# Stability of Solitary Waves of the NLS Equation

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# The NLS equation

We consider the Cauchy problem for the 1D nonlinear Schrödinger equation

$$i\partial_t v + \partial_x^2 v + F'(|v|^2)v = 0 \qquad \text{(NLS)}$$

with prescribed data

$$v(t=0) = v_0.$$

It is stemming from the Hamiltonian

$$H = \int [|\partial_x v|^2 - F(|v|^2)]\, dx.$$

# The NLS equation

The interaction potential $F$ will only be assumed to be smooth and have a local minimum at zero, $F(0) = 0$, $F'(0) = 0$, $F''(0) \geq 0$. Stationary waves of the type

$$v(t) = e^{it\omega}\Phi_\omega$$

are given by solutions of

$$\partial_x^2\Phi_\omega - \omega\Phi_\omega + F'(\Phi_\omega^2)\Phi_\omega = 0.$$

Under our assumptions on $F$, there exists a unique solution of the above equation on an interval $\omega \in (0, \omega_0)$, for some $\omega_0 > 0$. Furthermore, $\Phi_\omega$ is even, positive, decreasing on $x > 0$, and exponentially decreasing at infinity, along with its derivatives. See Strauss (1977) and Berestycki-Lions (1983) for these facts and a full characterization of the interval of existence.

# The NLS equation

For $\beta, \gamma, x_0 \in \mathbb{R}$, the Galilean symmetry

$$v(t,x) \mapsto e^{i(\beta x - \beta^2 t + \gamma)} v(t, x - 2\beta t - x_0)$$

leaves the set of solutions of (NLS) invariant. In particular, this gives the family of traveling waves

$$e^{i(\omega t + \beta x - \beta^2 t + \gamma)} \Phi_\omega(x - 2\beta t - x_0).$$

Our aim is to establish asymptotic stability of this family of traveling waves, under appropriate spectral assumptions on the linearized operator around them.

# Previous works

- M. Grillakis, J. Shatah, W. Strauss, Stability of solitary waves in the presence of symmetries, 1987
- A. Soffer, M. Weinstein, Multichannel nonlinear scattering for nonintegrable equations, 1990
- Modified scattering for unperturbed NLS: Using complete integrability: Deift-Zhou (2003); Without making use of complete integrability, and restricting the analysis to small solutions: Hayashi-Naumkin (1998), Lindblad-Soffer (2006), Kato-Pusateri (2011), Ifrim-Tataru (2015)

# Previous works

- V. S. Buslaev, G. S. Perelman, Scattering for the nonlinear Schrödinger equation: states close to a soliton, 1993
- J. Krieger and W. Schlag, Stable manifolds for all monic supercritical focusing nonlinear Schrödinger equations in one dimension, 2006
- S. Cuccagna and D. Pelinovsky, The asymptotic stability of solitons in the cubic NLS equation on the line, 2014
- S. Masaki, J. Murphy, J.-I. Segata, Modified scattering for the 1D cubic NLS with a repulsive delta potential, 2017
- P. Germain, F. Pusateri and F. Rousset, The nonlinear Schrödinger equation with a potential in dimension 1, 2018
- G. Chen and F. Pusateri, The 1D NLS with a weighted $L^1$ potential, 2019; The 1D cubic NLS with a non-generic potential, 2022

# Expanding around the soliton

Writing $v = e^{it}(\Phi + u)$, and expanding in $v$, we obtain

$$\begin{aligned}
&i\partial_t u + \partial_x^2 u - u \\
&+ V_+ u + V_- \overline{u} + V_{++} u^2 + V_{--} \overline{u}^2 + V_{+-} |u|^2 \\
&+ V_{++-} |u|^2 u + V_{+--} |u|^2 \overline{u} + V_{+++} u^3 + V_{---} \overline{u}^3 \\
&+ \{\text{higher order terms}\} = 0
\end{aligned}$$

# Expanding around the soliton

Here

$$V_+ = F'(\Phi^2) + F''(\Phi^2)\Phi^2, \;\; V_- = F''(\Phi^2)\Phi^2,$$

$$V_{++} = F''(\Phi^2)\Phi + \frac{1}{2}F'''(\Phi^2)\Phi^3, \;\; V_{--} = \frac{1}{2}F'''(\Phi^2)\Phi^3,$$

$$V_{+-} = 2F''(\Phi^2)\Phi + \frac{1}{2}F'''(\Phi^2)\Phi^3,$$

$$V_{++-} = F''(\Phi^2) + 2F'''(\Phi^2)\Phi^2 + \frac{1}{2}F''''(\Phi^2)\Phi^4,$$

$$V_{+--} = F'''(\Phi^2)\Phi^2 + \frac{1}{2}F''''(\Phi^2)\Phi^4,$$

$$V_{+++} = \frac{1}{2}F'''(\Phi^2)\Phi^2 + \frac{1}{6}F''''(\Phi^2)\Phi^4, \;\; V_{---} = \frac{1}{6}F''''(\Phi^2)\Phi^4.$$

Notice that all these potentials decay rapidly, except for $V_{++-}$.

# Expanding around the soliton

Writing

$$U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} u \\ \overline{u} \end{pmatrix}, \quad \begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = -\begin{pmatrix} V_+ \\ V_- \end{pmatrix},$$

the unknown $U$ satisfies the reality condition $\sigma_1 \overline{U} = U$ and the above becomes

$$i\partial_t U + \begin{pmatrix} \partial_x^2 - 1 & 0 \\ 0 & -\partial_x^2 + 1 \end{pmatrix} U - \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix} U = \{nonlinear\}.$$

Let us denote the vector Schrödinger operator

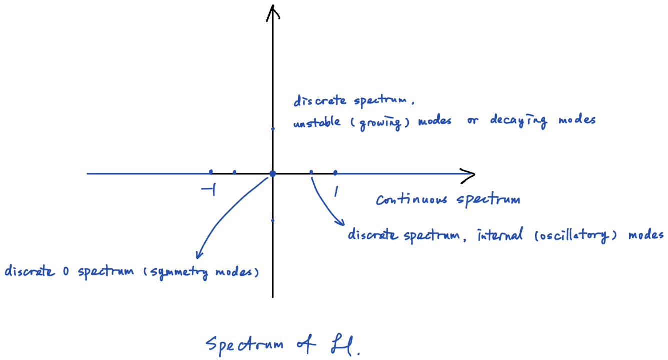
$$\mathcal{H} = \mathcal{H}_0 + V, \tag{1}$$

where

$$\mathcal{H}_0 = \begin{pmatrix} -\partial_x^2 + 1 & 0 \\ 0 & \partial_x^2 - 1 \end{pmatrix} \qquad V = \begin{pmatrix} V_1 & V_2 \\ -V_2 & -V_1 \end{pmatrix} \tag{2}$$

Since the soliton $\Phi$ is even, and it is exponentially decreasing, as are its derivatives, this remains true of $V$. Denote $f = e^{it\mathcal{H}} U$.

# Assumptions on the operator $\mathcal{H}$




We will assume that

(H1) The operator $\mathcal{H}$ has no internal modes.

(H2) The operator $\mathcal{H}$ is generic - no resonance at the endpoint of the continuous spectrum $E = \pm 1$.

# Main Result

Theorem (P. Germain, F. Pusateri, Z. Z., ongoing work)

*Under the above assumptions, there exists* $\epsilon_0 > 0$ *such that, if* $u_0$ *satisfies*

$$\|u_0\|_{H^1_x} + \|\widetilde{u_0}\|_{H^1_\xi} = \epsilon < \epsilon_0,$$

*then a global solution exists, which decays pointwisely and is globally bounded in* $L^2$ *type spaces: for any time* $t$,

$$\|\widetilde{f}\|_{L^\infty_t([0,T];L^\infty_\xi)} + \|t^{-\alpha}\widetilde{f}\|_{L^\infty_t([0,T];H^1_\xi)} \lesssim \epsilon \qquad (3)$$

# Main Result

## Theorem (P. Germain, F. Pusateri, Z. Z. (cont'd))

*Furthermore, we obtain a precise description of the asymptotic behavior of $u$, which undergoes modified scattering. To be more specific, the distorted Fourier transform of the profile $f$ satisfies the following: there exists an asymptotic profile $W^\infty = (W_+^\infty, W_-^\infty) \in \left(\langle\xi\rangle^{-3/2} L_\xi^\infty\right)^2$ such that, for $\xi > 0$,*

$$
\begin{aligned}
&(\widetilde{f}(t,\xi), \widetilde{f}(t,-\xi))^\top = \\
&S^{-1}(\xi) \exp\left(i \frac{\sqrt{\pi}}{2\sqrt{2}} diag\, (I_{+\infty}^2 |W_+^\infty(\xi)|^2, I_{-\infty}^2 |W_-^\infty(\xi)|^2) \log t\right) W^\infty(\xi) \\
&+ O(\epsilon^2 \langle t\rangle^{-\delta_0})
\end{aligned}
\tag{4}
$$

*as $t \to \infty$, for some $\delta_0 > 0$, and $S(\xi)$ is the scattering matrix (stated later).*

# Application: 1D cubic-quintic Schrödinger equation

Consider the 1D Schrödinger equation with focusing cubic and defocusing quintic nonlinearities:

$$i\partial_t v + \partial_x^2 v + |v|^2 v - |v|^4 v = 0, \quad (t,x) \in \mathbb{R} \times \mathbb{R}. \tag{5}$$

For this model, the linearized problem around the solitary waves does not have internal mode nor resonance. Therefore our result can be applied to this case. This case is also considered in

- Y. Martel, Asymptotic stability of solitary waves for the 1D cubic-quintic Schrödinger equation with no internal mode, 2021
- D. E. Pelinovsky, Y. S. Kivshar and V. V. Afanasjev, Internal modes of envelope solitons, 1998

# Spectra of linearized operators for NLS solitary waves

- S.-M. Chang, S. Gustafson, K. Nakanishi, T.-P. Tsai, Spectra of Linearized Operators for NLS Solitary Waves, 2008

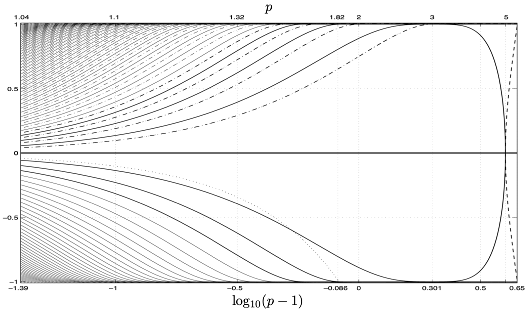



Figure 1: Spectra of $\mathcal{L}$, $L_+$ and $L_-$ for $n = 1$ with logarithmic axis for the values of $p - 1$. (solid line: purely imaginary eigenvalues of $\mathcal{L}$; dashed line: real eigenvalues of $\mathcal{L}$; dotted line: eigenvalues of $L_+$; dashdot line: eigenvalues of $L_-$)

Notice that this paper is talking about $i\mathcal{H}$ instead of $\mathcal{H}$, so the "purely imaginary eigenvalues" in the picture are the real eigenvalues of $\mathcal{H}$ (corresponding to the internal modes).

# Distorted Fourier transform

Our approach in this paper is based on the use of the distorted Fourier transform, that is, the so-called Weyl-Kodaira-Titchmarsh theory, which will allow us to apply some Fourier analytical techniques to nonlinear equations which involve some external potential.

# Distorted Fourier transform

- Jost functions: Bounded solutions $\mathcal{F}_+(x,\xi)$, $\mathcal{G}_+(x,\xi)$ of $\mathcal{H}f = (1+\xi^2)f$. We have $\mathcal{F}_+(x,0) = \mathcal{G}_+(x,0) = 0$, and

$$\begin{aligned}
\mathcal{F}_+(x,\xi) &\to s(\xi)e^{ix\xi}e_1 \text{ as } x \to +\infty, \\
\mathcal{F}_+(x,\xi) &\to [e^{ix\xi} + r(\xi)e^{-ix\xi}]e_1 \text{ as } x \to -\infty, \\
\mathcal{G}_+(x,\xi) &\to s(\xi)e^{-ix\xi}e_1 \text{ as } x \to -\infty, \\
\mathcal{G}_+(x,\xi) &\to [e^{-ix\xi} + r(\xi)e^{ix\xi}]e_1 \text{ as } x \to +\infty.
\end{aligned} \tag{6}$$

- Scattering matrix:

$$S(\xi) := \begin{pmatrix} s(\xi) & r(\xi) \\ r(\xi) & s(\xi) \end{pmatrix} \tag{7}$$

It satisfies

$$S(\xi)^* = S(\xi)^{-1} = S(-\xi) \tag{8}$$

for all $\xi \in \mathbb{R}$. Moreover, $s(0) = 0$, $r(0) = -1$, and

$$\begin{aligned}
|\partial_\xi^b(s(\xi) - s(0))| &= |\partial_\xi^b s(\xi)| \lesssim \langle \xi \rangle^{-1-b}, \\
|\partial_\xi^b(r(\xi) - r(0))| &= |\partial_\xi^b(r(\xi) + 1)| \lesssim \langle \xi \rangle^{-1-b}.
\end{aligned} \tag{9}$$

# Distorted Fourier transform

Denote the Pauli matrices by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (10)$$

Let

$$\mathcal{F}_-(x,\xi) := \sigma_1 \mathcal{F}_+(x,\xi), \ \mathcal{G}_-(x,\xi) := \sigma_1 \mathcal{G}_+(x,\xi), \quad (11)$$

$$\psi_\pm(x,\xi) := \begin{cases} \mathcal{F}_\pm(x,\xi) & \text{if } \xi \geq 0, \\ \mathcal{G}_\pm(x,-\xi) & \text{if } \xi \leq 0. \end{cases} \quad (12)$$

Denote $\psi(x,\xi) := (\psi_+(x,\xi), \psi_-(x,\xi))$. We define the *distorted* Fourier transform (dFT) in the *2D vector* case by

$$\begin{aligned} (\widetilde{\mathcal{F}} f)(\xi) &= \frac{1}{\sqrt{2\pi}} \int \overline{\psi(x,\xi)^\top} \sigma_3 f(x) dx = ((\widetilde{\mathcal{F}} f)_+, (\widetilde{\mathcal{F}} f)_-)^\top, \\ (\widetilde{\mathcal{F}}^{-1} \phi)(x) &= \frac{1}{\sqrt{2\pi}} \int \psi(x,\xi) \phi(\xi) d\xi. \end{aligned} \quad (13)$$

Formally,

$$\widetilde{\mathcal{F}}(e^{it\mathcal{H}} f) = (e^{it(1+\xi^2)} (\widetilde{\mathcal{F}} f)_+, e^{-it(1+\xi^2)} (\widetilde{\mathcal{F}} f)_-)^\top. \quad (14)$$

# Decay estimates

## Lemma (Pointwise decay)

*Under our assumptions on the potential,*

$$\|e^{it\mathcal{H}}h\|_{L^\infty} \lesssim \langle t\rangle^{-1/2}\|\widetilde{h}\|_{L^\infty} + \langle t\rangle^{-1/2-\alpha}\|\partial_\xi \widetilde{h}\|_{L^2} \tag{15}$$

*for* $\alpha \in (0, 1/4]$*.*

## Lemma (Enhanced local decay)

*Under our assumptions on the potential,*

$$\|\langle x\rangle^{-2}e^{it\mathcal{H}}h\|_{L^\infty_x} \lesssim \frac{1}{|t|}\|\widetilde{h}\|_{H^1_\xi}. \tag{16}$$

# Equation after Taking the Distorted Fourier transform

Recall that $f = e^{it\mathcal{H}}U$, and the equation on $U$ reads

$$i\partial_t U - \mathcal{H}U = \mathcal{N},$$

so that $f$ satisfies

$$\partial_t f = -ie^{it\mathcal{H}}\mathcal{N} \qquad \text{or} \qquad \partial_t \widetilde{f} = \begin{pmatrix} -ie^{it(1+\xi^2)}\widetilde{\mathcal{N}}_+ \\ -ie^{-it(1+\xi^2)}\widetilde{\mathcal{N}}_- \end{pmatrix}. \qquad (17)$$

The nonlinear term $-ie^{it\mathcal{H}}\mathcal{N}$ can be split into

$$\partial_t f = -ie^{it\mathcal{H}}\mathcal{N} = \mathcal{Q}^R + \mathcal{C}^S + \mathcal{C}^R + \mathcal{R}$$

where

- The regular quadratic terms $\mathcal{Q}^R$ include a rapidly decaying potential.
- The regular cubic terms $\mathcal{C}^R$ include a rapidly decaying potential.
- The singular cubic terms $\mathcal{C}^S$ do not include a decaying potential.
- The remainder terms $\mathcal{R}$ are higher order.

# Quadratic and cubic spectral distributions

## Proposition (Quadratic spectral distribution)

*Let $W(x)$ be a smooth coefficient function, such that $W(x)$ and its derivatives decay super-polynomially as $x \to \pm\infty$. There exists distributions $\mu^W_{jk\ell,\mu\nu}$ such that, if $f = (f_+, f_-)^\top$, $g = (g_+, g_-)^\top \in \mathcal{S}$, then*

$$\widetilde{\mathcal{F}}(W(x) f_j g_k e_\ell)(\xi) = \frac{1}{(2\pi)^{3/2}} \sum_{\lambda,\mu=\pm} \iint \widetilde{f}_\lambda(\eta) \widetilde{g}_\mu(\sigma) \mu^W_{jk\ell,\lambda\mu}(\xi,\eta,\sigma)\, d\eta\, d\sigma. \tag{18}$$

*Formally, $\mu^W_{jk\ell,\lambda\mu}$ is given by*

$$\mu^W_{jk\ell,\lambda\mu}(\xi,\eta,\sigma) := \int W(x)(\psi_\lambda(x,\eta)\cdot e_j)(\psi_\mu(x,\sigma)\cdot e_k)\overline{\psi(x,\xi)^\top}\sigma_3 e_\ell\, dx.$$

# Quadratic and cubic spectral distributions

## Proposition (Quadratic spectral distribution, cont'd)

*It satisfies the estimates*

$$|\partial_\xi^a \partial_\eta^b \partial_\sigma^c \mu^W_{jk\ell,\lambda\mu}(\xi,\eta,\sigma)| \lesssim \langle \inf_{\alpha,\beta} |\xi - \alpha\eta - \beta\sigma|\rangle^{-N} \frac{|\eta||\sigma|}{\langle\eta\rangle\langle\sigma\rangle}, \qquad (19)$$

$$|\partial_\xi^a \partial_\eta^b \partial_\sigma^c (\frac{1}{\eta\sigma} \mu^W_{jk\ell,\lambda\mu}(\xi,\eta,\sigma))| \lesssim \langle \inf_{\alpha,\beta} |\xi - \alpha\eta - \beta\sigma|\rangle^{-N} \frac{1}{\langle\eta\rangle\langle\sigma\rangle}. \qquad (20)$$

# Quadratic and cubic spectral distributions

## Proposition (Cubic spectral distribution)

*Let $W(x)$ be a smooth coefficient function, such that $W(x) - l_{\pm\infty}$ and its derivatives decay super-polynomially as $x \to \pm\infty$ for some constants $l_{\pm\infty}$. There exist distributions $\mu^W_{jk\ell m,\lambda\mu\nu}$ such that, if $f = (f_+, f_-)^\top$, $g = (g_+, g_-)^\top \in \mathcal{S}$, then*

$$\widetilde{\mathcal{F}}(Wf_j g_k h_\ell e_m)(\xi) = (4\pi^2)^{-1} \sum_{\lambda,\mu,\nu=\pm} \iiint \widetilde{f}_\lambda(\eta)\widetilde{g}_\mu(\sigma)\widetilde{h}_\nu(\zeta)\mu^W_{jk\ell m,\lambda\mu\nu}(\xi,\eta,\sigma,\zeta)\, d\eta\, d\sigma\, d\zeta. \tag{21}$$

*Formally, $\mu^W_{jk\ell m,\lambda\mu\nu} = (\mu^W_{jk\ell m,\lambda\mu\nu+}, \mu^W_{jk\ell m,\lambda\mu\nu-})^\top$, with*

$$\mu^W_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta) := (-1)^{m-1} \int W(x) \cdot (\psi_\lambda(x,\eta)\cdot e_j)(\psi_\mu(x,\sigma)\cdot e_k)(\psi_\nu(x,\zeta)\cdot e_\ell)(\overline{\psi_\rho(x,\xi)}\cdot e_m)\, dx.$$

# Quadratic and cubic spectral distributions

## Proposition (Cubic spectral distribution, cont'd)

*Moreover, it can be decomposed into*

$$\mu^W_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta) = \mu^S_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta) + \mu^R_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta)$$

*where the following holds:*
*1) The singular part* $\mu^S_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta)$ *can be written as*

$$\begin{aligned} &\mu^S_{jk\ell m,\lambda\mu\nu\rho}(\xi,\eta,\sigma,\zeta) \\ &:= \sum_{\epsilon=\pm} l_{\epsilon\infty} \sum_{\alpha,\beta,\gamma,\delta=\pm} a^{\epsilon}_{\substack{jk\ell m,\lambda\mu\nu\rho \\ \alpha\beta\gamma\delta}}(\xi,\eta,\sigma,\zeta)\Big[\sqrt{\frac{\pi}{2}}\delta(p) + \epsilon p.v.\frac{\hat{\phi}_3(p)}{ip}\Big], \\ &p := \alpha\xi - \beta\eta - \gamma\sigma - \delta\zeta, \end{aligned} \tag{22}$$

*where* $\phi_3$ *is a smooth, even, real-valued, compactly supported function with integral one. The coefficients are given by* $s(\xi)$ *and* $r(\xi)$*.*

# Quadratic and cubic spectral distributions

## Proposition (Cubic spectral distribution, cont'd)

*2) The regular part $\mu^R_{jk\ell m,\lambda\mu\nu}(\xi,\eta,\sigma,\zeta)$ can be written as*

$$\mu^R_{jk\ell m,\lambda\mu\nu}(\xi,\eta,\sigma,\zeta) = \sum_{\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4=\pm} 1_{\epsilon_1}(\xi)1_{\epsilon_2}(\eta)1_{\epsilon_3}(\sigma)1_{\epsilon_4}(\zeta)\boldsymbol{\tau}_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(\xi,\eta,\sigma,\zeta), \quad j,k,\ell,m=1,2,\ \lambda,\mu,\nu=\pm \tag{23}$$

*where for any $N$, $a$, $b$, $c$, $d$,*

$$|\partial_\xi^a\partial_\eta^b\partial_\sigma^c\partial_\zeta^d\boldsymbol{\tau}_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(\xi,\eta,\sigma,\zeta)| \lesssim \langle \inf_{\alpha,\beta,\gamma}|\xi-\alpha\eta-\beta\sigma-\gamma\zeta|\rangle^{-N}\frac{|\eta||\sigma||\zeta|}{\langle\eta\rangle\langle\sigma\rangle\langle\zeta\rangle}, \tag{24}$$

$$|\partial_\xi^a\partial_\eta^b\partial_\sigma^c\partial_\zeta^d(\frac{1}{\eta\sigma\zeta}\boldsymbol{\tau}_{\epsilon_1\epsilon_2\epsilon_3\epsilon_4}(\xi,\eta,\sigma,\zeta))| \lesssim \langle \inf_{\alpha,\beta,\gamma}|\xi-\alpha\eta-\beta\sigma-\gamma\zeta|\rangle^{-N}\frac{1}{\langle\eta\rangle\langle\sigma\rangle\langle\zeta\rangle}. \tag{25}$$

# 1D quadratic Klein-Gordon equation

We consider the 1D quadratic Klein-Gordon equation

$$(\partial_t^2 - \partial_x^2 + V + 1)u = a(x)u^2 + b(x)u^3 \tag{26}$$

under the following assumptions: the potential $V$ is even and rapidly decaying together with its derivatives:

$$|\partial_x^\alpha V(x)| \lesssim_{\alpha,N} \langle x \rangle^{-N}, \qquad \text{for all } \alpha, N,$$

and the functions $a$ and $b$ have limits at $\pm\infty$, which they reach rapidly:

$$|\partial_x^\alpha (a(x) - a_{\pm\infty})| \lesssim_{\alpha,N} \langle x \rangle^{-N}, \quad |\partial_x^\alpha (b(x) - b_{\pm\infty})| \lesssim_{\alpha,N} \langle x \rangle^{-N}, \\ \text{for } \pm x > 0. \tag{27}$$

We assume that $b$ is even, and will specify additional parity assumptions on $a$ below.

# 1D quadratic Klein-Gordon equation

Furthermore, the Schrödinger operator

$$H = -\partial_x^2 + V$$

is assumed to have no eigenvalues.
We provide this equation with data at time zero:

$$u(t = 0) = u_0, \quad \partial_t u(t = 0) = u_1.$$

In addition to regularity and decay conditions on these data, we will assume that one of the three following assumptions is satisfied

- The operator $H$ is generic.
- The operator $H$ is exceptional, with an even zero energy resonance, the data and $a(x)$ are odd.
- The operator $H$ is exceptional, with an odd zero energy resonance, the data and $a(x)$ are even.

# 1D quadratic Klein-Gordon equation

**Theorem (P. Germain, F. Pusateri, Z. Z., 2022)**

*Under the above assumptions, there exists* $\varepsilon_0 > 0$ *such that, if* $(u_0, u_1)$ *satisfy*

$$\|(\sqrt{H+1}u_0, u_1)\|_{H^4} + \|\langle x\rangle(\sqrt{H+1}u_0, u_1)\|_{H^1} = \varepsilon < \varepsilon_0,$$

*then a global solution exists, which decays pointwisely and is globally bounded in* $L^2$ *type spaces: for any time* $t$,

$$\|(\sqrt{H+1}u(t), \partial_t u(t))\|_{L^\infty} \lesssim \varepsilon\langle t\rangle^{-1/2}$$
$$\|u(t)\|_{H^5} + \|\partial_t u(t)\|_{H^4} \lesssim \varepsilon\langle t\rangle^{p_0},$$

*where* $p_0$ *is some small number.*

# Comparison to an earlier work

In an earlier work

- P. Germain and F. Pusateri, Quadratic Klein-Gordon equations with a potential in 1D, 2020

there is a singularity (full resonance) at the frequencies $\pm\sqrt{3}$. With additional assumptions in our result, additional cancellations at zero frequency are available.

This additional cancellation is responsible for stronger local decay, or for the vanishing of the quadratic spectral distribution at zero frequency. These effects are essentially equivalent; the former provides a more direct physical intuition, while the latter is the main tool which allows to simplify the proof.

## Decay estimates

Let

$$B := \sqrt{-\partial_x^2 + V + 1} = \sqrt{H+1}. \quad (28)$$

### Proposition (Pointwise decay)

*Under our assumptions on the potential,*

$$\|e^{itB} P_c f\|_{L^\infty} \lesssim \langle t \rangle^{-1/2} \|\langle \xi \rangle^{3/2} \widetilde{f}\|_{L^\infty} + \langle t \rangle^{-11/20} \|\langle \xi \rangle \partial_\xi \widetilde{f}\|_{L^2} + \langle t \rangle^{-7/12} \|f\|_{H^4}. \quad (29)$$

### Proposition (Enhanced local decay)

*Under our assumptions on the potential, on the parity of the zero energy resonance, and of $f$, we have*

$$\left\|\langle x \rangle^{-2} e^{itB} P_c f\right\|_{L^\infty_x} \lesssim \langle t \rangle^{-1} \left( \left\|\partial_\xi \widetilde{f}\right\|_{L^1_\xi} + \left\|\widetilde{f}\right\|_{L^1_\xi} + \|f\|_{H^1_x} \right). \quad (30)$$

# Quadratic spectral distribution

## Proposition (The odd case)

*Let $f, g \in \mathcal{S}$ be odd functions, and $a$ be odd and satisfying (27). There exists a distribution $\mu^{o}_{\iota_1\iota_2}$ such that, for $\xi \geq 0$,*

$$\widetilde{\mathcal{F}}\big(a\, f_{\iota_1} g_{\iota_2}\big)(\xi) = \iint_{(\mathbb{R}_+)^2} (\widetilde{f})_{\iota_1}(\eta)(\widetilde{g})_{\iota_2}(\sigma)\, \mu^{o}_{\iota_1\iota_2}(\xi,\eta,\sigma)\, d\eta\, d\sigma, \quad (31)$$

*with odd extension to $\xi < 0$, and such that $\mu^{o}_{\iota_1\iota_2}$ can be split into a singular and a regular part as follows:*

$$(2\pi)\mu^{o}_{\iota_1\iota_2} = \mu^{o,S}_{\iota_1\iota_2} + \mu^{o,R}_{\iota_1\iota_2},$$

# Quadratic spectral distribution

## Proposition (The odd case, cont'd)

- *The singular part $\mu^{o,S}_{\iota_1\iota_2}$ is given by*

$$\mu^{o,S}_{\iota_1\iota_2}(\xi,\eta,\sigma) = \sum_{\lambda,\mu,\nu} \overline{a_\lambda(\xi)}(a_\mu(\eta))_{\iota_1}(a_\nu(\sigma))_{\iota_2} \cdot \ell_{+\infty}\left[\sqrt{\frac{\pi}{2}}\delta(p) + i\,\varphi^*(p,\eta,\sigma)\,\mathrm{p.v.}\frac{\widehat{\phi}(p)}{p}\right]_{p=\lambda\xi-\mu\iota_1\eta-\nu\iota_2\sigma} \tag{32}$$

*where the coefficients are given by the scattering matrix.*
*$\phi \in \mathcal{S}$ is even with integral one, and*

$$\varphi^*(p,\eta,\sigma) = \varphi_{\le -D}\big(pR(\eta,\sigma)\big), \quad R(\eta,\sigma) = \frac{\langle\eta\rangle\langle\sigma\rangle}{\langle\eta\rangle + \langle\sigma\rangle}. \tag{33}$$

# Quadratic spectral distribution

## Proposition (The odd case, cont'd)

- *The regular part* $\mu^{o,R}_{\iota_1\iota_2}$ *satisfies, for* $\xi, \eta, \sigma > 0$

$$\mu^{o,R}_{\iota_1\iota_2}(\xi,\eta,\sigma) = \frac{\xi\cdot\eta\cdot\sigma}{\langle\xi\rangle\langle\eta\rangle\langle\sigma\rangle}\mathfrak{q}^{o}_{\iota_1\iota_2}(\xi,\eta,\sigma) \tag{34}$$

*where*

$$|\partial^a_\xi\partial^b_\eta\partial^c_\sigma\mathfrak{q}^o_{\iota_1\iota_2}(\xi,\eta,\sigma)| \lesssim \sup_{\mu,\nu}\frac{1}{\langle\xi+\mu\eta+\nu\sigma\rangle^N}|R(\eta,\sigma)|^{1+a+b+c},$$
$$|a|+|b|+|c| \leq N. \tag{35}$$

Thank you!