosity of the fluid
- κ, g are the permeability of the medium and the gravitational constant

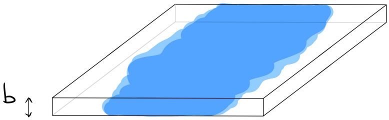


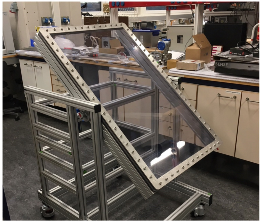
$-g\vec{e}_d$

example: Incompressible Porous Medium

Equations (Hele-Shaw cell)

$$
\begin{cases}
\partial_t \rho(x,t) + u(x,t) \cdot \nabla \rho(x,t) = 0, \quad x \in \mathbb{R}^2, \quad t \geq 0 \\[2em]
\nabla \cdot u(x,t) = 0, \\[2em]
\frac{12}{b^2} \mu u(x,t) = -\nabla p(x,t) + g\rho(x,t) \begin{pmatrix} 0 \\ -1 \end{pmatrix},
\end{cases}
$$





- Homogeneous medium (constant permeability, κ) $\equiv$ fixed width, b

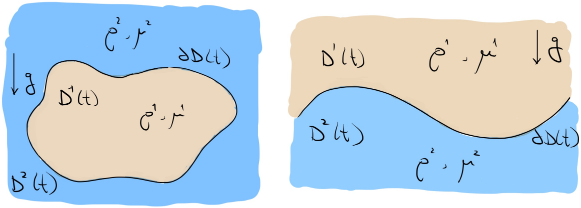
example: The Muskat Problem

Two incompressible and immiscible fluids:

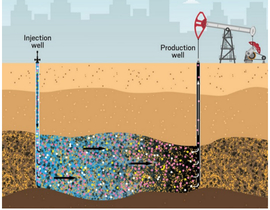
$$\mu(x,t) = \begin{cases} \mu^1, & x \in D^1(t), \\ \mu^2, & x \in D^2(t), \end{cases} \qquad \rho(x,t) = \begin{cases} \rho^1, & x \in D^1(t), \\ \rho^2, & x \in D^2(t), \end{cases}$$

- Two cases:

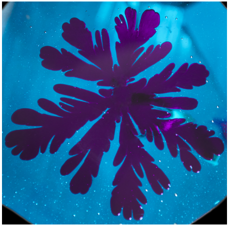
- *Circle case*: $\partial D(t)$ is a closed curve (or surface).
- *Flat case*: $\partial D(t)$ is a curve (or surface) vanishing at infinity.



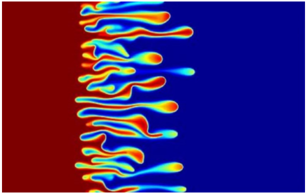
example: Muskat and Hele-Shaw problems



Oil pumping

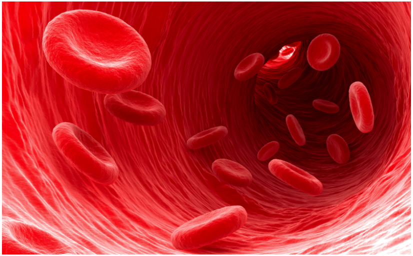


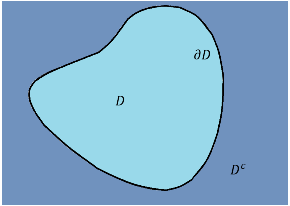
Hele-Shaw cell

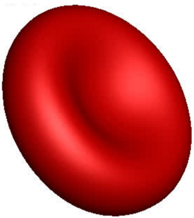


Viscous fingering

Peskin problem: Stokes immersed boundary problem







Peskin problem: Stokes immersed boundary problem

[BOOK] Flow patterns around heart valves: a digital computer method for solving the equations of motion
CS Peskin - 1972 - search.proquest.com
The flow pattern of blood in the heart is intimately connected with the performance of the heart valves. The motions of blood and valve leaflet interact strongly. Points of the valve …
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Flow patterns around heart valves: a numerical method
CS Peskin - Journal of computational physics, 1972 - Elsevier
The subject of this paper is the flow of a viscous incompressible fluid in a region containing immersed boundaries which move with the fluid and exert forces on the fluid. An example of such a boundary is the flexible leaflet of a human heart valve. It is the main achievement of the present paper that a method for solving the Navier-Stokes equations on a rectangular domain can now be applied to a problem involving this type of immersed boundary. This is accomplished by replacing the boundary by a field of force which is defined on the mesh …
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Numerical analysis of blood flow in the heart
CS Peskin - Journal of computational physics, 1977 - Elsevier
The flow pattern of blood in the heart is intimately connected with the performance of the heart valves. This paper extends previous work on the solution of the Navier-Stokes …
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The immersed boundary method
CS Peskin - Acta numerica, 2002 - cambridge.org
This paper is concerned with the mathematical structure of the immersed boundary (IB) method, which is intended for the computer simulation of fluid–structure interaction …
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MR2009378 (2004h:74029) Reviewed
Peskin, Charles S. (1-NY-X)
The immersed boundary method. (English summary)
Acta Numer. 11 (2002), 479–517.
74F10 (65M06 76D05 76M20 76M25 76Z05)
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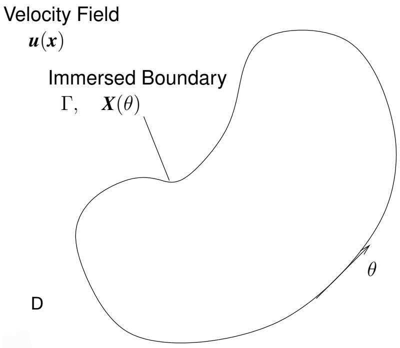
PENN TEXT

Peskin problem: Stokes immersed boundary problem



- Stokes flow in a 2D open domain denoted D and D^c where $D^c = \mathbb{R}^2 \backslash (D \cup \partial D)$.
- One dimensional elastic string $\Gamma = \partial D$ is parametrized by θ.
- Position on Γ is $\boldsymbol{X}(t, \theta)$ and the force is $\boldsymbol{F}(\theta)$.

Problem: Find the velocity field $\boldsymbol{u}(x)$ and the elastic string position $\boldsymbol{X}(t, \theta)$.

Peskin problem: Immersed elastic string (first formulation)

- Stokes fluid with viscosity μ satisfied in $D \cup D^c$:

$$\mu \Delta \boldsymbol{u} - \nabla p = 0, \quad \nabla \cdot \boldsymbol{u} = 0.$$

- Elasticity: jump conditions

$$[\![\Sigma \boldsymbol{n}]\!] = \boldsymbol{F}_{\mathrm{el}}(\boldsymbol{X})|\partial_\alpha \boldsymbol{X}|^{-1}, \quad [\![\boldsymbol{u}]\!] = 0, \text{ on } \partial D,$$

  where $\boldsymbol{n}$ is the unit normal vector on $\Gamma = \partial D$

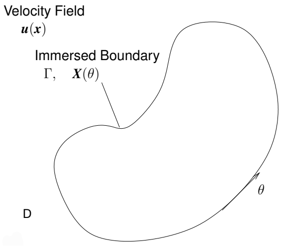
$$\Sigma = \mu \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}} \right) - p\mathbb{I} \ \text{ in } \ D \cup D^c.$$

  Above Σ is called the stress tensor.
- $[\![\cdot]\!]$ denotes the jump across $\Gamma = \partial D$
- The jump conditions represent the no-slip and force balance boundary conditions on the interface $\Gamma = \partial D$.
- General tension force law ($\mathcal{T}$ an arbitrary nonlinear function)

$$\boldsymbol{F}_{\mathrm{el}}(\boldsymbol{X}) = \partial_\theta \left(\mathcal{T}(|\partial_\theta \boldsymbol{X}|) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right)$$

- Simple tension force law: $F_{el}(\boldsymbol{X}) = k_0 \partial_\theta^2 \boldsymbol{X}$, $k_0 > 0$.

Second formulation: immersed boundary formulation



Immersed boundary method:

$$\Delta \boldsymbol{u} + \nabla p = \int_{\mathbb{T}} \partial_\theta \left(\mathcal{T}(|\partial_\theta \boldsymbol{X}|) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right) \delta(x - \boldsymbol{X}(\theta)) d\theta, \qquad \nabla \cdot \boldsymbol{u} = 0.$$

- The force balance condition at Γ is treated as an external singular force field supported on $\Gamma(t) = \{\boldsymbol{X}(\theta) : \theta \in \mathbb{T}\}$.
- The no-slip condition is satisfied automatically.

This formulation has proven to be very useful computationally.

Third formluation: Contour dynamic formulation

The string moves with the flow:

$$\partial_t \boldsymbol{X}(\theta, t) = \boldsymbol{u}(\boldsymbol{X}(\theta, t), t)$$

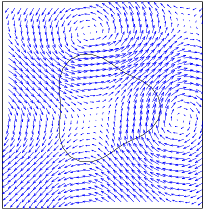
Then invert the Stokes operator in the whole space to obtain

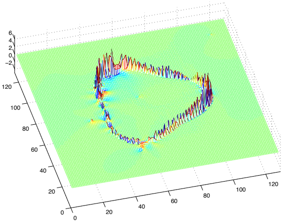
$$\partial_t \boldsymbol{X}(\theta, t) = \frac{1}{4\pi} \int_{\mathbb{T}} G(\boldsymbol{X}(\theta, t) - \boldsymbol{X}(\alpha, t)) \partial_\theta \left(\mathcal{T}(|\partial_\theta \boldsymbol{X}|) \frac{\partial_\theta \boldsymbol{X}}{|\partial_\theta \boldsymbol{X}|} \right) (\alpha, t) d\alpha,$$

where the kernel $G(x)$ is called the Stokeslet:

$$G(x) = -\log |x| \mathbb{I} + \frac{x \otimes x}{|x|^2}, \quad x \in \mathbb{R}^2.$$

Here also $\mathbb{I}$ is the identity matrix on $\mathbb{R}^2$.





- Low order numerical convergence rate for the immersed boundary problem: Mori (CPAM, 2008) Computations above done by Mori et al (2008).
- Applied mathematicians including Yoichiro Mori (University of Pennsylvania) and Johnny Guzman (Brown) expressed the importance of future analytical work.
- Low order numerical convergence rate means that existence and uniqueness results with low regularity can be very useful both for theoretical justifications but also perhaps for motivations to develop new numerical schemes.
- Lowest regularity that can be expected for well-posedness is possibly critical regularity.

Critical local well-posedness for the fully nonlinear Peskin problem

Brief references Muskat at critical regularity

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Boundary integral formulation for the Peskin problem

Boundary integral formulation for the general force law:

$$\partial_t \boldsymbol{X}(\theta) = \int_{\mathbb{T}} G(\delta_\alpha \boldsymbol{X}(\theta)) \partial_\alpha \left(\mathcal{T}(|\boldsymbol{X}'(\theta+\alpha)|) \frac{\boldsymbol{X}'(\theta+\alpha)}{|\boldsymbol{X}'(\theta+\alpha)|} \right) d\alpha.$$

Here $\boldsymbol{X}'(\theta) = \partial_\theta \boldsymbol{X}(\theta)$. Standard partial difference operator

$$\delta_\alpha \boldsymbol{X}(\theta) \stackrel{\text{def}}{=} \boldsymbol{X}(\theta+\alpha) - \boldsymbol{X}(\theta).$$

For $z \in \mathbb{R}^2$, $G(z)$ is the Stokeslet given by

$$G(z) = G_1(z) + G_2(z), \quad G_1(z) \stackrel{\text{def}}{=} -\frac{1}{4\pi} \log(|z|)\mathbb{I}, \quad G_2(z) \stackrel{\text{def}}{=} \frac{1}{4\pi} \frac{z \otimes z}{|z|^2}.$$

In the simple tension case $\mathcal{T}(r) = k_0 r$ the equation takes the form

$$\partial_t \boldsymbol{X}(\theta) = k_0 \int G(\delta_\alpha \boldsymbol{X}(\theta)) \partial_\alpha^2 \boldsymbol{X}(\theta+\alpha) d\alpha.$$

This looks very bad to study critical regularity due to $\partial_\alpha^2 \boldsymbol{X} = \boldsymbol{X}''$.

Boundary integral re-formulation

Propose to write the equation in an equivalent formulation that will cancel out the terms featuring an $\partial_\alpha^2 \boldsymbol{X} = \boldsymbol{X}''$. Integrate by parts against $G_1(z)$ while leaving $G_2(z)$ alone to obtain

$$
\begin{aligned}
\partial_t \boldsymbol{X}(\theta) &= \int \partial_\alpha \left(\frac{\mathcal{T}(|\boldsymbol{X}'|)}{|\boldsymbol{X}'|} \partial_\alpha (G_1(\delta_\alpha \boldsymbol{X})) \right) \delta_\alpha \boldsymbol{X}(\theta) d\alpha \\
&\quad + \int G_2(\delta_\alpha \boldsymbol{X}) \partial_\alpha \left(\mathcal{T}(|\boldsymbol{X}'(\theta+\alpha)|) \frac{\boldsymbol{X}'(\theta+\alpha)}{|\boldsymbol{X}'(\theta+\alpha)|} \right) d\alpha \\
&= \frac{1}{4\pi} \int \frac{2\left(\boldsymbol{X}'(\theta+\alpha) \cdot \frac{\delta_\alpha \boldsymbol{X}}{|\delta_\alpha \boldsymbol{X}|} \right)^2 - |\boldsymbol{X}'(\theta+\alpha)|^2}{|\delta_\alpha \boldsymbol{X}|^2} \frac{\mathcal{T}(|\boldsymbol{X}'(\theta+\alpha)|)}{|\boldsymbol{X}'(\theta+\alpha)|} \delta_\alpha \boldsymbol{X} \, d\alpha.
\end{aligned}
$$

This cancellation is very important for our analysis.

Equation for the derivative

The property of the cancellation of the highest order derivatives is also satisfied by the equation for $\partial_\theta \boldsymbol{X}(t,\theta) = \boldsymbol{X}'(t,\theta)$. We derive that $\boldsymbol{X}'(t,\theta)$ solves the equation

$$\partial_t \boldsymbol{X}'(\theta) = \int_{\mathbb{T}} \frac{d\alpha}{\alpha^2}\, \mathcal{K}[\boldsymbol{X}](\theta,\alpha)\delta_\alpha \mathbf{T}(\boldsymbol{X}'(\theta)),$$

where $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ is the tension map

$$\mathbf{T}(z) \overset{\text{def}}{=} \mathcal{T}(|z|)\hat{z}, \quad z \in \mathbb{R}^2.$$

Here the kernel $\mathcal{K}(\theta,\alpha) = \mathcal{K}[\boldsymbol{X}](\theta,\alpha)$ is given by

$$\begin{aligned}\mathcal{K}[\boldsymbol{X}](\theta,\alpha) \overset{\text{def}}{=}\ & \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot\mathcal{P}(D_\alpha\boldsymbol{X}(\theta))\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathbb{I} \\ & - \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot\mathcal{R}(D_\alpha\boldsymbol{X}(\theta))\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathcal{R}(D_\alpha\boldsymbol{X}(\theta)) \\ & + \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot(\mathcal{P}(D_\alpha\boldsymbol{X}(\theta))-\mathbb{I})\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathcal{P}(D_\alpha\boldsymbol{X}(\theta)).\end{aligned}$$

Kernel of the equation for the derivative

$$
\begin{aligned}
\mathcal{K}[\boldsymbol{X}](\theta,\alpha) \stackrel{\text{def}}{=} & \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot\mathcal{P}(D_\alpha\boldsymbol{X}(\theta))\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathbb{I} \\
& - \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot\mathcal{R}(D_\alpha\boldsymbol{X}(\theta))\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathcal{R}(D_\alpha\boldsymbol{X}(\theta)) \\
& + \frac{1}{4\pi}\frac{\boldsymbol{X}'(\theta+\alpha)\cdot(\mathcal{P}(D_\alpha\boldsymbol{X}(\theta))-\mathbb{I})\boldsymbol{X}'(\theta)}{|D_\alpha\boldsymbol{X}(\theta)|^2}\mathcal{P}(D_\alpha\boldsymbol{X}(\theta)).
\end{aligned}
$$

Then $\mathbb{I}$ is the identity matrix on $\mathbb{R}^2$, and the reflection matrices are

$$
\mathcal{R}(z) \stackrel{\text{def}}{=} \hat{z}\otimes\hat{z}^{\perp} + \hat{z}^{\perp}\otimes\hat{z}, \quad \mathcal{P}(z) \stackrel{\text{def}}{=} \hat{z}\otimes\hat{z} - \hat{z}^{\perp}\otimes\hat{z}^{\perp},
$$

where $\hat{z}^{\perp}\in\mathbb{R}^2$ is the unit vector perpendicular to the unit $\hat{z}$. And

$$
D_\alpha\boldsymbol{X}(\theta) \stackrel{\text{def}}{=} \frac{\delta_\alpha\boldsymbol{X}(\theta)}{\alpha} = \frac{\boldsymbol{X}(\theta+\alpha)-\boldsymbol{X}(\theta)}{\alpha}.
$$

Model: vector valued fractional porous medium equation

For small α, the integrand for the $\partial_t \boldsymbol{X}'$ equation is approximately

$$\frac{\mathcal{K}[\boldsymbol{X}](\theta, \alpha)}{\alpha^2} \delta_\alpha \mathbf{T}(\boldsymbol{X}'(\theta)) \approx \frac{\delta_\alpha \mathbf{T}(\boldsymbol{X}')}{4\pi\alpha^2}.$$

Thus the basic model equation for the general tension case should be a vector version of the fractional porous medium equation

$$\partial_t U = -(-\Delta)^{1/2} \mathbf{T}(U).$$

To the best of our knowledge, this equation has not been studied before.

The positivity and monotonicity assumptions that we make on the tension $\mathcal{T}$ are both physically motivated, as well as the same assumptions that typically appear on the porous media equation in order to ensure "ellipticity" for the problem.

Scalar fractional porous media equation

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Local vector valued porous media equation

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Scaling critical spaces

Here we give a list of some scaling critical spaces for the Peskin problem under the rescaling $\boldsymbol{X}_\lambda(t,\theta) = \lambda^{-1}\boldsymbol{X}(\lambda t, \lambda\theta)$ for $\lambda > 0$:

- the chord arc number

$$|\boldsymbol{X}|_* \stackrel{\text{def}}{=} \inf_{\theta,\alpha\in\mathbb{T},\alpha\neq 0} \left| \frac{\boldsymbol{X}(\theta+\alpha) - \boldsymbol{X}(\theta)}{\alpha} \right|.$$

- the Lipshitz space $\dot{W}^{1,\infty}(\mathbb{T})$
- the Wiener algebra $\mathcal{A}^1(\mathbb{T})$
- BMO^1 and VMO^1
- the homogeneous Besov spaces $\dot{B}^{1+\frac{1}{p}}_{p,r}(\mathbb{T})$ for all $p, r \in [1, \infty]$

We emphasize the spaces $\dot{B}^{\frac{3}{2}}_{2,r}(\mathbb{T})$ for $1 \leq r \leq \infty$ and $\dot{H}^{\frac{3}{2}}(\mathbb{T})$.

Solution Spaces

$L^q_T L^p_\theta$ mixed Lebesgue space norm:

$$||f||_{L^q_T(L^p_\theta)} \stackrel{\text{def}}{=} \left(\int_0^T \left(\int_{\mathbb{T}} |f(t,\theta)|^p \, d\theta \right)^{q/p} dt \right)^{1/q}, \quad 1 \leq p, q < \infty.$$

Besov spaces for $0 < s < 1$ and $p, q, r \in [1, \infty)$:

$$||f||_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}} \frac{d\beta}{|\beta|} \left(\frac{||\delta_\beta f||_{L^p(\mathbb{T})}}{|\beta|^s} \right)^r \right)^{1/r},$$

Chemin-Lerner mixed regularity spaces

$$||f||_{\widetilde{L}^q_T(\dot{B}^s_{p,r})} \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}} \frac{d\beta}{|\beta|^{1+sr}} ||\delta_\beta f||^r_{L^q_T(L^p_\theta)} \right)^{1/r}.$$

Solution Spaces with logarithmic regularity

Besov spaces with regularity on the logarithmic scale for $0 < s < 1$ and $p, r \in [1, \infty]$:

$$||f||_{\dot{B}^{s,\mu}_{p,r}} \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}} \frac{d\beta}{|\beta|} \left(\mu(|\beta|^{-1}) \frac{||\delta_\beta f||_{L^p(\mathbb{T})}}{|\beta|^s} \right)^r \right)^{1/r}.$$

Here the log scale derivative is defined as follows:

Definition

We consider functions $\mu \colon [0, \infty) \to [1, \infty)$ which satisfy the following three assumptions:

- μ is increasing and $\lim \mu(r) = \infty$ when r goes to $+\infty$;
- there is a positive constant c_0 such that $\mu(2r) \leq c_0 \mu(r)$ for any $r \geq 0$;
- the function $r \mapsto \mu(r)/\log(4 + r)$ is decreasing on $[0, \infty)$.

A Scaling Critical Besov Space for 2D Peskin

$$||\boldsymbol{X}'||_{\dot{B}^{1/2}_{2,1}} \stackrel{\text{def}}{=} \int_{\mathbb{T}} \frac{d\beta}{|\beta|^{3/2}} ||\delta_\beta \boldsymbol{X}'||_{L^2(\mathbb{T})}.$$

Definition of a weak solution

Let $\boldsymbol{X}_0' \in \dot{B}^{\frac{1}{2}}_{2,1}(\mathbb{T};\mathbb{R}^2)$ with $|\boldsymbol{X}_0|_* > 0$. We say that $\boldsymbol{X} : [0,T] \times \mathbb{T} \to \mathbb{R}^2$ is a weak solution of the Peskin problem with tension $\mathcal{T}$ and initial data $\boldsymbol{X}_0$ if $\mathbf{T}(\boldsymbol{X}') \in L^2_t(L^\infty_\theta \cap \dot{H}^{1/2}_\theta)$ with $\inf_{0\le t\le T} |\boldsymbol{X}(t)|_* > 0$, and for any function $\boldsymbol{Y} : [0,T] \times \mathbb{T} \to \mathbb{R}^2$ with $\boldsymbol{Y}' \in L^2_t(L^\infty_\theta \cap \dot{H}^{1/2}_\theta)$, we have

$$\int_{\mathbb{T}} d\theta \; \boldsymbol{Y}'(T,\theta)\cdot\boldsymbol{X}'(T,\theta) - \int_{\mathbb{T}} d\theta \; \boldsymbol{Y}_0'(\theta)\cdot\boldsymbol{X}_0'(\theta)$$

$$= \int_0^T dt \int_{\mathbb{T}} d\theta \; \partial_t \boldsymbol{Y}'(t,\theta)\cdot\boldsymbol{X}'(t,\theta)$$

$$-\frac{1}{2}\int_0^T dt \int_{\mathbb{T}} d\theta \int_{\mathbb{T}} \frac{d\alpha}{\alpha^2} \; \delta_\alpha \boldsymbol{Y}'(t)\cdot\mathcal{K}[\boldsymbol{X}](t,\theta,\alpha)\delta_\alpha \mathbf{T}(\boldsymbol{X}'(t)).$$

Definition of a strong solution

We say that $\boldsymbol{X} : [0, T] \times \mathbb{T} \to \mathbb{R}^2$ is a strong solution if $\boldsymbol{X} \in C^2((0, T] \times \mathbb{T} \to \mathbb{R}^2)$ solves the equation pointwise with $\inf_{0 \le t \le T} |\boldsymbol{X}(t)|_* > 0$ and

$$\lim_{t \to 0} ||\boldsymbol{X}'(t) - \boldsymbol{X}_0'||_{L^\infty} = 0.$$

Critical local existence theorem

Theorem (Cameron-S (2021) arXiv:2112.00692)

Let $\boldsymbol{X}_0 : \mathbb{T} \to \mathbb{R}^2$ with $\boldsymbol{X}_0' \in \dot{B}^{\frac{1}{2}}_{2,1}$ and $|\boldsymbol{X}_0|_* > 0$. Let the scalar tension $\mathcal{T} : (0,\infty) \to (0,\infty)$ be such that $\mathcal{T} \in C^{1,1}_{loc}(0,\infty)$ with $\mathcal{T}'(r) > 0$ for all $0 < r < \infty$. Then there is a $T > 0$ such that there exists a unique weak solution $\boldsymbol{X} : [0, T] \times \mathbb{T} \to \mathbb{R}^2$ to the Peskin problem, which is also a strong solution.
Furthermore for any $0 < \beta < 1$, $\boldsymbol{X} \in C^{2,\beta}_{loc}((0, T] \times \mathbb{T}; \mathbb{R}^2)$.
Additionally, if $\mathcal{T} \in C^{k,\gamma}_{loc}(0,\infty)$ for some $k \geq 2$ and $0 < \gamma < 1$ then we have that $\boldsymbol{X} \in C^{k+1,\gamma}_{loc}((0, T] \times \mathbb{T}; \mathbb{R}^2)$.

Due to the structure of equation, $\boldsymbol{X} \in C^{k+1,\gamma}_{loc}$ is the optimal regularity for $\mathcal{T} \in C^{k,\gamma}_{loc}$. We prove this theorem by proving a quantitative version under more restrictive assumptions on the tension.

Main quantiative critical local existence theorem

Theorem (Quantitative existence, Cameron-S (2021))

Consider initial data $\boldsymbol{X}_0 : \mathbb{T} \to \mathbb{R}^2$ *such that* $||\boldsymbol{X}_0'||_{\dot{B}^{\frac{1}{2},\mu}_{2,1}} \leq M$ *for some* μ *and for any* $M > 0$, *and* $|\boldsymbol{X}_0|_* > 0$. *Let the tension map* $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ *be such that* $D\mathbf{T} \in W^{1,\infty}(\mathbb{R}^2; \mathbb{R}^{2\times 2})$ *satisfying the ellipticity condition* $D\mathbf{T}(z) \geq \lambda \mathbb{I} > 0$.
Then there exists a time $T > 0$ *depending only on* M, μ, $|\boldsymbol{X}_0|_*$, λ *and* $||D\mathbf{T}||_{W^{1,\infty}}$ *such that there exists a strong solution,* $\boldsymbol{X} : [0, T] \times \mathbb{T} \to \mathbb{R}^2$ *to the Peskin problem with tension* $\mathbf{T}$ *and initial data* $\boldsymbol{X}_0$. *This solution satisfies for some universal constant* $c > 0$ *that*

$$\int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}} \mu(|\beta|^{-1}) \left(||\delta_\beta \boldsymbol{X}'||_{L^\infty_T L^2_\theta} + c\sqrt{\lambda} ||\delta_\beta \Lambda^{\frac{1}{2}} \boldsymbol{X}'||_{L^2_T L^2_\theta} \right) \leq 4 \int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}} \mu(|\beta|^{-1}) ||\delta_\beta \boldsymbol{X}_0'||_{L^2_\theta}.$$

Smoothing

Theorem (Quantitative smoothing, Cameron-S (2021))

Further for any small time $\tau > 0$ and any $0 < \beta < 1$, $\boldsymbol{X} \in C^{2,\beta}([\tau, T] \times \mathbb{T}; \mathbb{R}^2)$, with its norm depending only on τ, β, and the previously mentioned constants.
If we additionally have that $\mathbf{T} \in C^{k,\gamma}(\mathbb{R}^2; \mathbb{R}^2)$ for some $k \geq 2$ and $0 < \gamma < 1$. Then for any small time $\tau > 0$, $\boldsymbol{X} \in C^{k+1,\gamma}([\tau, T] \times \mathbb{T}; \mathbb{R}^2)$ with the $C^{k+1,\gamma}$ norm controlled by M, μ, $|\boldsymbol{X}_0|_*$, λ, γ, $||\mathbf{T}||_{C^{k,\gamma}}$, and τ.

Theorem (Uniqueness, Cameron-S (2021))

Consider $\boldsymbol{X}_0$ *and* $\boldsymbol{Y}_0$ *such that* $\boldsymbol{X}_0', \boldsymbol{Y}_0' \in \dot{B}_{2,1}^{\frac{1}{2},\mu}(\mathbb{T};\mathbb{R}^2)$ *with* $|\boldsymbol{X}_0|_* > 0$ *and* $|\boldsymbol{Y}_0|_* > 0$. *Let the tension map* $\mathbf{T} : \mathbb{R}^2 \to \mathbb{R}^2$ *satisfy the same conditions and consider the corresponding solutions* $\boldsymbol{X}, \boldsymbol{Y} : [0, T] \times \mathbb{T} \to \mathbb{R}^2$. *Choose any* $\nu(r)$ *satisfying the previous definition such that there exists* $r_* \geq 1$ *so that* $\frac{\nu(r)}{\mu(r)}$ *is decreasing for* $r \geq r_*$ *and in particular* $\lim_{r\to\infty} \frac{\nu(r)}{\mu(r)} = 0$. *Then for any* $\varepsilon > 0$, *there exists* $\delta_* > 0$ *such that for any* $0 < \delta \leq \delta_*$ *then* $||\boldsymbol{X}_0' - \boldsymbol{Y}_0'||_{L^2_\theta} < \delta$ *implies that*

$$\int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}} \nu(|\beta|^{-1}) ||\delta_\beta(\boldsymbol{X}' - \boldsymbol{Y}')||_{L^\infty_T L^2_\theta} < \varepsilon.$$

In particular if $||\boldsymbol{X}_0' - \boldsymbol{Y}_0'||_{L^2_\theta} = 0$ *then the solution is unique in* $\widetilde{L}^\infty_T(\dot{B}_{2,1}^{\frac{1}{2},\nu})$.

We can take for example $\nu(r) = \mu(r)^\gamma$ for any $0 < \gamma < 1$.

Theorem (Strong continuity, Cameron-S (2021))

We consider the two strong solutions $\boldsymbol{X}, \boldsymbol{Y} : [0, T] \times \mathbb{T} \to \mathbb{R}^2$ *to the Peskin problem with initial data* $\boldsymbol{X}_0$, $\boldsymbol{Y}_0$. *Suppose the tension map* $\mathbf{T}$ *satisfies* $D\mathbf{T} \in W^{2,\infty}(\mathbb{R}^2; \mathbb{R}^{2\times 2})$ *and the ellipticity condition* $D\mathbf{T}(z) \geq \lambda \mathbb{I} > 0$.
Then there exists $T_M > 0$ *depending only on* M, $|\boldsymbol{X}_0|_*$, $|\boldsymbol{Y}_0|_*$, μ, λ, *and* $||D\mathbf{T}||_{W^{2,\infty}}$ *such that for any* $0 < T \leq T_M$, *we have the following strong continuity estimate*

$$||\boldsymbol{X}' - \boldsymbol{Y}'||_{\mathcal{B}^\nu_T} + 2\lambda^{\frac{1}{2}}||\boldsymbol{X}' - \boldsymbol{Y}'||_{\mathcal{D}^\nu_T} \leq 8||\boldsymbol{X}'_0 - \boldsymbol{Y}'_0||_{\mathcal{B}^\nu}.$$

$$||\boldsymbol{X}'||_{\mathcal{B}^\nu_T} \approx \int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}} \mu(|\beta|^{-1}) ||\delta_\beta \boldsymbol{X}'||_{L^\infty_T L^2_\theta}.$$

$$||\boldsymbol{X}'||_{\mathcal{D}^\nu_T} \approx \int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}} \mu(|\beta|^{-1}) ||\delta_\beta \Lambda^{\frac{1}{2}} \boldsymbol{X}'||_{L^2_T L^2_\theta}.$$

Proof overview (Main apriori estimate)

The heart of our argument is the a priori estimate. We make use of our new equation for $\partial_t \boldsymbol{X}'$. Because $\mathcal{K}(\theta, \alpha)$ is symmetric in $\theta, \theta + \alpha$, our equation has divergence form symmetry making L^2 based energy estimates a natural choice. By making use of Besov spaces, we're interested then in keeping careful track of the time evolution of differences $||\delta_\beta \boldsymbol{X}'||_{L^2_\theta}(t)$ where $\beta \in \mathbb{T}$ is arbitrary. We have that $\delta_\beta \boldsymbol{X}'(\theta)$ solves

$$\partial_t \delta_\beta \boldsymbol{X}'(\theta) + \Lambda \delta_\beta \mathbf{T}(\boldsymbol{X}')(\theta) = \int_{\mathbb{T}} \frac{d\alpha}{\alpha^2} \delta_\beta \left(\mathcal{A}(\theta, \alpha) \delta_\alpha \mathbf{T}(\boldsymbol{X}')(\theta) \right).$$

Here $\mathcal{A}(\theta, \alpha)$ is part of the kernel $\mathcal{K}[\boldsymbol{X}]$. When we calculate $\partial_t ||\delta_\beta \boldsymbol{X}'||^2_{L^2_\theta}$, we then get one good diffusive term $-\lambda ||\delta_\beta \boldsymbol{X}'||^2_{\dot{H}^{1/2}}$ from the $\Lambda \delta_\beta \mathbf{T}(\boldsymbol{X}')$ (along with additional error terms).

Proof overview

We then treat the remaining terms as error, and are left to bound integrals of the form

$$\int_{\mathbb{T}}\int_{\mathbb{T}} d\theta d\alpha \frac{|\delta_\beta \delta_\alpha \boldsymbol{X}'|^2 \; |\delta_\alpha \boldsymbol{X}'|^q}{\alpha^2},$$

or

$$\int_{\mathbb{T}}\int_{\mathbb{T}} d\theta d\alpha \frac{|\delta_\beta \delta_\alpha \boldsymbol{X}'| \; |\delta_\beta \boldsymbol{X}'| \; |\delta_\alpha \boldsymbol{X}'|^{q+1}}{\alpha^2},$$

where $q = 1, 2$. If we were to bound the first term naively, we would get

$$||\delta_\beta \boldsymbol{X}'||^2_{\dot{H}^{1/2}} ||\boldsymbol{X}'||^q_{L^\infty_\theta},$$

which would make it impossible to close the estimate, as this is of the same order as our good diffusive term but with a possibly large coefficient in front for large data.

Proof overview

$$\int_{\mathbb{T}}\int_{\mathbb{T}} d\theta d\alpha \frac{|\delta_\beta \delta_\alpha \boldsymbol{X}'|^2 \ |\delta_\alpha \boldsymbol{X}'|^q}{\alpha^2},$$

However, the norm for $\dot{B}^{\frac{1}{2},\mu}_{2,1}$ both controls the size of the norm $\dot{B}^{\frac{1}{2}}_{2,1}$ and the rate of decay for

$$r \to \int_{|\alpha|<r} d\alpha \frac{||\delta_\alpha f||_{L^2_\theta}}{|\alpha|^{3/2}} \lesssim \frac{||f||_{\dot{B}^{1/2,\mu}_{2,1}}}{\mu(r^{-1})}.$$

Thus splitting the integral in our error term between $|\alpha| < \eta$ and $|\alpha| > \eta$ for some η sufficiently small depending on μ, $||\boldsymbol{X}_0'||_{\dot{B}^{1/2,\mu}_{2,1}}$, and other relevant constants, we are able to bound this error term as

$$\epsilon||\delta_\beta \boldsymbol{X}'||^2_{\dot{H}^{1/2}} + C||\delta_\beta \boldsymbol{X}'||^2_{L^2_\theta},$$

which we can handle.

Proof overview

$$\int_{\mathbb{T}}\int_{\mathbb{T}} d\theta d\alpha \frac{|\delta_\beta\delta_\alpha \boldsymbol{X}'|\ |\delta_\beta \boldsymbol{X}'|\ |\delta_\alpha \boldsymbol{X}'|^{q+1}}{\alpha^2},$$

For the second type of error term, the story is similar except that we are forced to bound the $|\delta_\beta \boldsymbol{X}'|$ in L^∞_θ, as it has no decay as $\alpha \to 0$. Thus we end up with an error term of the form

$$\epsilon ||\delta_\beta \boldsymbol{X}'||^2_{\dot{H}^{1/2}} + C||\delta_\beta \boldsymbol{X}'||^2_{L^2_\theta} + \epsilon||\delta_\beta \boldsymbol{X}'||^2_{L^\infty_\theta}.$$

This L^∞ error term at first seems very bad, as notably the Sobolev embedding fails and $||\delta_\beta \boldsymbol{X}'||^2_{L^\infty_\theta}$ is not controlled by our good diffusive piece $-\lambda||\delta_\beta \boldsymbol{X}'||^2_{\dot{H}^{\frac{1}{2}}}$. However, once we integrate in β against $\mu(|\beta|^{-1})|\beta|^{-3/2}$ the Sobolev embedding is again true:

$$\int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}}\mu(|\beta|^{-1})||\delta_\beta \boldsymbol{X}'||_{L^\infty_\theta} \leq C_\mu \int_{\mathbb{T}} \frac{d\beta}{|\beta|^{\frac{3}{2}}}\mu(|\beta|^{-1})||\Lambda^{\frac{1}{2}}\delta_\beta \boldsymbol{X}'||_{L^2_\theta}$$

and we can control this error term at the end of the estimate.

Higher regularity

Our higher regularity proofs are based on the proof of higher regularity for the scalar fractional porous medium equation:

Juan Luis Vázquez, Arturo de Pablo, Fernando Quirós, and Ana Rodríguez, *Classical solutions and higher regularity for nonlinear fractional diffusion equations*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 7, 1949–1975, doi:10.4171/JEMS/710

Global well-posedness for the Peskin problem with viscosity contrast

Peskin problem with viscosity contrast

- Incompressibility condition

$$\nabla \cdot \boldsymbol{u} = 0, \quad \text{in } D \cup D^c, \quad D^c = \mathbb{R}^2 \backslash (D \cup \partial D)$$

- Stokes with viscosities μ_1 and μ_2 ($\mu_1 \neq \mu_2$)

$$\mu_1 \Delta \boldsymbol{u} - \nabla p = 0, \quad \text{in } D,$$
$$\mu_2 \Delta \boldsymbol{u} - \nabla p = 0, \quad \text{in } D^c,$$

- Elasticity: jump conditions

$$[\![\Sigma \boldsymbol{n}]\!] = F_{el}(\boldsymbol{X}) |\partial_\alpha \boldsymbol{X}|^{-1}, \quad [\![\boldsymbol{u}]\!] = 0, \text{ on } \partial D,$$

where

$$\Sigma = \begin{cases} \mu_1 \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) - p\mathbb{I} & \text{in } D \\ \mu_2 \left(\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^{\mathrm{T}}\right) - p\mathbb{I} & \text{in } D^c \end{cases}.$$

- Simple tension force law: $F_{el}(\boldsymbol{X}) = k_0 \partial_\theta^2 \boldsymbol{X}$, $k_0 > 0$.
- The string moves with the flow:

$$\partial_t \boldsymbol{X}(\theta, t) = \boldsymbol{u}(\boldsymbol{X}(\theta, t), t)$$

Contour dynamic formulation with viscosity contrast

Inverting the Stokes operator in the whole space

$$\partial_t \boldsymbol{X}(\theta,t) = \frac{1}{4\pi}\int_{\mathbb{T}} G(\boldsymbol{X}(\theta,t)-\boldsymbol{X}(\alpha,t))F(\boldsymbol{X})(\alpha,t)d\alpha,$$
$$G(x) = -\log|x|\mathbb{I} + \frac{x\otimes x}{|x|^2},$$

and the net force (due to elasticity and viscosity contrast)

$$F_i(\theta) + 2A_\mu \partial_\theta \boldsymbol{X}_j^\perp \int_{\mathbb{T}} \mathcal{H}_{ijk}(\boldsymbol{X}(\theta)-\boldsymbol{X}(\alpha))F_k(\alpha)d\alpha = 2A_e F_{el,i}(\theta).$$

where $i, j, k \in \{1,2\}$ and

$$A_\mu = \frac{\mu_2-\mu_1}{\mu_1+\mu_2}, \quad A_e = \frac{k_0}{\mu_2+\mu_1}, \quad \mathcal{H}_{ijk} = -\frac{1}{\pi}\frac{x_i x_j x_k}{|\boldsymbol{x}|^4},$$

Linearization

Unique equilibria: uniformly parametrized circles,

$$\boldsymbol{X}_c(\alpha) = a\begin{pmatrix}1\\0\end{pmatrix} + b\begin{pmatrix}0\\1\end{pmatrix} + c\begin{pmatrix}\cos\alpha\\ \sin\alpha\end{pmatrix} + d\begin{pmatrix}-\sin\alpha\\ \cos\alpha\end{pmatrix}$$

Linearization around equilibrium $\boldsymbol{Z} = \boldsymbol{X} - \boldsymbol{X}_c$:

$$\partial_t \boldsymbol{Z}(\alpha) = -\frac{A_e}{2}\Big(\Lambda \boldsymbol{Z}(\alpha) - H\boldsymbol{Z}(\alpha)^{\perp}\Big) + N(\boldsymbol{Z}, \boldsymbol{X}_c)(\alpha).$$

Here H is the Hilbert transform, and $\Lambda = (-\Delta)^{1/2}$.

Obstacle: no dissipation for $0, \pm 1$ frequencies. Here, we cannot choose the parametrization.

Functional spaces

Generalizing the Wiener algebra of functions with absolutely convergent Fourier series, we define the homogeneous $\dot{\mathcal{F}}^{s,1}_{\nu}$ and nonhomogeneous $\mathcal{F}^{s,1}_{\nu}$ norms as

$$\|\boldsymbol{X}\|_{\dot{\mathcal{F}}^{s,1}_{\nu}} = \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\nu(t)|k|}|k|^s|\widehat{\boldsymbol{X}}(k)|, \qquad s\in\mathbb{R},$$

and

$$\|\boldsymbol{X}\|_{\mathcal{F}^{s,1}_{\nu}} = |\widehat{\boldsymbol{X}}(0)| + \sum_{k\in\mathbb{Z}\setminus\{0\}} e^{\nu(t)|k|}|k|^s|\widehat{\boldsymbol{X}}(k)|, \qquad s\geq 0,$$

with $\nu_\infty > 0$ is chosen sufficiently small we define

$$\nu(t) = \nu_\infty \frac{t}{1+t} \geq 0,$$

When $\nu\equiv 0$, we denote $\dot{\mathcal{F}}^{s,1}_0 = \dot{\mathcal{F}}^{s,1}$ and $\mathcal{F}^{s,1}_0 = \mathcal{F}^{s,1}$.
When $s=1$ the $\dot{\mathcal{F}}^{s,1}$ norm is critical for the Peskin problem.

Strong solution

Definition (Strong solution)

Let

$$\boldsymbol{\mathcal{X}} \in C([0,T];\mathcal{F}^{1,1}) \cap C^1((0,T];\mathcal{F}^{0,1}),$$

and

$$|\boldsymbol{\mathcal{X}}|_*(t) = \inf_{\theta,\eta\in\mathbb{S},\theta\neq\eta} \frac{|\boldsymbol{\mathcal{X}}(\theta,t)-\boldsymbol{\mathcal{X}}(\eta,t)|}{|\theta-\eta|} > 0,$$

for $0 \leq t \leq T$. Then, $\boldsymbol{\mathcal{X}}$ is a *strong solution* to the viscosity-contrast Peskin problem with initial value $\boldsymbol{\mathcal{X}}(0) = \boldsymbol{\mathcal{X}}_0$ if it satisfies the contour dynamic formlation pointwise for $0 < t \leq T$ and $\boldsymbol{\mathcal{X}}(t) \to \boldsymbol{\mathcal{X}}_0$ in $\mathcal{F}^{1,1}$ as $t \to 0$.

Theorem (García-Juárez, Mori, S (2021) Anal. PDE)

Let $A_\mu \in (-1,1)$ *and* $\boldsymbol{X}_0 \in \mathcal{F}^{1,1}$*. Let* $\boldsymbol{Z}_0 = \boldsymbol{X}_0 - \boldsymbol{X}_{0,c}$*. Assume that initially the deviation* $\boldsymbol{X}_0$ *satisfies the medium-size condition*

$$\|\boldsymbol{Z}_0\|_{\dot{\mathcal{F}}^{1,1}} < k(A_\mu),$$

where $k(A_\mu) > 0$*, and the area enclosed by* $\boldsymbol{X}_0$ *is* π*. Then, for any* $T > 0$*, there exists a constant* $\nu_\infty > 0$ *such that there exists a* unique global strong solution $\boldsymbol{X}(t)$*, which lies in the space*

$$\boldsymbol{X} \in C([0,T];\mathcal{F}^{1,1}_\nu) \cap C^1((0,T];\mathcal{F}^{0,1}_\nu) \cap L^1([0,T];\dot{\mathcal{F}}^{2,1}_\nu),$$

In particular, it becomes instantaneously analytic. Moreover,

$$\|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t) + \frac{A_e}{4}\mathcal{C}\int_0^t \|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{2,1}_\nu}(\tau)d\tau \le \|\boldsymbol{Z}_0\|_{\dot{\mathcal{F}}^{1,1}}, \quad 0 \le t \le T.$$

with $\mathcal{C} = \mathcal{C}\big(\|\boldsymbol{Z}_0\|_{\dot{\mathcal{F}}^{1,1}}, A_\mu, \nu_\infty\big) > 0$.

Theorem (García-Juárez, Mori, S (2021) Anal. PDE)

In addition,

$$\|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{1,1}_{\nu}}(t) \leq \|\boldsymbol{Z}_0\|_{\dot{\mathcal{F}}^{1,1}} e^{-\frac{A_e}{4}\mathcal{C}t}.$$

The zero frequency $\widehat{\boldsymbol{Z}_c}(0)$ *remains uniformly bounded for all times as follows*

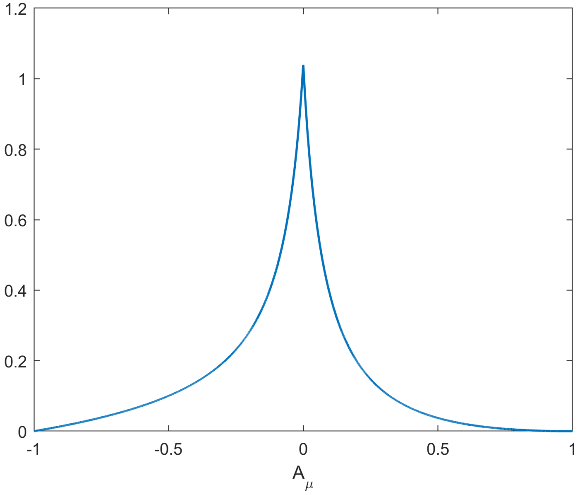
$$|\widehat{\boldsymbol{Z}_c}(0)| \leq |\widehat{\boldsymbol{Z}}_{0,c}(0)| + \tilde{C}\|\boldsymbol{Z}_0\|^2_{\dot{\mathcal{F}}^{1,1}},$$

with $\tilde{C} = \tilde{C}\big(\|\boldsymbol{X}_0\|_{\dot{\mathcal{F}}^{1,1}}, A_\mu\big) > 0$, *while*

$$1 - \frac{1}{2}\|\boldsymbol{Z}\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}} \leq |\widehat{\boldsymbol{Z}}_c(1)|^2 \leq 1 + \frac{1}{2}\|\boldsymbol{Z}\|^2_{\dot{\mathcal{F}}^{1,1}_{\nu}}.$$

The decay to zero of the deviation $\boldsymbol{Z}$ shows the exponentially fast convergence to a uniformly parametrized circle with the same area as the initial one.

Allowable size of the inital data: $10^3 k(A_\mu)$



Closely related recent references

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- Francisco Gancedo, Eduardo García-Juárez, Neel Patel, and Robert M. Strain, *Global regularity for gravity unstable Muskat bubbles*, Mem. Amer. Math. Soc. (2021) **in press**, 87 pages, arXiv:1902.02318.

Results

Ideas:

- Linearize around a time-dependent uniformly parametrized circle, $a(t), b(t), c(t), d(t)$
- Diagonalize the linear operator
- Use the incompressibility condition
- Project the equation into the space of equilibria and its orthogonal complement

Result: For any value of $A_\mu \in (-1, 1)$, and a perturbation from the uniformly parametrized circles not too far in $\mathcal{F}^{1,1}$ (critical regularity), global well-posedness, instant analyticity, and exponential convergence.

Proof idea: Fourier

Basic idea to estimate $\|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{1,1}_\nu}(t)$: $\quad (\boldsymbol{Z}_t = -\mathcal{L}\boldsymbol{Z} + N(\boldsymbol{Z}))$

$$
\begin{aligned}
\frac{d}{dt}\|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{1,1}_\nu} &= \frac{d}{dt}\left(\int |\xi| e^{\nu t|\xi|}|\hat{\boldsymbol{Z}}(\xi)| d\xi\right) \\
&= \nu \int |\xi|^2 e^{t\nu|\xi|}|\hat{\boldsymbol{Z}}(\xi)| d\xi \\
&\quad + \int |\xi| e^{\nu t|\xi|} \frac{1}{2} \frac{\hat{\boldsymbol{Z}}_t(\xi)\overline{\hat{\boldsymbol{Z}}}(\xi) + \hat{\boldsymbol{Z}}(\xi)\overline{\hat{\boldsymbol{Z}}}_t(\xi)}{|\hat{\boldsymbol{Z}}(\xi)|} d\xi \\
&\leq (\nu - \frac{1}{2}A_e) \int |\xi|^2 e^{\nu t|\xi|}|\hat{\boldsymbol{Z}}(\xi)| d\xi + \int |\xi| e^{\nu t|\xi|}|\widehat{N(\boldsymbol{Z})}(\xi)| d\xi \\
&= \underbrace{(\nu - \frac{1}{2}A_e)}_{<0} \|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{2,1}_\nu} + \|N\|_{\dot{\mathcal{F}}^{1,1}_\nu}
\end{aligned}
$$

Goal: Find $\|N\|_{\dot{\mathcal{F}}^{1,1}_\nu} \leq c(\|f\|_{\dot{\mathcal{F}}^{1,1}_\nu}, A_\mu) A_e \|\boldsymbol{Z}\|_{\dot{\mathcal{F}}^{2,1}_\nu}$.

Thank you!

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