On the stability of a point charge for the Vlasov-Poisson system

SITE online conference: Long Time Behavior and Singularity Formation in PDEs

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June 1, 2022

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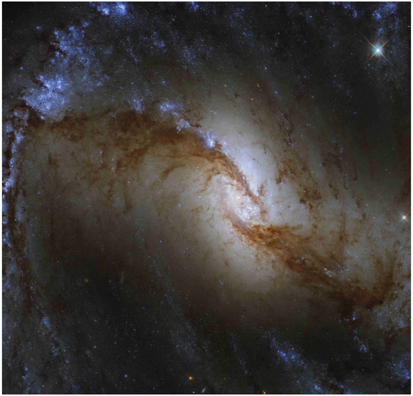


The Vlasov-Poisson equations

Continuum description of classical N-body problem as $N \to \infty$:

particle distribution $f(x, v, t) \geq 0$, as a function of time $t \in \mathbb{R}$, position $x \in \mathbb{R}^3$ and velocity $v \in \mathbb{R}^3$

$$\partial_t f + v \cdot \nabla_x f - \lambda \nabla_x \phi \cdot \nabla_v f = 0, \quad \Delta_x \phi(x,t) = \int f(x,v,t) dv,$$

- $\lambda > 0$: attractive interactions / gravitational case,
  - ▶ stationary states: many,
- $\lambda < 0$: repulsive interactions / plasma case,
  - ▶ stationary states: no smooth, localized.

▶ Global solutions? Yes.

[Batt, Horst, Bardos-Degond, Pfaffelmoser, Schaeffer, Lions-Perthame,. . .]

▶ Asymptotic behavior? Largely open.

Asymptotic dynamics on $\mathbb{R}^3$

- linear / orbital stability of stationary solutions,

  [Jeans, Bernstein-Greene-Kruskal, Guo, Lin, Rein, Lemou-Méhats-Raphaël, Hadžić-Rein-Straub, Bedrossian-Masmoudi-Mouhot, Han-Kwan-Nguyen-Rousset. . .]

Asymptotic behavior / stability only known near:

1. vacuum for small, dilute gases – modified scattering

   [Choi-Kwon, Hwang-Rendall-Velazquez, Smulevici,. . . , Ionescu-Pausader-Wang-W., Pankavich, Flynn-Ouyang-Pausader-W.]

2. homogeneous "Poisson" equilibrium – linear scattering ("Landau damping") [Ionescu-Pausader-Wang-W.]

   [$\mathbb{T}^d$: Mouhot-Villani, Bedrossian-Masmoudi-Mouhot, Grenier-Nguyen-Rodnianski]

3. **repulsive point charge** – modified scattering

   [Pausader-W., Pausader-W.-Yang]

Mechanism of stability on $\mathbb{R}^3$: dispersion

In linear approximation, a small distribution streams freely

$$(\partial_t + v \cdot \nabla_x) f = 0 \qquad \Rightarrow \quad f(x,v,t) = f_0(x - tv, v).$$

A smooth distribution of particles gets increasingly diluted:

$$\begin{aligned} \rho(x,t) := \int f(x,v,t) dv = t^{-3} \int f_0(a, \frac{x-a}{t}) da \\ = t^{-3} \int f_0(a, \frac{x}{t}) da + O(t^{-4+}). \end{aligned}$$

Expect: $\boldsymbol{F} = \pm \nabla \Delta^{-1} \rho \to 0$. (False for a point particle $f = \delta_{(\mathcal{X}(t), \mathcal{V}(t))}$.)

However: Nonlinear effects remain relevant throughout evolution

Point Charge in Vlasov-Poisson

A point charge in Vlasov-Poisson

▶ Question: **Stability** of a point charge $f_{eq} = q_c \delta_{(0,0)}(x, v)$?

Track solution as

$$f(x, v, t) = q_c \delta_{(\mathcal{X}(t), \mathcal{V}(t))} + q_g \mu^2(x, v, t) dx dv.$$

→ yields:

$$\left(\partial_t + v \cdot \nabla_x + \frac{q}{2} \frac{x - \mathcal{X}(t)}{|x - \mathcal{X}(t)|^3} \cdot \nabla_v \right) \mu + \lambda \nabla_x \psi \cdot \nabla_v \mu = 0,$$

$$\frac{d\mathcal{X}}{dt} = \mathcal{V}, \qquad \frac{d\mathcal{V}}{dt} = \bar{q} \nabla_x \psi(\mathcal{X}), \qquad \Delta_x \psi = \int_{\mathbb{R}^3_v} \mu^2 dv, \tag{VP}$$

with $\lambda, q, \bar{q} > 0$ – **repulsive**.

- [Marchioro-Miot-Pulvirenti '11]: global strong solutions under support restriction
- [Desvillettes-Miot-Saffirio '15]: global weak solutions under less support restriction
- [Crippa-Ligabue-Saffirio '18]: global "Lagrangian" solutions under less support restriction

Main Result

Theorem **[Pausader-W.-Yang '22, in progress]**

Given $(\mathcal{X}_0, \mathcal{V}_0) \in \mathbb{R}^3 \times \mathbb{R}^3$ and $\mu_0 \in C^1_c(\mathbb{R}^3 \setminus \{\mathcal{X}_0\} \times \mathbb{R}^3)$, there exists $\varepsilon^* > 0$ such that for any $0 < \varepsilon < \varepsilon^*$, there exists a unique global strong solution of (VP) with repulsive interactions and initial data

$$(\mathcal{X}(t=0), \mathcal{V}(t=0)) = (\mathcal{X}_0, \mathcal{V}_0), \qquad \mu(t=0) = \varepsilon\mu_0.$$

Moreover, we have precise asymptotics as $t \to \infty$:

$$\nabla_x \psi(t) \sim \frac{1}{t^2}\mathcal{E}_\infty,\ \mu(Y,W,t) \sim \mu_\infty(x,v),\ \mathcal{X}(t) \sim \mathcal{X}_\infty + t\mathcal{V}_\infty + \ln(t)\mathcal{C}_\infty.$$

1. More precise and less restrictive in "action-angle" variables. E.g.:
   - can allow for μ_0 supported near $(\mathcal{X}_0, \mathcal{V}_0)$,
   - can allow for unbounded support $\mathrm{supp}(\mu_0) = \mathbb{R}^3 \times \mathbb{R}^3$.
2. If $\mu_0 \in C^0_c$, then get a global solution with almost sharp decay.

Main Result (cont'd)

❸ **Radial case** **[Pausader-W. '20]**:
If $\mathcal{X}_0 = \mathcal{V}_0 = 0$ and μ_0 radial, then $\mathcal{X}(t) = \mathcal{V}(t) = 0$.

▷ Radial phase space $(r, v) \in \mathbb{R}_+^* \times \mathbb{R}$: For $\boldsymbol{\mu}(r, v, t) := rv\mu(r, v, t)$ get

$$\left(\partial_t + v\partial_r + \frac{q}{2r^2}\partial_v\right)\boldsymbol{\mu} = \lambda \boldsymbol{E}\partial_v \boldsymbol{\mu},$$

$$\boldsymbol{E}(r, t) := -\partial_r \psi(r, t) = \frac{1}{r^2}\int_{s=0}^{r} \boldsymbol{\varrho}(s, t)\, ds,$$

$$\boldsymbol{\varrho}(s, t) := \int \boldsymbol{\mu}^2(s, v, t)\, dv.$$

Main Result (cont'd)

3. **Radial case** **[Pausader-W. '20]**:
If $\mathcal{X}_0 = \mathcal{V}_0 = 0$ and μ_0 radial, then $\mathcal{X}(t) = \mathcal{V}(t) = 0$.

Then the electric field decays pointwise and there exists an asymptotic profile $\mu_\infty \in L^2(\mathbb{R}_+^* \times \mathbb{R})$ and a Lagrangian map (Y, W) such that

$$\mu(Y, W, t) \to \mu_\infty(r, v), \qquad t \to \infty.$$

Here, in terms of asymptotic "electric field profile" $\mathcal{E}_\infty$:

$$Y(r, v, t) \sim t\sqrt{v^2 + \frac{q}{r}} - \frac{rq}{2(q + rv^2)} \ln(t) + \lambda \mathcal{E}_\infty(\sqrt{v^2 + \frac{q}{r}}) \ln(t),$$

$$W(r, v, t) \sim \sqrt{v^2 + \frac{q}{r}} - \frac{rq}{2(q + rv^2)} \frac{1}{t}.$$

Proof strategy: method of asymptotic actions

Based on Hamiltonian structure:

$$(\mathrm{VP}) \quad \Leftrightarrow \quad \partial_t \mu + \{\mathcal{H}_0 + \mathcal{H}_{pert}, \mu\} = 0,$$

with $\mathcal{H}_0$ linearized Hamiltonian, $\mathcal{H}_{pert}$ from electrostatic potential.

1. **Lagrangian** analysis of linearized equation: can integrate flow of $\mathcal{H}_0$ exactly via "action-angle" variables,
2. **Eulerian** analysis of nonlinear equation:
   bootstrap in PDE framework (L^2 based, dispersive)
   - global solutions with almost sharp decay via energy estimates / propagation of moments,
   - sharp decay via propagation of derivative control,
   - asymptotic behavior via "mixing" mechanism.

Some guiding principles

to abide by:

- Use symplectic structure (Poisson brackets…) as much as possible. In particular, only use canonical transformations.
- Only integrate over all phase space $\iint d\boldsymbol{x}d\boldsymbol{v}$.
  (No role for density $\rho(t,\boldsymbol{x})$ or scattering mass $m(t,\boldsymbol{v})\ldots$)
- Rely on conversation laws of the linearized ODE as much as possible.

Linearized Equation & Action-Angle Coordinates

Linearized Equation

Linearization of (VP):

$$(\partial_t + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} + q\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \cdot \nabla_{\boldsymbol{v}})\mu = 0 \quad \Leftrightarrow \quad \partial_t \mu + \{\mathcal{H}_0, \mu\} = 0, \quad (\mathrm{VP}_{lin})$$

with $\mathcal{H}_0 = \frac{|\boldsymbol{v}|^2}{2} + \frac{q}{|\boldsymbol{x}|}$ linear Hamiltonian.

▶ transport by flow of repulsive two-body problem [Newton 1687]

$$\dot{\boldsymbol{x}} = \boldsymbol{v}, \qquad \dot{\boldsymbol{v}} = q\frac{\boldsymbol{x}}{|\boldsymbol{x}|^3}, \qquad (\mathrm{ODE})$$

▶ super-integrable (!): 5 scalar conserved quantities

$$\mathcal{H}_0 = \frac{|\boldsymbol{v}|^2}{2} + \frac{q}{|\boldsymbol{x}|}, \qquad \boldsymbol{L} = \boldsymbol{x} \times \boldsymbol{v}, \qquad \boldsymbol{R} = \boldsymbol{v} \times \boldsymbol{L} + q\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$$

▶ trajectories easy to parameterize in the plane; more difficult in general.

Asymptotic action-angle

We are looking for a set of **asymptotic action-angle** coordinates $\mathcal{T} : (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\vartheta, \boldsymbol{a})$ such that

1. $\mathcal{T}$ is canonical $d\boldsymbol{x} \wedge d\boldsymbol{v} = d\vartheta \wedge d\boldsymbol{a}$,
2. $\mathcal{T}$ integrates linearized equation: for ODE trajectory $(\vartheta(t), \mathbf{a}(t))$

$$\dot{\vartheta} = \boldsymbol{a}, \quad \dot{\boldsymbol{a}} = 0 \quad \Leftrightarrow \quad (\boldsymbol{x}, \boldsymbol{v})(t) = (\boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}), \boldsymbol{V}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}))$$

or

$$\Theta(\boldsymbol{x}(t), \boldsymbol{v}(t)) = \Theta(\mathbf{x}_0, \boldsymbol{v}_0) + t\mathcal{A}(\mathbf{x}_0, \boldsymbol{v}_0), \quad \mathcal{A}(\boldsymbol{x}(t), \boldsymbol{v}(t)) = \mathcal{A}(\mathbf{x}_0, \boldsymbol{v}_0),$$

3. $\mathcal{T}$ satisfies the asymptotic action property as $t \to +\infty$:

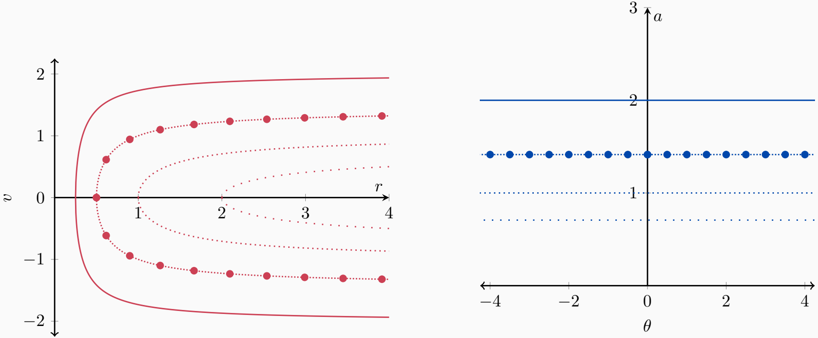
$$|\boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}) - t\boldsymbol{a}| = o(t), \qquad |\boldsymbol{V}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}) - \boldsymbol{a}| = o(1).$$

Linearized Equation: radial case $(1+1$ dim$)$

- Trajectories

$$\dot{r} = v, \quad \dot{v} = \frac{q}{2r^2}, \qquad \mathcal{H}_0 = v^2 + \frac{q}{r}$$

phase portrait:



Linearized Equation: radial case $(1+1$ dim$)$ (2)

Lemma [Canonical Transformation]

Let

$$\mathcal{A}(r,v) = \sqrt{\mathcal{H}_0}, \quad \Theta(r,v) = \text{ clock along trajectory.}$$

The map $(r,v) \mapsto (\Theta(r,v), \mathcal{A}(r,v))$ is a *canonical* diffeomorphism which linearizes the flow $\Phi^t(r,v)$ of the Kepler ODE, i.e.

$$\Theta(\Phi^t(r,v)) - \Theta(r,v) = t\mathcal{A}(r,v), \qquad \mathcal{A}(\Phi^t(r,v)) = \mathcal{A}(r,v).$$

Proof: We have

$$\dot{r} = \sqrt{\mathcal{A}^2 - \frac{q}{r}}$$

$\to$ integrate; with $r_{min} = \frac{q}{v^2 + \frac{q}{r}} = \frac{q}{\mathcal{A}^2}$, define

$$\Theta(r,v) = \frac{v}{|v|} r_{min} G(\frac{r}{r_{min}}),$$

where $G : (1,\infty) \to \mathbb{R}$ satisfies $G(1) = 0$, $G'(s) = \left[1 - \frac{1}{s}\right]^{-\frac{1}{2}}$. □

Linearized equation: solved

With $(\boldsymbol{X}(\vartheta, \boldsymbol{a}), \boldsymbol{V}(\vartheta, \boldsymbol{a}))$ inverse of $(\Theta(\boldsymbol{x}, \boldsymbol{v}), \mathcal{A}(\boldsymbol{x}, \boldsymbol{v}))$, define

$$\nu(\vartheta, \boldsymbol{a}, t) = \mu(\boldsymbol{X}(\vartheta, \boldsymbol{a}), \boldsymbol{V}(\vartheta, \boldsymbol{a}), t),$$
$$\gamma(\vartheta, \boldsymbol{a}, t) = \nu(\vartheta + t\boldsymbol{a}, \boldsymbol{a}, t) = \mu(\boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}), \boldsymbol{V}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}), t)$$

- integrates the linearized equation:

$$\begin{aligned}\left(\partial_t + \boldsymbol{v} \cdot \nabla_x - q \frac{\boldsymbol{x}}{|\boldsymbol{x}|^3} \cdot \nabla_v\right) \mu &= \partial_t \mu + \{\frac{|\boldsymbol{v}|^2}{2} + \frac{q}{|\boldsymbol{x}|}, \mu\} \\ &= \partial_t \nu + \{\frac{|\boldsymbol{a}|^2}{2}, \nu\} = (\partial_t + \boldsymbol{a} \cdot \nabla_\vartheta)\, \nu \\ &= \partial_t \gamma\end{aligned}$$

Nonlinear Dynamics & Asymptotics via "Mixing"

Nonlinear equation

Then[1] since coordinate change is symplectic

$$\partial_t \mu + \{\mathcal{H}_0 + \mathcal{H}_{pert}, \mu\} = 0 \qquad \Leftrightarrow \qquad \partial_t \gamma = \lambda \{\Psi, \gamma\}, \qquad \text{(VP')}$$

with

$$\begin{aligned} \Psi(\vartheta, \boldsymbol{a}, t) &= \phi(\boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}), t) \\ &= \iint \frac{1}{|\boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a}) - \boldsymbol{y}|} \mu^2(\boldsymbol{y}, \boldsymbol{v}, t) d\boldsymbol{v} d\boldsymbol{y} \\ &= \iint \frac{1}{|\widetilde{\boldsymbol{X}}(\vartheta, \boldsymbol{a}) - \widetilde{\boldsymbol{X}}(\theta, \alpha)|} \gamma^2(\theta, \alpha, t) d\theta d\alpha \end{aligned}$$

and $\widetilde{\boldsymbol{X}}(\vartheta, \boldsymbol{a}) = \boldsymbol{X}(\vartheta + t\boldsymbol{a}, \boldsymbol{a})$.

- nonlinear analysis works with this purely nonlinear equation

[1] ignoring point mass dynamics for now

Asymptotic dynamics (heuristics)

$$\partial_t \gamma + \lambda\{\Psi, \gamma\} = 0, \qquad \Psi = \iint \frac{1}{|\widetilde{\boldsymbol{X}}(\vartheta, \boldsymbol{a}) - \widetilde{\boldsymbol{X}}(\theta, \alpha)|} \gamma^2(\theta, \alpha) d\theta d\alpha$$

By asymptotic action property $\widetilde{\boldsymbol{X}}(\vartheta, \boldsymbol{a}) = t\boldsymbol{a} + o(t)$, hence with

$$\Phi(\boldsymbol{a}, t) = \iint \frac{1}{|\boldsymbol{a} - \alpha|} \gamma^2(\theta, \alpha, t) d\theta d\alpha$$

obtain asymptotic shear equation

$$0 = \partial_t \gamma + \frac{\lambda}{t}\{\Phi, \gamma\} + O(t^{-1-}) = \partial_t \gamma - \frac{\lambda}{t} \nabla_a \Phi(\boldsymbol{a}, t) \cdot \nabla_\vartheta \gamma + O(t^{-1-})$$

$$\Downarrow$$

$$\frac{d}{dt}\left(\gamma(\vartheta + \lambda \ln(t) \mathcal{E}_\infty(\boldsymbol{a}), \boldsymbol{a}, t)\right) = O(t^{-1-}), \quad \mathcal{E}_\infty(\boldsymbol{a}) = \lim_{t\to\infty} \nabla_a \Phi(\boldsymbol{a}, t).$$

Bootstrap analysis

Starting from the nonlinear equation

$$\partial_t \gamma + \{\Psi, \gamma\} = 0, \qquad \Psi = \phi(\widetilde{\boldsymbol{X}}),$$

want to propagate bounds on moments and smoothness of γ.

Key property:

$$\forall g \iint \{\Psi, g\} d\vartheta d\boldsymbol{a} = 0$$

$$\text{(e.g.)} \Rightarrow \quad \frac{d}{dt} \iint \gamma^2 d\vartheta d\boldsymbol{a} = -\iint \{\Psi, \gamma^2\} d\vartheta d\boldsymbol{a} = 0.$$

Bootstrap Analysis (2)

1. Moments: for scalar ω

$$\partial_t(\omega\gamma) + \{\Psi, \omega\gamma\} = \gamma\{\Psi, \omega\} = \gamma \cdot \mathcal{E}(\widetilde{\boldsymbol{X}}) \cdot \{\widetilde{\boldsymbol{X}}, \omega\},$$

$$\mathcal{E}(\boldsymbol{y}, t) = \nabla\phi(\boldsymbol{y}, t) = c \iint \frac{\boldsymbol{y} - \widetilde{\boldsymbol{X}}(\theta, \alpha)}{|\boldsymbol{y} - \widetilde{\boldsymbol{X}}(\theta, \alpha, t)|^3} \gamma^2(\theta, \alpha, t) d\theta d\alpha$$

$\Rightarrow$ global solutions with almost optimal decay $|\mathcal{E}| \lesssim \langle t\rangle^{-2} \ln\langle t\rangle$.

2. Derivatives via symplectic gradients:

$$\begin{aligned}\partial_t\{f, \gamma\} + \{\Psi, \{f, \gamma\}\} &= -\{\{f, \Psi\}, \gamma\} \\ &= -\mathcal{F}_{jk}\{\widetilde{\mathbf{X}}^j, \gamma\}\{f, \widetilde{\mathbf{X}}^k\} - \mathcal{E}_j\{\{f, \widetilde{\mathbf{X}}^j\}, \gamma\}\end{aligned}$$

with $\mathcal{F}(\boldsymbol{y}, t) = \nabla^2\phi(\boldsymbol{y}, t)$.
$\Rightarrow$ sharp decay $|\mathcal{E}| \lesssim \langle t\rangle^{-2}$ and precise asymptotics.

Radial Bootstrap: Moments and Derivatives

Proposition

There exists ε^* such that for all $0 < \varepsilon_0 \le \varepsilon_1 \le \delta < \varepsilon^*$, the following holds. Let γ be a solution to (VP') with initial data γ_0 on $0 \le t \le T$ and assume that for $0 \le t \le T$,

$$\| \left(a^{-20} + \theta^{20} + a^{20}\right) \gamma(t)\|_{L^2_{\theta,a}} + \|(a + a^{-1})\partial_\theta \gamma(t)\|_{L^2_{\theta,a}} + \|a\partial_a \gamma(t)\|_{L^2_{\theta,a}} \le \varepsilon_1 \langle t\rangle^\delta,$$

then in fact

$$\| \left(a^{-20} + a^{20}\right) \gamma\|_{L^2_{\theta,a}} + \|(a + a^{-1})\partial_\theta \gamma\|_{L^2_{\theta,a}} \le \varepsilon_0 + \varepsilon_1^{\frac{3}{2}},$$
$$\|\theta^{20}\gamma\|_{L^2_{\theta,a}} + \|a\partial_a \gamma\|_{L^2_{\theta,a}} \le \varepsilon_0 + \varepsilon_1^{\frac{3}{2}} t^\delta.$$

Note: Moments can be propagated by themselves

Radial Bootstrap (2)

Proof sample: We have

$$\partial_t(a^q\gamma) - \lambda\left\{\widetilde{\Psi}, a^q\gamma\right\} = -\lambda\{\widetilde{\Psi}, a^q\}\gamma = q\lambda\partial_\theta\widetilde{\Psi}\cdot a^{q-1}\gamma.$$

Recall that $\iint\{\widetilde{\Psi}, f\}f = 0$, so

$$\frac{1}{2}\frac{d}{dt}\|a^q\gamma\|^2_{L^2_{\theta,a}} \lesssim \|a^{-1}\partial_\theta\widetilde{\Psi}\|_{L^\infty}\|a^q\gamma\|^2_{L^2_{\theta,a}}.$$

One can show that $\|a^{-1}\partial_\theta\widetilde{\Psi}\|_{L^\infty_{\theta,a}} \lesssim t^{-\frac{3}{2}}\|a^{-\frac{3}{4}}\gamma\|^2_{L^2_{\theta,a}} + l.o.t.$

$$\Rightarrow \|a^q\gamma(t)\|^2_{L^2_{\theta,a}} \lesssim \|a^q\gamma(0)\|^2_{L^2_{\theta,a}} \lesssim \varepsilon_0^2.$$

For derivatives, need more: e.g. $\partial_{\alpha\beta}\widetilde{\Psi}$ for $\alpha, \beta \in \{\theta, a\}$,
$\to$ set $\{R(\theta + ta, a) = r\}$ ... (see paper) □

Remarks on the non-radial problem

1. much more involved geometry:
   - construction of asymptotic action-angle variables,
   - super-integrability really useful,
   - "bad derivatives" for kinematic quantities such as $\widetilde{\boldsymbol{X}}$,

     $\Rightarrow$ need to adapt coordinates.
2. motion of point charge:
   - additional complication for nonlinear problem,
   - various types of linearized trajectories (w.r.t. point charge)

     $\Rightarrow$ need formulations in adapted frames,
   - simpler if $\mathrm{supp}(\mu_0) \subset\subset \mathbb{R}^3_{\boldsymbol{x}} \times \mathbb{R}^3_{\boldsymbol{v}}$.

Sample: Construction of Asymptotic Actions

Via generating function: to find $\mathcal{T} : (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\vartheta, \mathbf{a})$, fix a function $S(\boldsymbol{x}, \boldsymbol{a})$ and define

$$\boldsymbol{v} = \nabla_x S(\boldsymbol{x}, \boldsymbol{a}), \qquad \vartheta = \nabla_a S(\boldsymbol{x}, \boldsymbol{a}).$$

If this (implicitly) defines a change of coordinates, then it is guaranteed to be canonical.

How to choose $\boldsymbol{a}$? Want (inter alia):

- $\boldsymbol{a}$ constant along trajectories of linearized problem,
- asymptotic action property.

We take $\boldsymbol{a} = \boldsymbol{v}_\infty$.

Sample: Scattering problem

Given $\boldsymbol{x}_0 \in \mathbb{R}^3$ and $\boldsymbol{a} \in \mathbb{R}^3$, find trajectory through $\boldsymbol{x}_0$ with asymptotic velocity $\boldsymbol{v}_\infty = \boldsymbol{a}$:

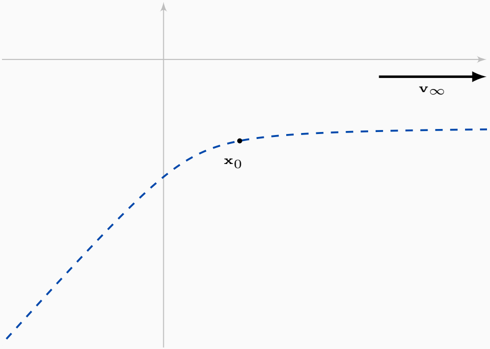


Figure 1: The scattering problem

Sample: Scattering problem

Given $\boldsymbol{x}_0 \in \mathbb{R}^3$ and $\boldsymbol{a} \in \mathbb{R}^3$, find trajectory through $\boldsymbol{x}_0$ with asymptotic velocity $\boldsymbol{v}_\infty = \boldsymbol{a}$:

1. compute $\boldsymbol{v} = \nabla_{\boldsymbol{x}} \mathcal{S}(\boldsymbol{x}, \boldsymbol{a})$,
2. integrate to recover $\mathcal{S}(\boldsymbol{x}, \boldsymbol{a})$,
3. let $\vartheta = \nabla_{\boldsymbol{a}} \mathcal{S}(\boldsymbol{x}, \boldsymbol{a})$,
4. invert to get $\Theta(\boldsymbol{x}, \boldsymbol{v})$.
   ($\mathcal{A}(\boldsymbol{x}, \boldsymbol{v})$ is recovered more easily from a direct study of the trajectories via conservation laws.)

Sample: Scattering problem

The scattering problem can be solved elegantly using the "velocity circle" method of Hamilton [Milnor '83].

The association $(\boldsymbol{x}, \boldsymbol{a}) \mapsto (\boldsymbol{x}, \boldsymbol{v})$ has a fold: to most choices $(\boldsymbol{x}, \boldsymbol{a})$, there correspond 2 trajectories (one "incoming" and one "outgoing"), but it is possible to find a smooth map $(\Theta, \mathcal{A}) : (\boldsymbol{x}, \boldsymbol{v}) \mapsto (\vartheta, \boldsymbol{a})$.