Regularity structure, global-in-time existence and uniqueness of conservative solutions to the Hunter-Saxton equation

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Joint work with Yu Gao and Hao Liu

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Outline

Background

The Hunter-Saxton equation was introduced in (J. K. Hunter and R. Saxton, *SIAM J. Appl. Math.*, 51(6), 1991): for $(x, t) \in \mathbb{R} \times \mathbb{R}$,

$$u_t + uu_x = \frac{1}{2} \int_{-\infty}^{x} u_y^2(y, t)\, \mathrm{d}y,$$

or

$$u_t + uu_x = \frac{1}{4} \left(\int_{-\infty}^{x} u_y^2(y, t)\, \mathrm{d}y - \int_{x}^{\infty} u_y^2(y, t)\, \mathrm{d}y \right).$$

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- Arises in the theoretical study of nematic liquid crystals.
- Both equations are *similar*, but **NOT** completely equivalent!!
  - The major difference is the asymptotic behavior of RHS at ∞.
  - Differentiating both equations with respect to x, we obtain the SAME equation:

$$(u_t + uu_x)_x = \frac{1}{2} u_x^2.$$

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where $[L, P] := LP - PL$.

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$$\frac{du}{dt} = \tan u \quad \xleftrightarrow{x = \sin u} \quad \frac{dx}{dt} = x.$$

Formal calculations: finite-time blowup

Differentiating the Hunter-Saxton equation

$$u_t + uu_x = \frac{1}{2}\int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y$$

with respect to x yields

$$u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2.$$

This is a *Riccati type equation*, so smooth solutions may blow up in finite time.

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For $u_t + uu_x = 0$, a direct differentiation (in x) yields

$$u_{xt} + uu_{xx} = -u_x^2,$$

so for any smooth solution $u := u(x,t)$ subject to the initial data $u|_{t=0} = \bar{u}$,

$$u_x(\xi + t\bar{u}(\xi), t) = \frac{\bar{u}_x(\xi)}{1 + t\bar{u}_x(\xi)}.$$

u is classical for all $t \geq 0$ if and only if $\bar{u}$ is C^1 and $\bar{u}_x > 0$ in $\mathbb{R}$.

Energy conservation and structural similarity

- **Energy conservation**: Multiplying $u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2$ by u_x yields

$$(u_x^2)_t + (uu_x^2)_x = 0.$$

Question: Is $\|u_x(\cdot, t)\|_{L^2}$ actually *conserved*?

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- **Structural similarity**: Differentiating the Hunter-Saxton equation with respect to x twice yields

$$u_{xxt} + 2u_x u_{xx} + uu_{xxx} = 0.$$

Set $m := u_{xx}$. Then

$$m_t + 2mu_x + m_x u = 0, \quad m = u_{xx}.$$

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It resembles the Camassa-Holm equation:

$$m_t + 2mu_x + m_x u = 0, \quad m = u - u_{xx}.$$

Remark: *Both of these equations are integrable equations.*

Outline

Flow map

Let the initial datum $\bar{u}$ be smooth and $\bar{u}_x \in L^2(\mathbb{R})$. Consider the **flow map**:

$$\frac{\partial}{\partial t} X(\xi, t) = u(X(\xi, t), t), \quad X(\xi, 0) = \xi.$$

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Then

$$\frac{\partial^2}{\partial t^2} X(\xi, t) = (u_t + u u_x)(X(\xi, t), t),$$

and by $(u_x^2)_t + (u u_x^2)_x = 0$,

$$\frac{\partial}{\partial t} \int_{-\infty}^{X(\xi,t)} u_x^2(y, t)\,\mathrm{d}y = u_x^2(X(\xi, t), t) \underbrace{\frac{\partial}{\partial t} X(\xi, t)}_{=u(X(\xi,t),t)} + \int_{-\infty}^{X(\xi,t)} \underbrace{(u_x^2)_t(y, t)}_{=-(u u_x^2)_x(y,t)} \,\mathrm{d}y = 0.$$

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This implies that the acceleration $\int_{-\infty}^{X(\xi,t)} u_x^2(y,t)\,\mathrm{d}y$ *is conserved along the flow map!!*

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This implies that the acceleration $\int_{-\infty}^{X(\xi,t)} u_x^2(y, t)\,\mathrm{d}y$ *is conserved along the flow map!!* Using the Hunter-Saxton equation $u_t + u u_x = \frac{1}{2} \int_{-\infty}^{x} u_y^2(y, t)\,\mathrm{d}y$ yields

$$\frac{\partial^2}{\partial t^2} X(\xi, t) = \frac{1}{2} \int_{-\infty}^{X(\xi,t)} u_x^2(y, t)\,\mathrm{d}y \equiv \frac{1}{2} \int_{-\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y.$$

Method of characteristics

Solving

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} X(\xi, t) = \dfrac{1}{2} \displaystyle\int_{-\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y, \\ \quad X(\xi, 0) = \xi, \quad \text{and} \quad \dfrac{\partial}{\partial t} X(\xi, 0) = \bar{u}(\xi), \end{cases}$$

we have

$$X(\xi, t) = \xi + \bar{u}(\xi)t + \frac{t^2}{4} \int_{\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y$$

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and hence,

$$u(X(\xi, t), t) = \frac{\partial}{\partial t} X(\xi, t) = \bar{u}(\xi) + \frac{t}{2} \int_{\infty}^{\xi} \bar{u}_x^2(y)\, \mathrm{d}y.$$

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Furthermore, a direct computation yields

$$X_\xi(\xi, t) = \left[1 + \frac{t}{2} \bar{u}_x(\xi)\right]^2 \geq 0.$$

Remark: These formulae make sense, as long as $\xi \mapsto X(\xi, t)$ is invertible (e.g., $X_\xi > 0$).

Finite time blow up

Recall that $u_{xt} + uu_{xx} = -\frac{1}{2}u_x^2$, so

$$\frac{\partial}{\partial t}u_x(X(\xi,t),t) = -\frac{1}{2}u_x^2(X(\xi,t),t).$$

Using the method of characteristics again, we also have

$$u_x(X(\xi,t),t) = \frac{2\bar{u}_x(\xi)}{2 + t\bar{u}_x(\xi)},$$

where $X(\xi,t) = \xi + \bar{u}(\xi)t + \frac{t^2}{4}\int_{\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y$ was obtained on the last slide.

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where $X(\xi,t) = \xi + \bar{u}(\xi)t + \frac{t^2}{4}\int_\infty^\xi \bar{u}_x^2(y)\,\mathrm{d}y$ was obtained on the last slide. Hence, if $\inf\limits_{x\in\mathbb{R}} \bar{u}_x(x) < 0$, then

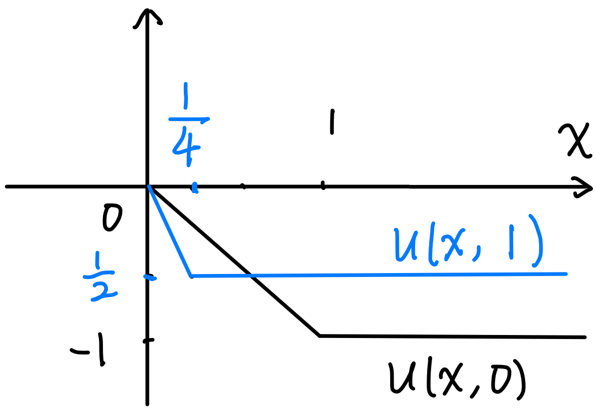
$$\inf_x u_x(x,t) \to -\infty \quad \text{as} \quad t \to T_0 := -\frac{2}{\inf_{x\in\mathbb{R}} \bar{u}_x(x)}.$$

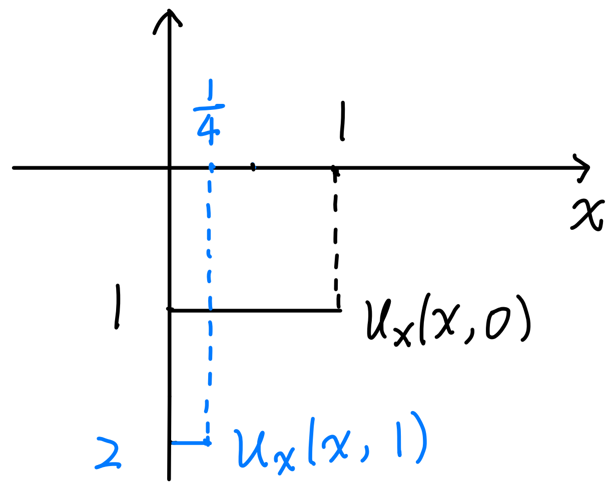
Question: How about $\|u_x(\cdot,t)\|_{L^2}$?
Question: Can we further extend the solution after the blowup time?

Initial data for an explicit example

Consider

$$\bar{u}(x) = \begin{cases} 0, & x \leq 0, \\ -x, & 0 < x < 1, \\ -1, & x \geq 1, \end{cases} \qquad \bar{u}_x(x) = \begin{cases} 0, & x \leq 0, \\ -1, & 0 < x < 1, \\ 0, & x \geq 1. \end{cases}$$





Explicit example

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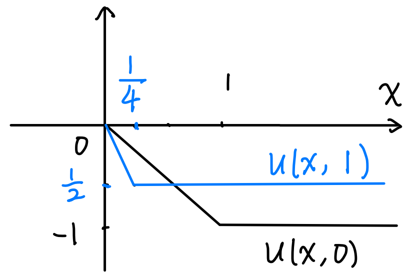
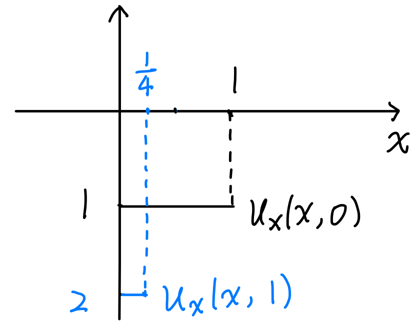
For $t \in [0, 2)$, we have

$$u(x,t) = \begin{cases} 0, & x \le 0, \\ -\dfrac{x}{1 - t/2}, & 0 < x < (1 - t/2)^2, \\ -(1 - t/2), & x > (1 - t/2)^2, \end{cases}$$

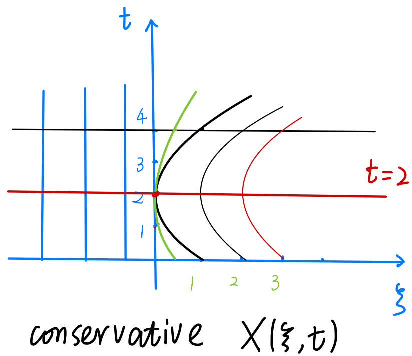
$$u_x(x,t) = \begin{cases} -\dfrac{1}{1 - t/2}, & 0 < x < (1 - t/2)^2, \\ 0, & \text{otherwise}, \end{cases} \qquad \|u_x(\cdot, t)\|_{L^2} = 1.$$

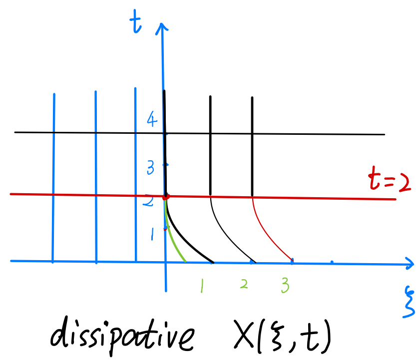
Question: What will happen after $t = 2$? Conservative or dissipative?





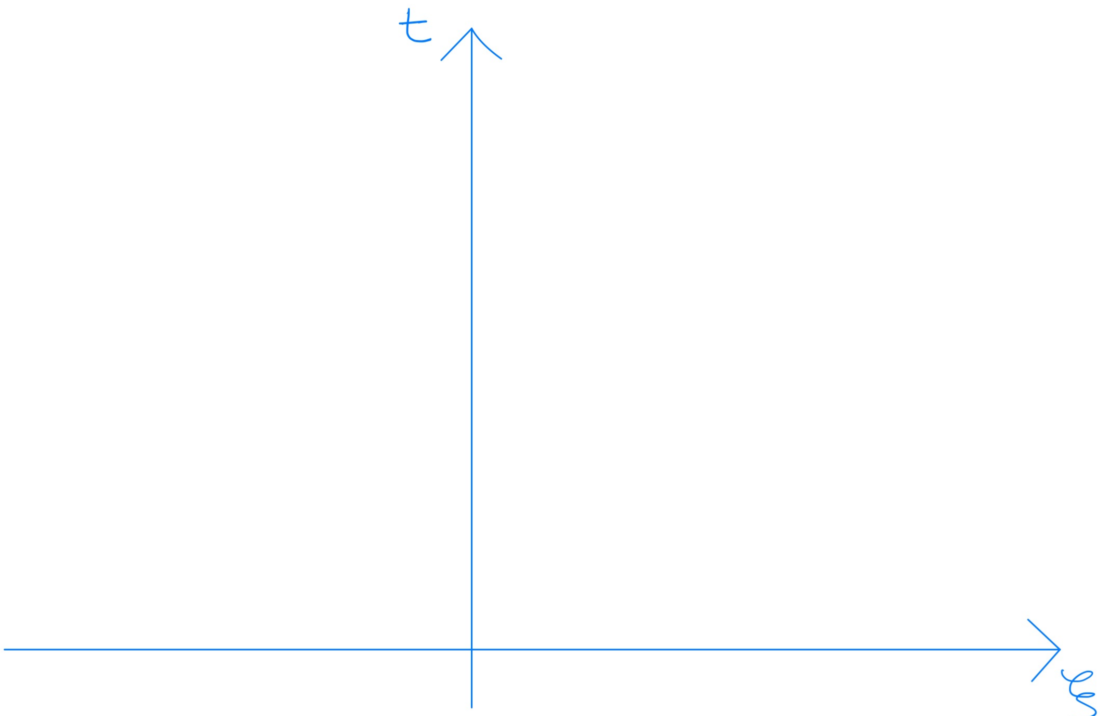
$$u(x,t) \xrightarrow{C_b(\mathbb{R})} 0, \quad u_x^2(\cdot,t)\,\mathrm{d}x \overset{*}{\rightharpoonup} \delta_0 \quad \text{as} \quad t \to 2^-.$$





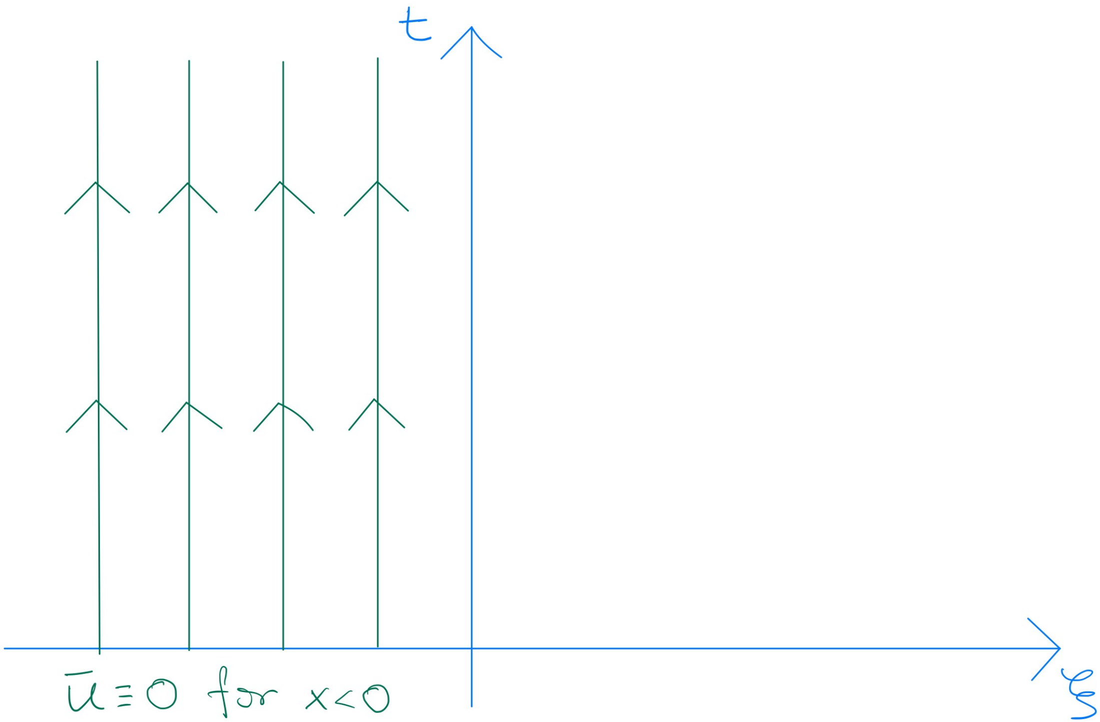
What was happening (for the conservative solution)?

Due to the *semi-group property*, WLOG, we can **RESTART** the problem at the problematic time (i.e., $t = 2$ on the previous slide). Now, **imagine** we can take an initial data $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$. **Question**: *What does the conservative solution look like?*



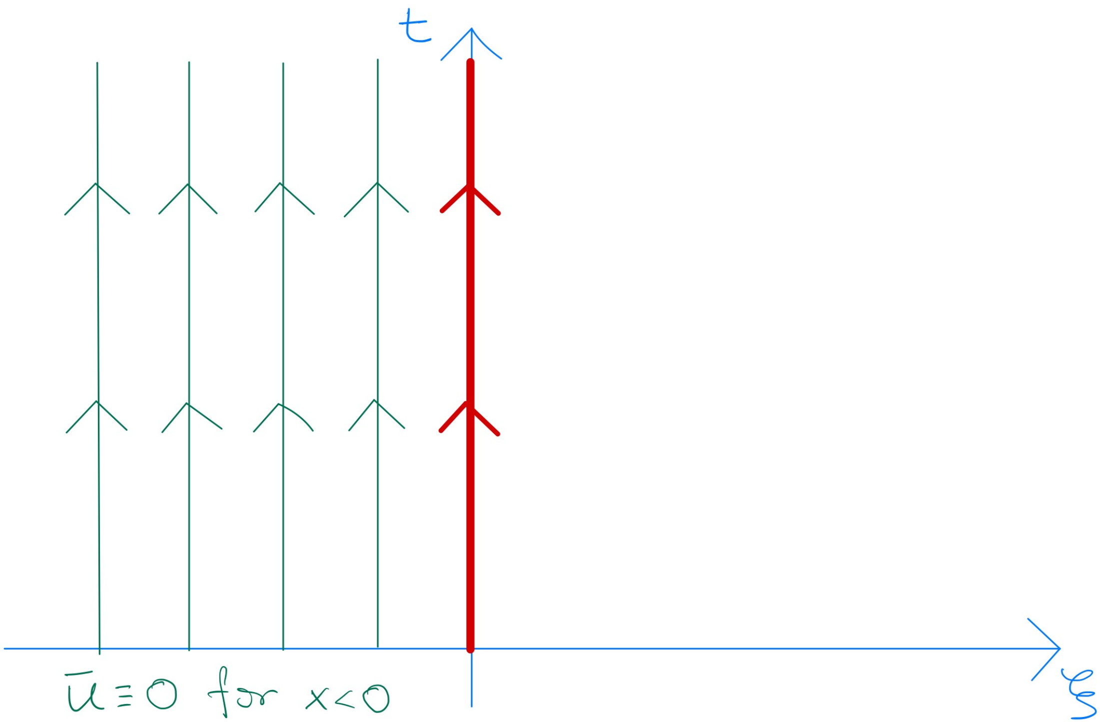
Initial data: $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$.

Since $\bar{u} \equiv 0$ and $\int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y = 0$ for $x < 0$, all the characteristics on the left are vertical straight lines. (Recall: the acceleration at x is $\frac{1}{2}\int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y$.)



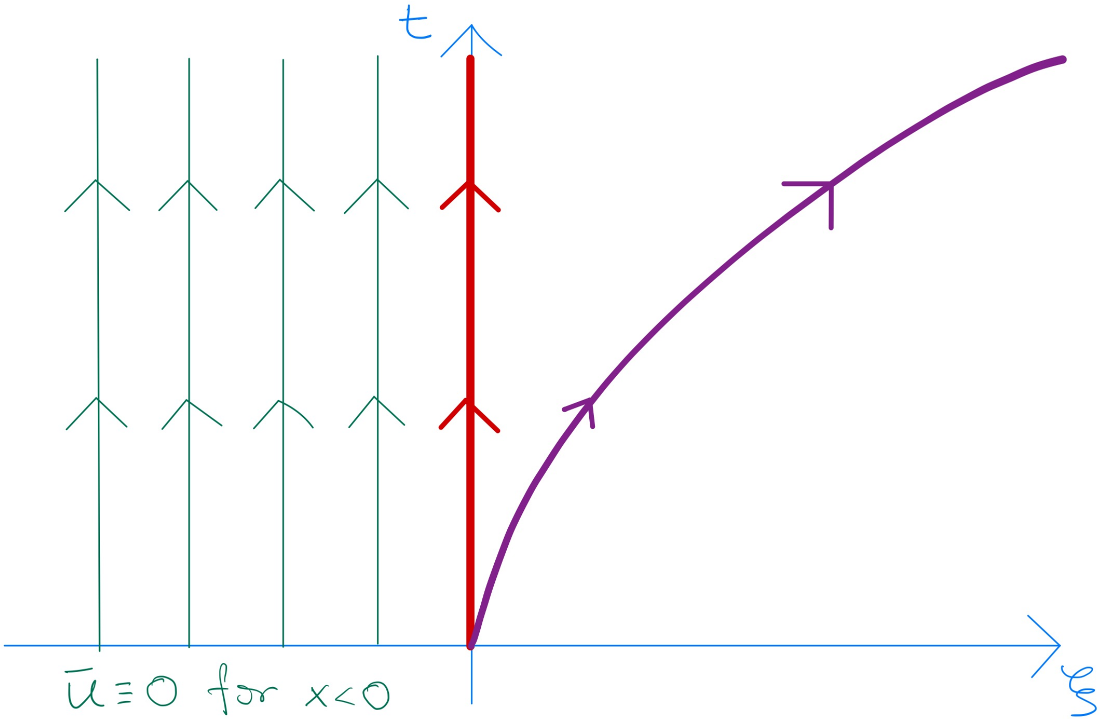
Initial data: $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$.

Since $\bar{u}(0) = 0$ and $\int_{(-\infty,0)} u_y^2(y,t)\,\mathrm{d}y = 0$, the leftmost characteristic curve at $x = 0$ is also a vertical line. (Recall: the acceleration is $\frac{1}{2}\int_{-\infty}^{0} u_y^2(y,t)\,\mathrm{d}y$.)



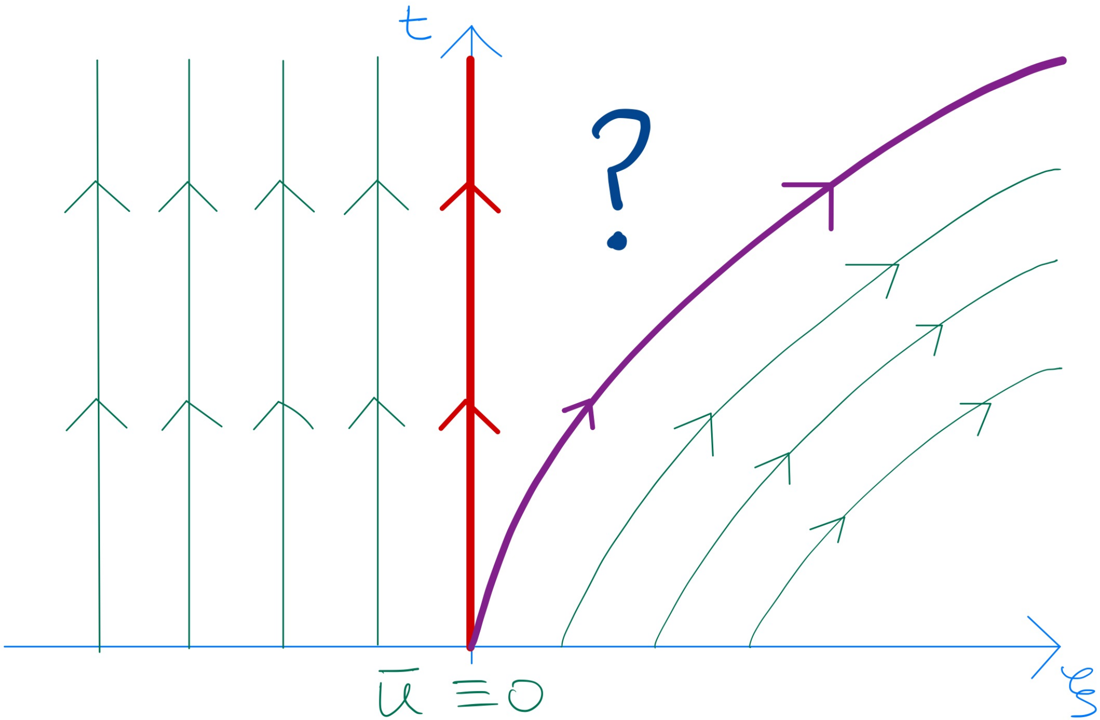
Initial data: $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$.

Since $\bar{u}(0) = 0$ and $\int_{(-\infty,0]} u_y^2(y,t)\,\mathrm{d}y = 1$, the rightmost characteristic curve at $x = 0$ is the purple curve below. (Recall: the acceleration is $\frac{1}{2}\int_{-\infty}^{0} u_y^2(y,t)\,\mathrm{d}y$.)



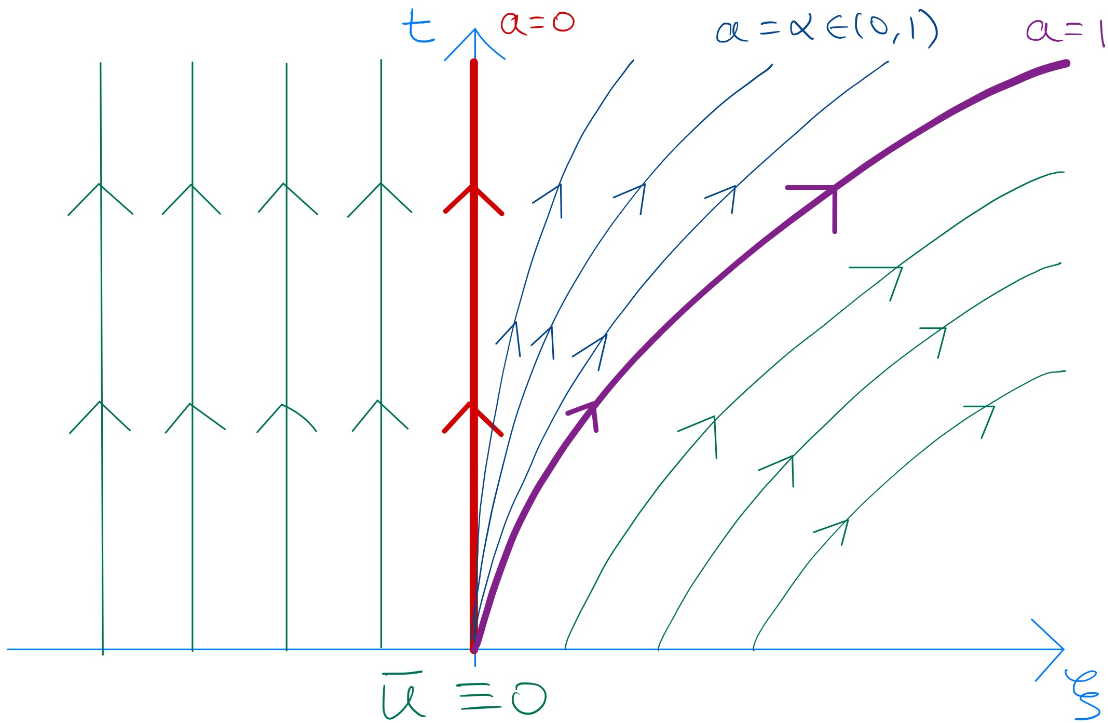
Initial data: $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$.

Since $\bar{u} \equiv 0$ and $\int_{-\infty}^{x} u_y^2(y,t)\,\mathrm{d}y = 1$ for all $x > 0$, all the characteristics on the right are the green curves below. **Question**: What should be in the middle?



Initial data: $\bar{u}$ such that $\bar{u}_x^2 dx = \delta_0$.

Since $\bar{u}(0) = 0$, if we "define" $a := \int_{-\infty}^{0} u_y^2(y,t)\,\mathrm{d}y$ to be any number $\alpha \in (0,1)$, then all the characteristics in the middle are the blue curves below.



Conservative and dissipative solutions

Dissipative and conservative:
Hunter-Zheng (ARMA 1995, ARMA 1995, $\bar{u}_x \in BV(\mathbb{R}^+)$; chopped-BV, method of regularized characteristics, zero-viscosity/dispersion limit for special solutions)
Zhang-Zheng (1998, 1999; $0 \le \bar{u}_x \in L^p(\mathbb{R}^+)$, $p \ge 2$; ARMA 2000; $\bar{u}_x \in L^2(\mathbb{R}^+)$; methods of Young measures)

Dissipative:
Bressan-Constantin (SIMA 2005; existence, uniqueness, stability)
Dafermos (JHDE 2011, 2012; uniqueness, monotone increasing case)
Tieślak-Jamróz (Adv. Math. 2016; uniqueness, general case)

Conservative:
Bressan-Zhang-Zheng (ARMA 2007; existence, uniqueness, stability)
Bressan-Holden-Raynaud (JMPA 2010; Lipschitz stability)
Carrillo-Grunert-Holden (CPDE 2019; Lipschitz stability)
Grunert-Holden (Res. Math. Sci. 2022; uniqueness)

λ-dissipative ($\lambda \in [0,1]$):
Grunert-Tandy (ArXiv 2021; Lipschitz stability)

Outline

Energy measure and generalized framework

Space for (energy) conservative solutions:

Definition

Let $\mathcal{D}$ be the set of pairs (u, μ) satisfying

(i) $u \in C_b(\mathbb{R})$, $u_x \in L^2(\mathbb{R})$;

(ii) $\mu \in \mathcal{M}_+(\mathbb{R})$;

(iii) $\mathrm{d}\mu_{ac} = u_x^2 \,\mathrm{d}x$, where μ_{ac} is the absolutely continuous part of measure μ with respect to the Lebesgue measure $\mathcal{L}$.

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<u>**Generalized Framework**</u>:

$$\begin{cases} u_t + u u_x = \dfrac{1}{2} \displaystyle\int_{-\infty}^{x} \mathrm{d}\mu(t), \\ \mu_t + (u\mu)_x = 0, \\ \mathrm{d}\mu_{ac}(t) = u_x^2(\cdot, t) \, \mathrm{d}x. \end{cases}$$

Outline

Technical problem for flow map $X(\xi, t)$

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ be an initial datum. When $\bar{u}_x^2 \,\mathrm{d}x = \mathrm{d}\bar{\mu}$, the flow map is

$$X(\xi, t) = \xi + \bar{u}(\xi)t + \frac{t^2}{4} \int_{\infty}^{\xi} \bar{u}_x^2(y)\,\mathrm{d}y.$$

For a generic initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, the flow map $X(\xi, t)$ is no longer suitable when $\mathrm{d}\bar{\mu} \neq \mathrm{d}\bar{\mu}_{ac}$.

Technical Problem: how to define the cumulative energy distribution:

$$\int_{-\infty}^{x} \mathrm{d}\bar{\mu}?$$

α-variable system

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Assume $\bar{\mu} = \bar{\mu}_{ac} + \bar{\mu}_{pp} + \bar{\mu}_{sc}$. Here, $\bar{\mu}_{ac}$: absolutely continuous part, $\bar{\mu}_{pp}$: pure point part, $\bar{\mu}_{sc}$: singular continuous part.

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"Flatten" the singular part of $\bar{\mu}$ by **defining** $\bar{x} := \bar{x}(\alpha)$ *as follows*:

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \alpha \leq \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

Remark: *See (Bressan-Zhang-Zhang DCDS 2014; for uniqueness of conservative solutions to Camassa-Holm equaiton).*

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In addition, we define two pseudo-inverses:

$$z_1(x) := \inf\{\alpha : \ \bar{x}(\alpha) = x\}, \quad z_2(x) := \sup\{\alpha : \ \bar{x}(\alpha) = x\},$$

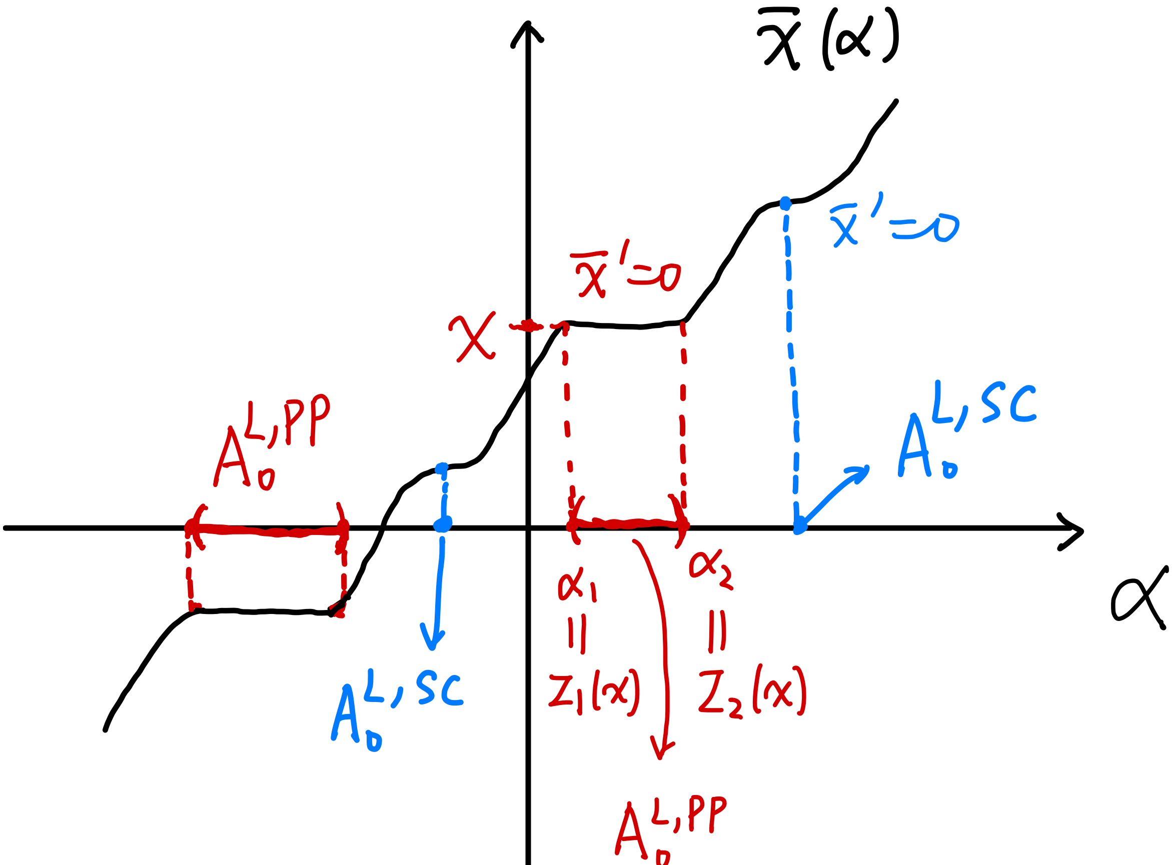
and the following three sets:

$$B_0^L := \{\alpha : \ \ \bar{x}'(\alpha) > 0\},$$

$$A_0^{L,pp} := \{\alpha : \ \bar{x}'(\alpha) = 0, \ z_1(\bar{x}(\alpha)) < z_2(\bar{x}(\alpha))\},$$

$$A_0^{L,sc} := \{\alpha : \ \bar{x}'(\alpha) = 0, \ z_1(\bar{x}(\alpha)) = z_2(\bar{x}(\alpha))\}.$$

Definitions of $\bar{x}$ and related concepts



Proposition

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Then
(i) Lipschitz continuity: $|\bar{x}(\alpha) - \bar{x}(\beta)| \leq |\alpha - \beta|$, for any α, $\beta \in \mathbb{R}$.
(ii) Define $f(\alpha) := 1 - \bar{x}'(\alpha)$. Then $\bar{x}\#(f \, \mathrm{d}\alpha) = \bar{\mu}$, and $\|f\|_{L^1} = \bar{\mu}(\mathbb{R})$.
(iii) Decomposition of $\bar{\mu}$:

$$\bar{\mu}_{pp} = \bar{x}\#(f|_{A_0^{L,pp}} \, \mathrm{d}\alpha), \; \bar{\mu}_{sc} = \bar{x}\#(f|_{A_0^{L,sc}} \, \mathrm{d}\alpha), \; \bar{\mu}_{ac} = \bar{x}\#(f|_{B_0^L} \, \mathrm{d}\alpha).$$

$$\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha) = f(\alpha), \quad \alpha \in B_0^L := \{\alpha : \quad \bar{x}'(\alpha) > 0\}.$$

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Proof of (i).

According to $\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \le \alpha \le \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)])$, for any $\alpha_1 < \alpha_2$,

$$\begin{aligned} 0 \le \bar{x}(\alpha_2) - \bar{x}(\alpha_1) &\le \alpha_2 - \bar{\mu}((-\infty, \bar{x}(\alpha_2))) - \alpha_1 + \bar{\mu}((-\infty, \bar{x}(\alpha_1)]) \\ &\le \alpha_2 - \alpha_1 - \bar{\mu}((\bar{x}(\alpha_1), \bar{x}(\alpha_2))) \le \alpha_2 - \alpha_1. \end{aligned}$$

Hence, $\bar{x}$ is Lipschitz continuous with Lipschitz constant bounded by 1. $\square$

Proposition

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(i) Lipschitz continuity: $|\bar{x}(\alpha) - \bar{x}(\beta)| \le |\alpha - \beta|$, for any $\alpha, \beta \in \mathbb{R}$.
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Proof of $\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha) = f(\alpha)$, for all $\alpha \in B_0^L$.

Let φ be a test function. Then the identify follows immediately by comparing

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\bar{\mu}_{ac} = \int_{\mathbb{R}} \varphi(\bar{x}(\alpha)) f|_{B_0^L}(\alpha)\,\mathrm{d}\alpha, \quad \text{and}$$

$$\int_{\mathbb{R}} \varphi(x)\,\mathrm{d}\bar{\mu}_{ac} = \int_{\mathbb{R}} \varphi(x)\bar{u}_x^2(x)\,\mathrm{d}x = \int_{\mathbb{R}} \varphi(\bar{x}(\alpha))\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha)\,\mathrm{d}\alpha.$$

$\square$

A useful lemma for push forward measures

Lemma (Structure for push forward measures)

Let $X : \mathbb{R} \to \mathbb{R}$ be continuous, increasing, and surjective, and $0 \leq g \in L^1(\mathbb{R})$. Define the measure

$$\mu := X\#(g\,\mathrm{d}\xi),$$

two pseudo-inverses:

$$Z_1(x) := \inf\{\xi : \ X(\xi) = x\}, \ Z_2(x) := \sup\{\xi : \ X(\xi) = x\},$$

the following three sets:

$$\begin{aligned} A^{pp} &:= \{\xi : \ X_\xi(\xi) = 0, \ Z_1(X(\xi)) < Z_2(X(\xi))\}, \\ A^{sc} &:= \{\xi : \ X_\xi(\xi) = 0, \ Z_1(X(\xi)) = Z_2(X(\xi))\}, \\ B &:= \{\xi : \ X_\xi(\xi) > 0\}, \end{aligned}$$

and $g_1 := g \cdot 1_B$, $g_2 := g \cdot 1_{A^{pp}}$, $g_3 := g \cdot 1_{A^{sc}}$. Then

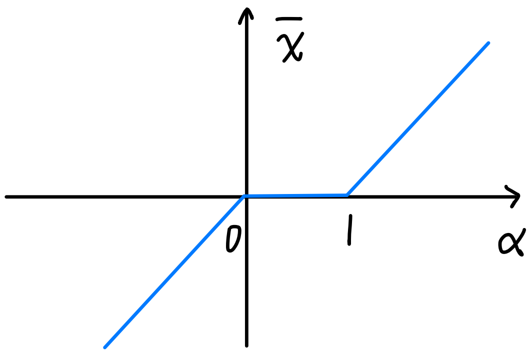
(i) $\mathcal{L}(X(A^{pp} \cup A^{sc})) = 0$, $X(A^{pp})$ *is a countable set;*

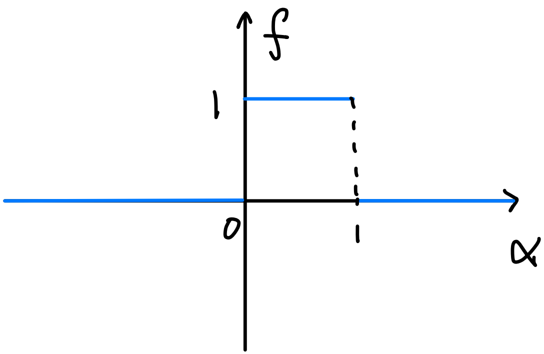
(ii) $\mathrm{d}\mu_{ac} = X\#(g_1\,\mathrm{d}\xi)$, $\mathrm{d}\mu_{pp} = X\#(g_2\,\mathrm{d}\xi)$, $\mathrm{d}\mu_{sc} = X\#(g_3\,\mathrm{d}\xi)$.

An example

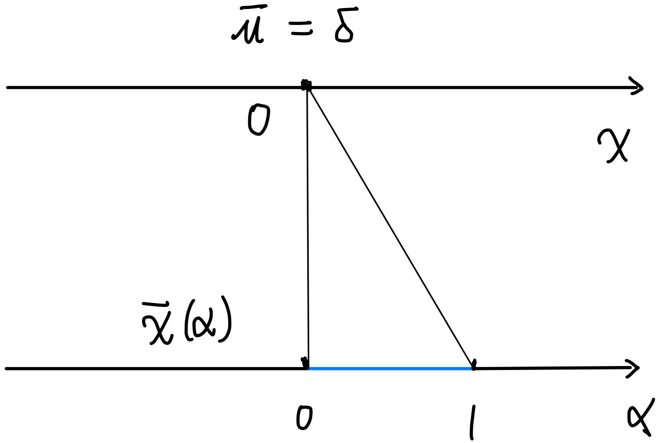
If $\bar{\mu} = \delta_0$, then

$$\bar{x}(\alpha) = \begin{cases} \alpha, & \alpha < 0, \\ 0, & 0 \le \alpha \le 1, \\ \alpha - 1, & \alpha > 1, \end{cases} \qquad f(\alpha) = 1 - \bar{x}'(\alpha) = \begin{cases} 0, & \alpha < 0, \\ 1, & 0 \le \alpha \le 1, \\ 0, & \alpha > 1. \end{cases}$$



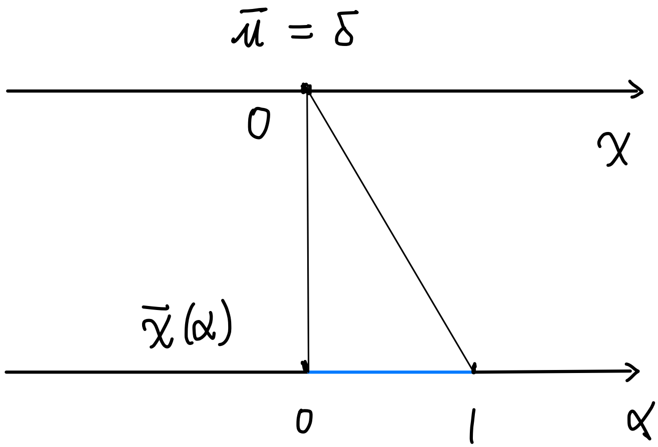


Energy density



$$\begin{cases} f(\alpha) = 1 - \bar{x}'(\alpha) \\ \bar{u} = \bar{x} \# f \end{cases}$$

Energy density



$$\begin{cases} f(\alpha) = 1 - \bar{x}'(\alpha) \\ \bar{\mu} = \bar{x} \# f \end{cases}$$

Recall Technical Problem: how to define the cumulative energy distribution: $\int_{-\infty}^{x} \mathrm{d}\bar{\mu}$?

Function f: the energy density in the α-variable, and the cumulative energy distribution is given by $\int_{(-\infty,\alpha)} f(\alpha)\,\mathrm{d}\alpha = \alpha - \bar{x}(\alpha)$.

In this example, for $\alpha \in [0,1]$, we have $\int_{(-\infty,\alpha)} f(\alpha)\,\mathrm{d}\alpha = \alpha$.

Ideas for global characteristics in α-variable

Formal calculations for a smooth solution u:

For any $\beta \in \mathbb{R}$, we define $x(\beta, t)$ by

$$x(\beta, t) + \int_{(-\infty, x(\beta,t))} u_x^2(y, t)\,\mathrm{d}y = \beta, \quad t \in \mathbb{R}.$$

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Then a direct computation yields

$$\partial_t x(\beta, t) = \frac{(uu_x^2)(x(\beta, t), t)}{1 + u_x^2(x(\beta, t), t)}, \qquad \partial_\beta x(\beta, t) = \frac{1}{1 + u_x^2(x(\beta, t), t)},$$

Define functions:

$$\begin{cases} U(\beta, t) = u(x(\beta, t), t) \\ H(\beta, t) = \beta - x(\beta, t). \end{cases}$$

Ideas for global characteristics in α-variable

Define

$$\begin{cases} U(\beta,t) = u(x(\beta,t),t) \\ H(\beta,t) = \beta - x(\beta,t). \end{cases}$$

Formally, this provides the following system (Grunert-Holden, 2022):

$$\begin{cases} x_t(\beta,t) + Ux_\beta(\beta,t) = U(\beta,t), \\ H_t(\beta,t) + UH_\beta(\beta,t) = 0, \\ U_t(\beta,t) + UU_\beta(\beta,t) = \dfrac{1}{2}H(\beta,t). \end{cases}$$

Ideas for global characteristics in α-variable

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Drawbacks:

- No explicit formula,
- non-uniqueness,
- $$x_t(\beta, t) \neq u(x(\beta, t), t) \Rightarrow \int_{(-\infty, x(\beta, t))} u_x^2(y, t) \, \mathrm{d}y \text{ is not conserved.}$$

Global characteristics in α-variable

Crucial Idea: *Allow β to also move with respect to the time t!!*

Introduce a reformulation function $\beta(t)$ with $\beta(0) = \alpha$ such that

$$x(\beta(t), t) + \int_{-\infty}^{x(\beta(t),t)} u_x^2(y, t)\,\mathrm{d}y = \beta(t), \quad t \in \mathbb{R},$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t} x(\beta(t), t) = u(x(\beta(t), t), t).$$

Global characteristics in α-variable

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and

$$\frac{\mathrm{d}}{\mathrm{d}t} x(\beta(t), t) = u(x(\beta(t), t), t).$$

Then

$$\frac{\mathrm{d}}{\mathrm{d}t}[\beta(t) - x(\beta(t), t)] = \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^{x(\beta(t),t)} u_x^2(y, t)\,\mathrm{d}y = 0,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\beta(t) = \frac{\mathrm{d}}{\mathrm{d}t} u(x(\beta(t), t), t) = \frac{1}{2} \int_{-\infty}^{x(\beta(t),t)} u_x^2(y, t)\,\mathrm{d}y = \frac{1}{2}\left[\beta(t) - x(\beta(t), t)\right].$$

Global characteristics in α-variable

Hence,

$$\frac{\mathrm{d}^3}{\mathrm{d}t^3}x(\beta(t),t) = \frac{\mathrm{d}^3}{\mathrm{d}t^3}\beta(t) = 0.$$

Furthermore, we also have the following initial data:

$$x(\beta(0),0) = \bar{x}(\alpha), \quad \frac{\mathrm{d}}{\mathrm{d}t}x(\beta(t),t)\Big|_{t=0} = \bar{u}(\bar{x}(\alpha)),$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}x(\beta(t),t)\Big|_{t=0} = \frac{1}{2}\int_{-\infty}^{\bar{x}(\alpha)} \bar{u}_x^2(y)\,\mathrm{d}y = \frac{1}{2}\left[\alpha - \bar{x}(\alpha)\right].$$

Global characteristics:

$$y(\alpha,t) := x(\beta(t),t) = \bar{x}(\alpha) + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)).$$

Advantages:

1. We only need information of initial datum;
2. The formula (for y) can be generalized to any $(\bar{u},\bar{\mu}) \in \mathcal{D}$.

From global $y(\alpha, t)$ to global $(u(t), \mu(t))$:

$$\begin{cases} u(x,t) = \dfrac{\partial}{\partial t} y(\alpha, t) = \bar{u}(\bar{x}(\alpha)) + \dfrac{t}{2}(\alpha - \bar{x}(\alpha)) \ \text{ for } \ x = y(\alpha, t), \\ \mu(t) = y(\cdot, t)\#(f \, \mathrm{d}\alpha), \quad f(\alpha) = 1 - \bar{x}'(\alpha). \end{cases}$$

From global $y(\alpha, t)$ to global $(u(t), \mu(t))$:

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Moral

For a generic initial data $(\bar{u}, \bar{\mu}) \in \mathcal{D}$, we "define" the conservative solution as follows:

1. We can still define $\bar{x}$ via

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \alpha \leq \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

2. Using this $\bar{x}$, we can define $\beta(t)$ and y by

$$\beta(t) := \alpha + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)),$$

$$y(\alpha, t) := \bar{x}(\alpha) + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)).$$

3. Finally, we can obtain u and μ by the above formulae.

Recovery of solution $(u(t), \mu(t))$

From global $y(\alpha, t)$ to global $(u(t), \mu(t))$:

$$\begin{cases} u(x,t) = \dfrac{\partial}{\partial t} y(\alpha,t) = \bar{u}(\bar{x}(\alpha)) + \dfrac{t}{2}(\alpha - \bar{x}(\alpha)) \ \text{ for } \ x = y(\alpha,t), \\ \mu(t) = y(\cdot,t)\#(f\,\mathrm{d}\alpha), \quad f(\alpha) = 1 - \bar{x}'(\alpha). \end{cases}$$

Two pseudo-inverses:

$$z_1(x,t) = \inf\{\alpha : \ y(\alpha,t) = x\}, \ z_2(x,t) = \sup\{\alpha : \ y(\alpha,t) = x\}.$$

Three sets:

$$\begin{aligned} A_t^{L,pp} &= \{\alpha : \ y_\alpha(\alpha,t) = 0, \ z_1(y(\alpha,t),t) < z_2(y(\alpha,t),t)\}, \\ A_t^{L,sc} &= \{\alpha : \ y_\alpha(\alpha,t) = 0, \ z_1(y(\alpha,t),t) = z_2(y(\alpha,t),t)\}, \\ B_t^L &= \{\alpha : \ y_\alpha(\alpha,t) > 0\}. \end{aligned}$$

Outline

Properties of $\mu(t)$

Theorem (Gao-Liu-W. 2022, *SIAM J. Math. Anal.*)

(i) Energy conservation: $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$ *and* $\mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R})$, $t \in \mathbb{R}$.
(ii) Decomposition: $\mu_{pp}(t) = y(\cdot, t)\#(f|_{A_t^{L,pp}}\,\mathrm{d}\alpha)$, $\mu_{sc}(t) = y(\cdot, t)\#(f|_{A_t^{L,sc}}\,\mathrm{d}\alpha)$,

$$\mu_{ac}(t) = y(\cdot, t)\#(f|_{B_t^L}\,\mathrm{d}\alpha).$$

Coordinates cause singular sets:

$$A_t^L := A_t^{L,pp} \cup A_t^{L,sc} = \left\{\alpha \in B_0^L : \quad \bar{u}_x(\bar{x}(\alpha)) = -\frac{2}{t}\right\}, \quad t \in \mathbb{R}.$$

Support of singular parts:

$$\mathrm{supp}\Big(\mu_{pp}(t) + \mu_{sc}(t)\Big) \subset \left\{x + \bar{u}(x)t + \frac{t^2}{4}\bar{\mu}((-\infty, x)) : \ x \in A_t^E = \bar{x}(A_t^L)\right\}.$$

(iii) Countably many time $t \in \mathbb{R}$ *for singular measures:* $T_p := \{t : \ \mu_{pp}(t) \neq 0\}$ *and* $T_s := \{t : \ \mu_{sc}(t) \neq 0\}$ *are countable.*

Key ingredients for proof

1. Energy density in the α-variable:

$$f(\alpha) = 1 - \bar{x}'(\alpha) = \begin{cases} \bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha), & \alpha \in B_0^L, \\ 1, \quad \alpha \in A_0^L = A_0^{L,pp} \cup A_0^{L,sc}, \end{cases}$$

2. Derivative of $y(\alpha, t)$:

$$y_\alpha(\alpha, t) = \begin{cases} \bar{x}'(\alpha)\left[1 + \frac{t}{2}\bar{u}_x(\bar{x}(\alpha))\right]^2, & \alpha \in B_0^L, \\ \frac{t^2}{4}, \quad \alpha \in A_0^L. \end{cases}$$

3. The real line $\mathbb{R}$ cannot be written as the union of uncountably many disjoint subsets with positive measures.

Mass for singular measures

Recall: $\bar{u}_x^2(\bar{x}(\alpha))\bar{x}'(\alpha) = f(\alpha)$ for $\alpha \in B_0^L$.

Remark

- Mass of pure point parts: $\bar{x}'(\alpha) = \frac{t^2}{t^2+4}$ and $f(\alpha) = \frac{4}{t^2+4}$ on A_t^L. We consider a point $x_0 \in y(A_t^{L,pp}, t)$ and $\alpha_1 := z_1(x_0, t) < z_2(x_0, t) =: \alpha_2$. Then, $\bar{u}_x(x) = -\frac{2}{t}$ for all $x \in [x_1, x_2] := [\bar{x}(\alpha_1), \bar{x}(\alpha_2)]$. The mass concentrated at x is calculated by

$$\mu(t)(\{x_0\}) = \int_{[\alpha_1,\alpha_2]} f(\alpha)\,\mathrm{d}\alpha = \frac{4}{t^2+4}(\alpha_2 - \alpha_1) = \frac{4}{t^2}(x_2 - x_1).$$

- Mass of singular continuous part: define $A_t^{E,sc} = \bar{x}(A_t^{L,sc})$ and then $\bar{u}_x(x) = -\frac{2}{t}$ for $x \in A_t^{E,sc}$. We have

$$\mu_{sc}(t)(\mathbb{R}) = \frac{4}{t^2}\mathcal{L}(A_t^{E,sc}).$$

Properties of $u(t)$

Theorem (Gao-Liu-W. 2022, *SIAM J. Math. Anal.*)

(iv) For all time $t \in \mathbb{R}$, the function $u(\cdot, t)$ is globally absolutely continuous and

$$\mathrm{d}\mu_{ac}(t) = u_x^2(x, t)\,\mathrm{d}x.$$

Moreover,

$$u \in C(\mathbb{R}; C_b(\mathbb{R})) \cap C^{1/2}_{loc}(\mathbb{R} \times \mathbb{R}), \quad u_x \in L^\infty(\mathbb{R}; L^2(\mathbb{R})), \quad u_t \in L^2_{loc}(\mathbb{R} \times \mathbb{R}).$$

(v) If $\bar{u}(-\infty) := \lim_{x \to -\infty} \bar{u}(x)$ exists, then we have

$$\lim_{x \to -\infty} u(x, t) = \bar{u}(-\infty).$$

On the other hand, if $\bar{u}(+\infty) := \lim_{x \to +\infty} \bar{u}(x)$ exists, then we also have

$$\lim_{x \to +\infty} u(x, t) = \bar{u}(+\infty) + \frac{1}{2}\bar{\mu}(\mathbb{R})t.$$

Key ingredients for proof

Using the formulae for $u(y(\alpha, t), t)$ and $y(\alpha, t)$, one can easily show that

$$u_x^2(y(\alpha, t), t)y_\alpha(\alpha, t) = f(\alpha), \quad \alpha \in B_t^L.$$

Remark

Usually $u_x \notin C(\mathbb{R}; L^2(\mathbb{R}))$, since

$$\int_{\mathbb{R}} u_x^2(x, t)\,\mathrm{d}x = \mu_{ac}(t)(\mathbb{R}) < \mu(t)(\mathbb{R}) = \bar{\mu}(\mathbb{R}), \quad \text{for all } t \in T_s \cup T_p.$$

Relation between $y(\alpha, t)$ and $X(\xi, t)$

Theorem (Gao-Liu-W. 2022, *SIAM J. Math. Anal.*)

(vi) Consider a time $s \in \mathbb{R}$ such that $\mu(s)$ is absolutely continuous with respect to the Lebesgue measure. Let $\tilde{u}(x) = u(x, s)$, and $X(\xi, t)$ be defined by $\tilde{u}$ via

$$X(\xi, t) = \xi + \tilde{u}(\xi)t + \frac{t^2}{4}\int_{-\infty}^{\xi} \tilde{u}_x^2(y)\,\mathrm{d}y.$$

Then we have

$$\tilde{u} \in C_b(\mathbb{R}), \quad \tilde{u}_x \in L^2(\mathbb{R}), \quad \|\tilde{u}_x^2\|_{L^1} = \bar{\mu}(\mathbb{R}).$$

For any $t \in \mathbb{R}$, we also have

$$y(\cdot, t) = X(\cdot, t - s) \circ y(\cdot, s), \quad \mu(t) = X(\cdot, t - s)\#(\tilde{u}_x^2\,\mathrm{d}x),$$

and (denoting $\tilde{F}(\xi) := \int_{-\infty}^{\xi} \tilde{u}_x^2(y)\,\mathrm{d}y$)

$$u(x, t) = \frac{\partial}{\partial t}X(\xi, t - s) = \tilde{u}(\xi) + \frac{(t - s)}{2}\tilde{F}(\xi), \quad \textit{for } x = X(\xi, t - s).$$

Outline

Generalized Framework:

$$\begin{cases} u_t + uu_x = \dfrac{1}{2}\displaystyle\int_{-\infty}^{x} \mathrm{d}\mu(t), \\ \mu_t + (u\mu)_x = 0, \\ \mathrm{d}\mu_{ac}(t) = u_x^2(\cdot,t)\,\mathrm{d}x. \end{cases}$$

Definition (Conservative solutions)

For initial datum $(\bar{u},\bar{\mu}) \in \mathcal{D}$, $(u(t),\mu(t))$ is said to be a conservative solution if

(i) $u \in C(\mathbb{R}; C_b(\mathbb{R})) \cap C^{1/2}_{loc}(\mathbb{R}\times\mathbb{R})$, $u_t \in L^2_{loc}(\mathbb{R}\times\mathbb{R})$, $u_x(\cdot,t) \in L^2(\mathbb{R})$ for all $t \in \mathbb{R}$, and $\mu \in C(\mathbb{R}; \mathcal{M}_+(\mathbb{R}))$;

(ii) $(u(\cdot,0),\mu(0)) = (\bar{u},\bar{\mu})$, and $\mathrm{d}\mu(t) = u_x^2(x,t)\,\mathrm{d}x$ for a.e. $t \in \mathbb{R}$;

(iii) the equations

$$\int_{\mathbb{R}}\int_{\mathbb{R}} u\phi_t - \phi\left(uu_x - \frac{1}{2}F\right)\mathrm{d}x\,\mathrm{d}t = 0, \quad \int_{\mathbb{R}}\int_{\mathbb{R}} (\phi_t + u\phi_x)\,\mathrm{d}\mu(t)\,\mathrm{d}t = 0$$

hold for all $\phi \in C_c^\infty(\mathbb{R}\times\mathbb{R})$ and $F(x,t) := \int_{-\infty}^{x} \mathrm{d}\mu(t)$;

(iv) $\mathrm{d}\mu_{ac}(t) = u_x^2(\cdot,t)\,\mathrm{d}x$ for all $t \in \mathbb{R}$.

Existence

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Define $\bar{x}$ via

$$\bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha))) \leq \alpha \leq \bar{x}(\alpha) + \bar{\mu}((-\infty, \bar{x}(\alpha)]).$$

Then define

$$y(\alpha, t) := \bar{x}(\alpha) + \bar{u}(\bar{x}(\alpha))t + \frac{t^2}{4}(\alpha - \bar{x}(\alpha)),$$

and

$$\begin{cases} u(x,t) := \dfrac{\partial}{\partial t} y(\alpha, t) = \bar{u}(\bar{x}(\alpha)) + \dfrac{t}{2}(\alpha - \bar{x}(\alpha)) \ \text{ for } \ x = y(\alpha, t), \\ \mu(t) := y(\cdot, t)\#(f \, \mathrm{d}\alpha), \quad f(\alpha) := 1 - \bar{x}'(\alpha). \end{cases}$$

Theorem (Existence; Gao-Liu-W. 2022, *SIAM J. Math. Anal.*)

Let $(\bar{u}, \bar{\mu}) \in \mathcal{D}$ be an initial datum. Let (u, μ) be defined as above. Then, $(u(t), \mu(t))$ is a global-in-time conservative solution to the generalized framework of Hunter-Saxton equation subject to the initial datum $(\bar{u}, \bar{\mu})$.

Outline

Uniqueness via characteristics

Theorem (Uniqueness of characteristics and conservative solutions; Gao-Liu-W. 2022, *SIAM J. Math. Anal.*)

Let (v, ν) be a conservative solution to the generalized framework of Hunter-Saxton equation subject to an initial datum $(\bar{u}, \bar{\mu}) \in \mathcal{D}$. Then there exists a unique characteristic $y_1(\alpha, t)$ satisfying

$$\frac{\partial}{\partial t} y_1(\alpha, t) = v(y_1(\alpha, t), t), \quad y_1(\alpha, 0) = \bar{x}(\alpha),$$

and

$$\nu(t)((-\infty, y_1(\alpha, t))) \leq \alpha - \bar{x}(\alpha) \leq \nu(t)((-\infty, y_1(\alpha, t)]),$$

for any $\alpha \in \mathbb{R}$ and a.e. $t \in \mathbb{R}$. The uniqueness of characteristics and conservative solutions follows, i.e., $(v, \nu) = (u, \mu)$, where (u, μ) is constructed in the existence theorem.

Difficulty: $v \in C_b(\mathbb{R})$, $v_x \in L^2(\mathbb{R})$, function v is not Lipschitz.

Key lemma

Lemma

Let (v, ν) be a conservative solution to the HS equation. Consider the time t and τ such that ν is absolutely continuous. Then for any fixed $y \in \mathbb{R}$ and $\varepsilon_0 > 0$,

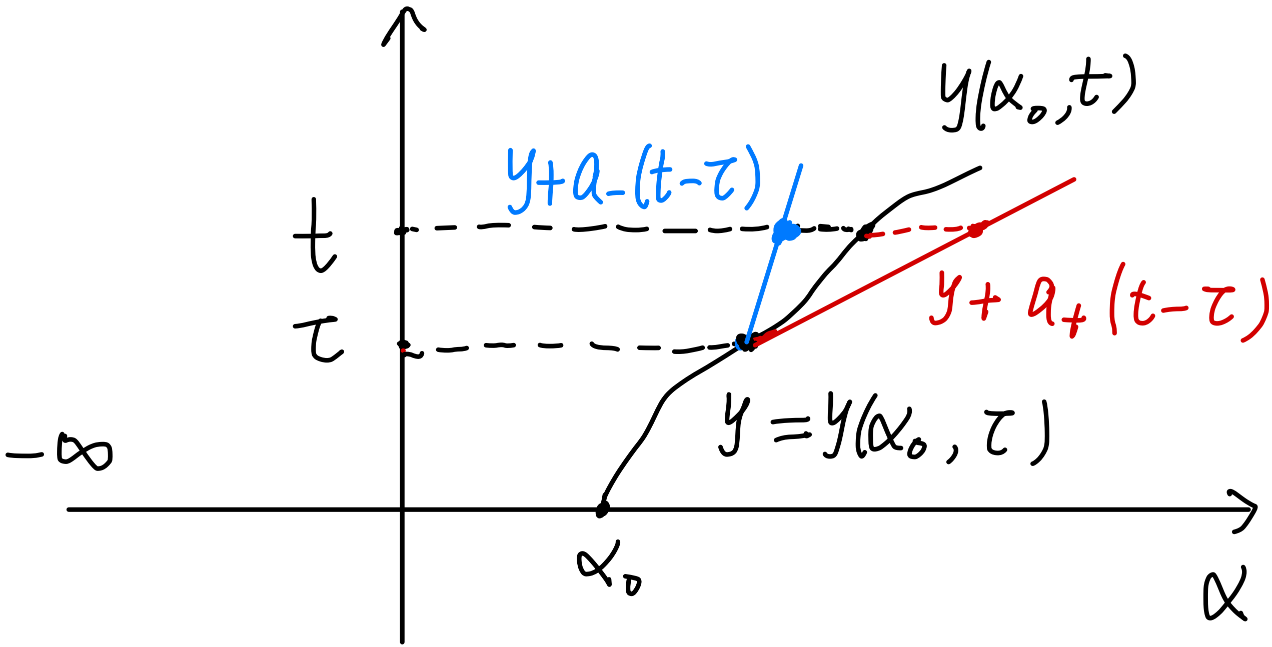
$$\int_{(-\infty, y+a_-(t-\tau))} v_x^2(x,t)\,\mathrm{d}x \leq \int_{(-\infty,y)} v_x^2(x,\tau)\,\mathrm{d}x \leq \int_{(-\infty, y+a_+(t-\tau))} v_x^2(x,t)\,\mathrm{d}x,$$

provided that $t - \tau > 0$ is small enough (depending on v, y and ε_0), where $a_\pm := v(y,\tau) \pm \varepsilon_0$. Moreover, for any $T > 0$ and any $-T \leq \tau < t \leq T$,

$$\int_{(-\infty, y-C_T(t-\tau))} v_x^2(x,t)\,\mathrm{d}x \leq \int_{(-\infty,y)} v_x^2(x,\tau)\,\mathrm{d}x \leq \int_{(-\infty, y+C_T(t-\tau))} v_x^2(x,t)\,\mathrm{d}x,$$

for all C_T satisfying $\|v\|_{C_b(\mathbb{R}\times[-T,T])} \leq C_T$.

Ideas for the proof of the lemma:



$$a_\pm = v(y, \tau) \pm \varepsilon_0$$

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Thank you!