Exponential mixing for random flows

Long Time Behavior and Singularity Formation in PDEs-Part V

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Passive scalars

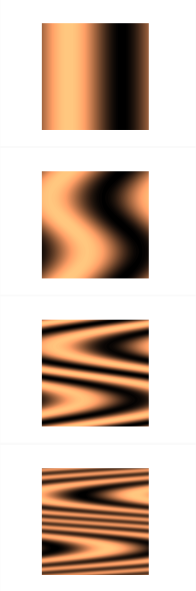
The passive scalar problem

We aim to study of the long-time dynamics of the transport equation

$$\partial_t \rho + \boldsymbol{u} \cdot \nabla \rho = 0,$$

- d-dimensional periodic domain $\mathbb{T}^d = [0, 2\pi)^d$;
- $\boldsymbol{u} = \boldsymbol{u}(t, x) : [0, \infty) \times \mathbb{T}^d \to \mathbb{R}^d$ is a given divergence-free velocity field;
- consider only mean-free solutions $\int_{\mathbb{T}^d} \rho(t, x) \mathrm{d}x = 0, \ \forall t \geq 0$.

Shear flows



Take $\boldsymbol{u} = (u(y), 0)$ in $\mathbb{T}^2$:

$$\partial_t \rho + u(y)\partial_x \rho = 0, \qquad \rho(0) = \rho^{in}.$$

The x-average $\langle \rho \rangle_x$ is conserved

$$\partial_t \langle \rho(t) \rangle_x = 0.$$

However,

$$\rho(t) - \langle \rho(t) \rangle_x \to 0 \quad \text{as } t \to \infty \text{ weakly}.$$

Chaotic flows

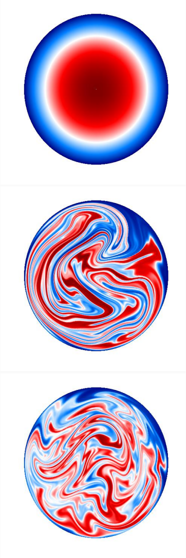


Figure 1: From J. Vanneste

The high and low concentrations of a scalar in a disc when stirred by a random flow.

- If $\boldsymbol{u}$ is chaotic, then all mean-zero solutions can be mixed
- How mixed is ρ at a given time?
- How fast is ρ mixed?

Mixing measures

Mixing can be thought of as a cascading process in which information travels to smaller and smaller spatial scales. This can be quantified by negative Sobolev norms (see Lin, Thiffeault, Doering '11)

$$\|\rho(t)\|_{\dot{H}^{-s}}^2 = \sum_{k\in\mathbb{Z}^d\setminus\{0\}} |k|^{-2s}|\rho_k(t)|^2 \to 0, \quad \text{as } t \to \infty. \tag{1}$$

This is related to the following: consider $X(t,x)$ the solution to the ODE

$$\partial_t X(t,x) = \boldsymbol{u}(t, X(t,x)), \quad X(0,x) = x.$$

Then (1) is related to the decay of correlations

$$\int_{\mathbb{T}^d} f(x)g(X(t,x)^{-1})\mathrm{d}x = \int_{\mathbb{T}^d} f(X(t,x))g(x)\mathrm{d}x \to 0, \quad \text{as } t \to \infty. \tag{2}$$

Some deterministic results

Consider $\|\rho(t)\|_{\dot{H}^{-1}}$

- smooth shear flows (Bedrossian, Coti Zelati '15): $t^{-1/n}$, where n is the order of critical points ($n = 2$ for $u(y) = \sin y$).

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- Regular autonomous velocities in 2D cannot mix faster than t^{-1} (Bonicatto, Marconi '19).
- More complicated flows can be exponential mixers: (Alberti, Crippa, Mazzucato '14, Yao, Zlatos '14), also universal (Elgindi, Zlatos '18), but for $u \in L^\infty_t W^{s,p}_x, s < \frac{1+\sqrt{5}}{2}, p \in [1, \frac{2}{2s+1-\sqrt{5}})$.

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- Exponential is the best possible rate: (Crippa, De Lellis. '08, Seis '13, Iyer, Kiselev, Xu '14).

Some probabilistic results

Consider $\|\rho(t)\|_{\dot{H}^{-1}}$

- if $\boldsymbol{u}$ is the solution of the stochastic NSE: $\mathrm{e}^{-\lambda t}$ almost surely (Bedrossian, Blumenthal, Punshon-Smith '19)

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- if $\boldsymbol{u}$ is Gaussian, decorrelated in time and specific scaling properties in space built in (Kraichnan model): $\mathrm{e}^{-\lambda t}$ almost surely (Flandoli, Galeati, Luo '21, Gess, Yaroslavtsev '21).

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Open question: does there exists a smooth universal exponential mixers on $\mathbb{T}^d$? $(L_t^\infty \cap C_t^\infty)C_x^\infty$

The Pierrehumbert model

In '94, R. Pierrehumbert looked at a velocity field u that alternates, after every time interval of size $\tau > 0$, between two transversal shears with a randomly and independently chosen phase shift.

- Random phases: $\{\omega_j = (\omega_j^1, \omega_j^2)\}_{j\in\mathbb{N}} \subset [0, 2\pi)^2$ be a sequence of i.i.d random variables uniformly distributed in $[0, 2\pi)^2$.

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- Horizontal shearing:

$$u(t, x_1, x_2) = \begin{pmatrix} \sin(x_2 - \omega_n^1) \\ 0 \end{pmatrix}$$

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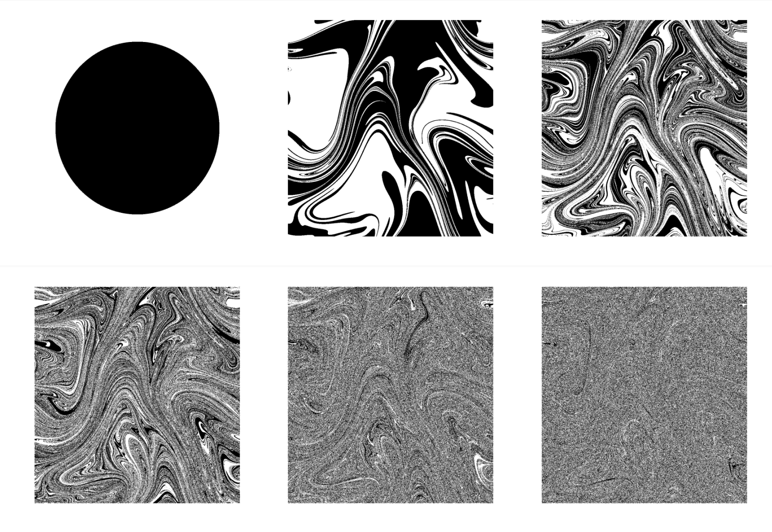
- Vertical shearing:

$$u(t, x_1, x_2) = \begin{pmatrix} 0 \\ \sin(x_1 - \omega_n^2) \end{pmatrix}$$

  for $t \in [(2n-1)\tau, 2n\tau)$.



The Pierrehumbert model as an RDS

Define

$$f_\beta^H(x) = \begin{pmatrix} x_1 + \tau \sin(x_2 - \beta) \\ x_2 \end{pmatrix}, \qquad f_\beta^V(x) = \begin{pmatrix} x_1 \\ x_2 + \tau \sin(x_1 - \beta) \end{pmatrix}.$$

Then, the position $X(2\tau, x) \in \mathbb{T}^2$ of the particle at time $t = 2\tau$ is given by $X(2\tau, x) = f_{\omega_1}(x)$, where

$$f_{\omega_1}(x) = f^V_{\omega_1^2} \circ f^H_{\omega_1^1}(x)\,.$$

- any sequence of possible random phase shifts is written as $\underline{\omega} = (\omega_1, \omega_2, \dots) \in \Omega := \left([0, 2\pi)^2\right)^{\mathbb{N}}$.
- $X(2\tau n, x) = f^n_{\underline{\omega}}(x)$

$$f^n_{\underline{\omega}}(x) := f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x).$$

- By Yao, Zlatos '14, enough to look at the discrete dynamics.

Random dynamical systems

Random dynamical systems

Let $(\Omega_0, \mathcal{F}_0, \mathbb{P}_0)$ be a fixed probability space, X a complete metric space.

- $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_0, \mathcal{F}_0, \mathbb{P}_0)^{\mathbb{N}}$, and

$$f^n_{\underline{\omega}} = f_{\omega_n} \circ \cdots \circ f_{\omega_1}, \qquad n \in \mathbb{N},$$

- The transition kernels are

$$P(x, A) = \mathbb{P}_0(f_\omega(x) \in A), \qquad P^{n+1}(x, A) = \int P^n(y, A) P(x, \mathrm{d}y).$$

Here:

- X is a finite-dimensional compact manifold ($X = \mathbb{T}^2$).
- $\mathbb{P}_0 \ll \mathrm{Leb}$.
- f_ω is measure-preserving.

Some key properties

The RDS should satisfy certain non-degeneracy conditions

- Irreducibility: For every $x \in X$ and open set $U \subset X$, there exists $N = N(x, U) \geq 0$ such that $P^N(x, U) > 0$;
- Aperiodicity: no cyclic behavior allowed;
- Minorization property: there exist a set A and a positive measure ν_n on X such that, for all $x \in A$, we have that

$$P^n(x, B) \geq \nu_n(B) \quad \text{for all Borel } B \subset X\,.$$

- Drift condition: There exists $V : X \to [1, \infty)$, $\alpha \in (0, 1)$, $b > 0$ and a compact set $C \subset X$ such that

$$PV \leq \alpha V + b\chi_C.$$

Geometric ergodicity

Theorem (Abstract Harris Theorem)

Let P be a Feller transition kernel and assume the following:

1. *Minorization property*
2. *Topological irreducibility*
3. *Strong aperiodicity*
4. *Drift condition*

Then, P is V-uniformly geometrically ergodic, i.e., P admits a unique stationary measure π, and has the property that there exists $D > 0$ and $\gamma \in (0,1)$ such that for all $x \in X$ and $\varphi \in L_V^\infty(X)$, we have

$$\left| P^n\varphi(x) - \int \varphi \, \mathrm{d}\pi \right| \leq DV(x)\|\varphi\|_V \gamma^n \, .$$

True for the one point process with $V \equiv 1$.

Lyapunov exponents

- By ergodic theory: the asymptotic exponential growth rate

$$\lambda_1 := \lim_{n\to\infty} \frac{1}{n} \log |D_x f^n_{\underline{\omega}}|$$

  exists and is constant over $\mathbb{P} \times$ Leb-a.e. $(\underline{\omega}, x)$.

- (A version of) Furstenberg criterion: Denote $\Phi_x : \Omega_0^n \to X$ defined for $\underline{\omega}^n = (\omega_1, \cdots, \omega_n)$ by

$$\Phi_x(\underline{\omega}^n) = f_{\omega_n} \circ \cdots \circ f_{\omega_1}(x)\,.$$

  If there exist $n \geq 1$ and $(\underline{\omega}^n_\star, x_\star)$ s.t.

  1. $D_{\underline{\omega}^n_\star} \Phi_{x_\star}$ is surjective.
  2. The restriction of $D_{\underline{\omega}^n_\star} D_{x_\star} f^n_{\underline{\omega}_\star}$ to $\ker D_{\underline{\omega}^n_\star} \Phi_{x_\star}$ is surjective as a linear operator onto $T_{\widehat{\Phi}_{x_\star}(\underline{\omega}^n_\star)} SL_d(\mathbb{R})$.

  Then $\lambda_1 > 0$.

Checking Furstenberg for Pierrehumbert

In the case of Pierrehumbert: $\underline{\omega}^n \in (\mathbb{T}^2)^n$ (resp. $\underline{\omega}^n \in (\mathbb{T}^\ell)^n$).

- $D_{\underline{\omega}^n_\star} \Phi_{x_\star}$ is a $2 \times 2n$ matrix (resp. $d \times \ell n$).
- $\ker D_{\underline{\omega}^n_\star} \Phi_{x_\star}$ is $2n-2$ dimensional by condition 1 (resp. $\ell n - d$).
- $T_{\widehat{\Phi}_{x_\star}(\underline{\omega}^n_\star)} SL_2(\mathbb{R})$ is 3 dimensional (resp. $d^2 - 1$).

Necessary condition: $2n - 2 \geq 3$ hence $n \geq 5/2$
(resp. $\ell n - d \geq d^2 - 1$ hence $n \geq (d^2 + d - 1)/\ell$).

- Condition 2 can be checked by a computer (but also by hand!).
- Problem becomes costly with dimension and with less noise.

Chaos for Pierrehumbert

Theorem (Blumenthal, CZ, Gvalani '22)

The Pierrehumbert model $f_{\omega_1}(x) = f^V_{\omega_1^2} \circ f^H_{\omega_1^1}(x)$ *has a positive Lyapunov exponent.*

- True also without changing phases every "half" time: $f_{\omega_1}(x) = f^V_{\omega_1} \circ f^H_{\omega_1}(x)$
- True also with fixed phases and random switching times: $f_{\tau_1}(x) = f^V_{\tau_1^2} \circ f^H_{\tau_1^1}(x)$ or $f_{\tau_1}(x) = f^V_{\tau_1} \circ f^H_{\tau_1}(x)$
- no need of sin
- Completely open: true also without any randomness?

Exponential mixing

Decay of correlations

The Pierrehumbert model $f_{\omega_1}(x) = f^V_{\omega_1^2} \circ f^H_{\omega_1^1}(x)$ is mixing.

Theorem (Blumenthal, CZ, Gvalani '22)

Let $q, s > 0$. There exists a function $D : \Omega \to [1, \infty)$ and a constant $\alpha > 0$ such that for all mean-free functions $\varphi, \psi \in H^s(\mathbb{T}^2)$, we have the almost sure estimate

$$\left| \int_{\mathbb{T}^2} \varphi(x) \psi \circ f^n_{\underline{\omega}}(x) \mathrm{d}x \right| \leq D(\underline{\omega}) \mathrm{e}^{-\alpha n} \|\varphi\|_{H^s} \|\psi\|_{H^s} ,$$

while the function D satisfies the moment bound $\mathbb{E}|D|^q < \infty$.

More Markov chains

Setting

$$\mathrm{Cor}_n(\varphi, \psi) = \left| \int \varphi(x) \psi \circ f^n_{\underline{\omega}}(x) \mathrm{d}\pi(x) \right| ,$$

we know that

$$\mathbb{P} \left\{ \mathrm{Cor}_n(\varphi, \psi) > \varepsilon \right\} \leq \varepsilon^{-2} \, \mathbb{E}_{\mathbb{P}} \left| \int \varphi\psi \circ f^n_{\underline{\omega}} \mathrm{d}\pi \right|^2 .$$

Since

$$\mathbb{E}_{\mathbb{P}} \left| \int \varphi\psi \circ f^n_{\underline{\omega}} \mathrm{d}\pi \right|^2 = \mathbb{E}_{\mathbb{P}} \int \varphi(x)\varphi(y)\psi \circ f^n_{\underline{\omega}}(x) \psi \circ f^n_{\underline{\omega}}(y) \mathrm{d}\pi(x) \mathrm{d}\pi(y) \, ,$$

we can re-write this expression using the two-point process $(x_n, y_n) = (f^n_{\underline{\omega}}(x_0), f^n_{\underline{\omega}}(y_0))$ on $X \times X$. Hence

$$\mathbb{E}_{\mathbb{P}} \left| \int \varphi\psi \circ f^n_{\underline{\omega}} \mathrm{d}\pi \right|^2 = \int \psi^{(2)} P^{(2)} \varphi^{(2)} \mathrm{d}\pi^{(2)} \, .$$

Here $\varphi^{(2)}(x, y) := \varphi(x)\varphi(y)$, $\mathrm{d}\pi^{(2)}(x, y) = \mathrm{d}\pi(x)\mathrm{d}\pi(y)$ and $P^{(2)}$ is the Markov semigroup for (x_n, y_n).

The two-point process

The two-point process

$$P^{(2)}\left((x,y),K\right) = \mathbb{P}_0\left((f_\omega(x), f_\omega(y)) \in K\right),$$

is defined on $X \times X \setminus \Delta$, with $\Delta = \{(x,x),\ x \in X\}$.

- The kernel $P^{(2)}$ is V-uniformly geometrically ergodic, where $V : X \times X \setminus \Delta \to \mathbb{R}$ is of the form

$$V(x,y) = d(x,y)^{-p}\psi(x,y).$$

- if $d(x,y) \ll 1$, V depends most on its values near the diagonal Δ. We can approximate two point motion by its linearization:

$$\mathbb{E}_{\mathbb{P}_0}\varphi(f_\omega(x), D_x f_\omega v), \qquad x \in X,\ |v| = 1.$$

- this motivates the study of the projective dynamics

$$\hat{f}_\omega(x,v) := \left(f_\omega(x), \frac{D_x f_\omega v}{|D_x f_\omega v|}\right).$$

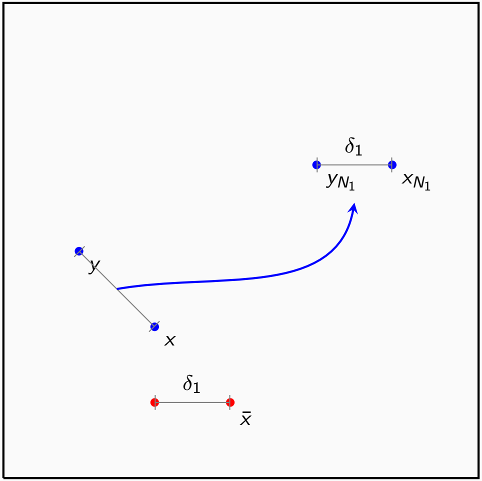
A checklist

Mixing for this kind of RDS can be proven if (besides continuity and regularity conditions):

- The one-point motion is uniformly geometrically ergodic with a unique invariant measure.
- The top Lyapunov exponent is strictly positive.
- The projective motion is uniformly geometrically ergodic with a unique invariant measure.
- The two-point motion is V-uniformly geometrically ergodic with a unique invariant measure.

Topological irreducibility

Irreducibility can be obtained by combining regularity of the chain and a form of approximate controllability:



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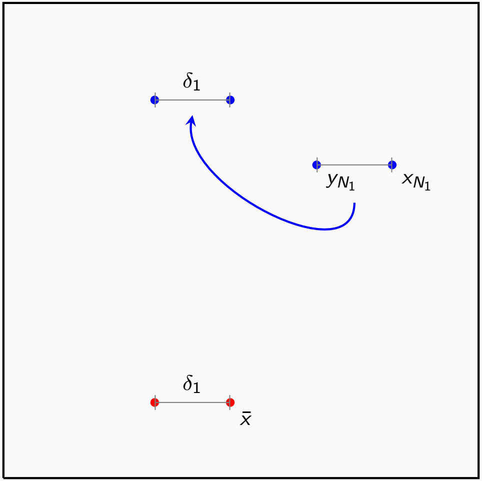

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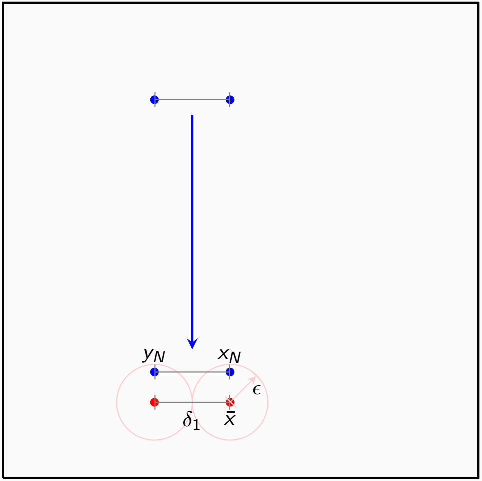

Some further directions

- Construct smooth mixers in any dimension.
- Reduce to the minimum the degrees of freedom (less randomness. . .).
- Enhanced dissipation, Batchelor spectrum, intermittency . . .
- Construct time-periodic mixers in any dimension.
- Quantify the mixing rate and the Lyapunov exponent (in terms of switching time?).

THANK YOU