A class of asymptotic preserving numerical schemes for low Mach number flows.

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Context and motivation

Institute for Radio-protection and Nuclear Safety (IRSN).

An industrial CFD code:

- Stable schemes independently of the time and space steps.
- Accurate schemes for both compressible and incompressible flows.

The numerical methods described in this talk are implemented in the open source CALIF^{3}S software, designed by the IRSN.

Outline

- The incompressible limit
- Co-localized discretizations *v.s.* staggered discretizations
- The incompressible limit for staggered discretizations
- Conclusion and perspectives

Non-dimensionalised Navier-Stokes equations

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) + \boldsymbol{\nabla} p = 0.$$

Characteristic quantities of the flow :

L_∞ :	characteristic length,	U_∞ :	mean velocity,
ρ_∞ :	mean density,	c_∞ :	mean speed of sound,

Non-dimensional quantities :

$$\bar{\boldsymbol{x}} = \frac{\boldsymbol{x}}{L_\infty}, \quad \bar{t} = t\,\frac{U_\infty}{L_\infty}, \quad \bar{\rho} = \frac{\rho}{\rho_\infty}, \quad \bar{\boldsymbol{u}} = \frac{\boldsymbol{u}}{U_\infty}, \quad \bar{p} = \frac{p}{\rho_\infty c_\infty^2}.$$

Non-dimensionalised Navier-Stokes equations

Non-dimensionalized equations :

$$\partial_{\bar{t}}\bar{\rho} + \mathrm{div}_{\bar{x}}(\bar{\rho}\bar{\boldsymbol{u}}) = 0,$$

$$\partial_{\bar{t}}(\bar{\rho}\bar{\boldsymbol{u}}) + \mathrm{div}_{\bar{x}}(\bar{\rho}\bar{\boldsymbol{u}} \otimes \bar{\boldsymbol{u}}) - \frac{1}{\mathrm{Re}}\Big(\boldsymbol{\Delta}_{\bar{x}}\bar{\boldsymbol{u}} + \frac{2}{3}\boldsymbol{\nabla}_{\bar{x}}(\mathrm{div}\bar{\boldsymbol{u}})\Big) + \frac{1}{\mathcal{M}^2}\boldsymbol{\nabla}_{\bar{x}}\bar{p} = 0.$$

with

$\mathrm{Re} = \dfrac{\rho_\infty L_\infty U_\infty}{\mu}$: Reynolds number, $\mathcal{M} = \dfrac{U_\infty}{c_\infty}$: Mach number.

We study the limit $\mathcal{M} \to 0$, assuming that $\dfrac{1}{\mathrm{Re}(\mathcal{M})} = \bar{\mu} + o(1)$ when $\mathcal{M} \to 0$.

Zero Mach number limit : the continuous problem

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{1, 2, 3\}$.

- Non dimensionalised Navier-Stokes: on $\Omega \times (0, T)$:

$$\partial_t \rho^\varepsilon + \mathrm{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon) = 0,$$

$$\partial_t (\rho^\varepsilon \boldsymbol{u}^\varepsilon) + \mathrm{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon) + \frac{1}{\varepsilon^2} \boldsymbol{\nabla} (\rho^\varepsilon)^\gamma = 0,$$

ε = Mach number, $\mathrm{div}(\boldsymbol{\tau}^\varepsilon) = \mu \boldsymbol{\Delta} \boldsymbol{u}^\varepsilon + (\lambda + \mu) \boldsymbol{\nabla} (\mathrm{div} \boldsymbol{u}^\varepsilon)$.

- BC: $\boldsymbol{u}^\varepsilon = 0$ on $\partial\Omega$.
- « Ill prepared » initial condition:

$$\|\boldsymbol{u}_0^\varepsilon\|_{\mathrm{L}^2(\Omega)^d} + \frac{1}{\varepsilon} \|\rho_0^\varepsilon - 1\|_{\mathrm{L}^\infty(\Omega)} \leqslant C.$$

- « Well prepared » initial condition:

$$\|\boldsymbol{u}_0^\varepsilon\|_{\mathrm{H}^1(\Omega)^d} + \frac{1}{\varepsilon} \|\mathrm{div}\, \boldsymbol{u}_0^\varepsilon\|_{\mathrm{L}^2(\Omega)} + \frac{1}{\varepsilon^2} \|\rho_0^\varepsilon - 1\|_{\mathrm{L}^\infty(\Omega)} \leqslant C.$$

Zero Mach number limit : the continuous problem

Let $T > 0$ and $\Omega \subset \mathbb{R}^d$, $d \in \{1,2,3\}$.

$$\partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon) = 0,$$

$$\partial_t(\rho^\varepsilon \boldsymbol{u}^\varepsilon) + \operatorname{div}(\rho^\varepsilon \boldsymbol{u}^\varepsilon \otimes \boldsymbol{u}^\varepsilon) - \mu \boldsymbol{\Delta} \boldsymbol{u}^\varepsilon - (\lambda + \mu) \boldsymbol{\nabla}(\operatorname{div} \boldsymbol{u}^\varepsilon) + \frac{1}{\varepsilon^2} \boldsymbol{\nabla}(\rho^\varepsilon)^\gamma = 0,$$

$$\boldsymbol{u}_{\varepsilon}{}_{|\partial\Omega} = 0.$$

Formally, when $\varepsilon \to 0$:

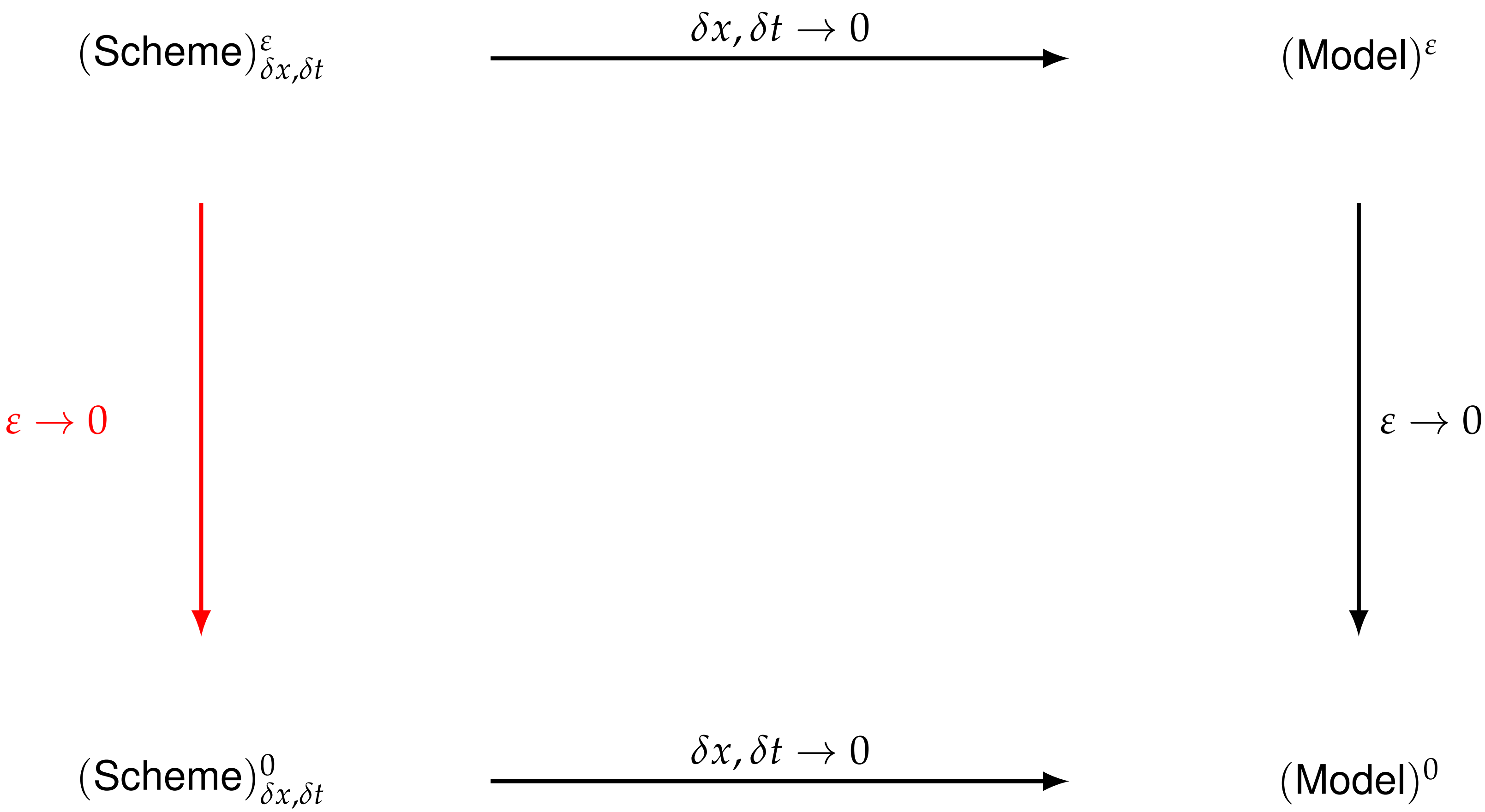
- Assuming $\rho^\varepsilon \to \rho$ and $\boldsymbol{u}^\varepsilon \to \boldsymbol{u}$.
- $\boldsymbol{\nabla} \wp(\rho^\varepsilon) \to \boldsymbol{\nabla} \wp(\rho) = 0 \quad \Longrightarrow \quad \rho = \rho(t) \quad \overset{\text{BC}}{\Longrightarrow} \quad \rho = cst = \rho_0.$
- At the limit we obtain *formally*:

$$\operatorname{div}(\boldsymbol{u}) = 0,$$

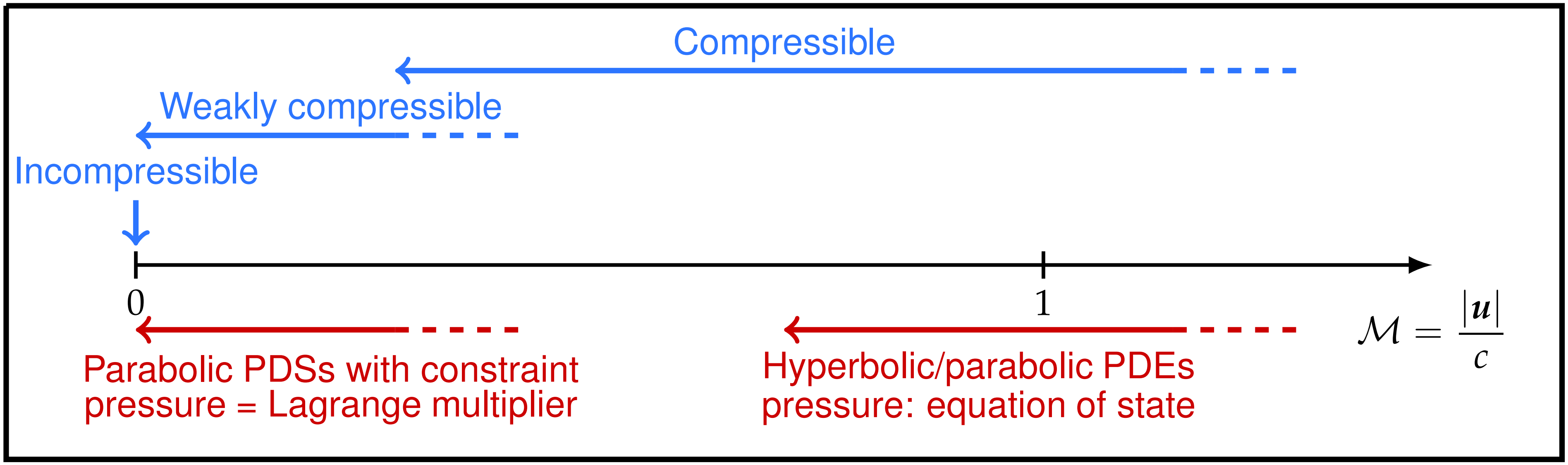
$$\rho_0 \partial_t \boldsymbol{u} + \rho_0 \operatorname{div}(\boldsymbol{u} \otimes \boldsymbol{u}) - \mu \boldsymbol{\Delta} \boldsymbol{u} + \boldsymbol{\nabla} \pi = 0$$

with $\pi = \lim\limits_{\varepsilon \to 0} \frac{1}{\varepsilon^2}((\rho^\varepsilon)^\gamma - \rho_0^\gamma)$.

Asymptotic preserving numerical schemes



Accurate schemes for all Mach number regimes



Difficulties:

- Different numerical schemes for different PDEs.
- Different discretization constraints (pressure oscillations, CFL,...).

Goal: Design numerical schemes for all Mach number regimes.

Co-localized discretizations for compressible flows

$$\partial_t \rho + \operatorname{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t (\rho \boldsymbol{u}) + \operatorname{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \operatorname{div} \boldsymbol{\tau}(\boldsymbol{u}) + \frac{1}{\varepsilon^2} \boldsymbol{\nabla} p(\rho) = 0.$$

Advantages:

- Two unknowns $(\rho, \boldsymbol{u})$: it is easier to discretize them at the same location.
- Classical Finite Volume techniques (Riemann solvers...).
- Discrete energy and entropy inequalities are easy to prove.
- Discrete jump relations, shock waves are correctly approximated.

Difficulties: ($\varepsilon =$ Mach number)

- Propagation speeds: $u - u/\varepsilon$, u, $u + u/\varepsilon$.
- If explicit scheme $\Longrightarrow$ unstable because the CFL is too restrictive: $\delta t \leqslant \varepsilon\big(\delta x + \delta x^2/\mu\big)$.
- Odd-even decoupling of the pressure in the limit $\varepsilon \to 0$.

Staggered discretizations for incompressible flows

$$\operatorname{div} \boldsymbol{u} = 0,$$

$$-\mu \boldsymbol{\Delta u} + \boldsymbol{\nabla} p = \boldsymbol{f},$$

$$\boldsymbol{u}_{|\partial\Omega} = 0, \quad \int_\Omega p = 0.$$

- Step 1 : Eliminate the pressure :

  Solve for $\boldsymbol{u} \in V = \{\boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d, \operatorname{div} \boldsymbol{v} = 0\}$:

$$\mu \int_\Omega \boldsymbol{\nabla u} : \boldsymbol{\nabla v} - \int_\Omega p \operatorname{div} \boldsymbol{v} =< \boldsymbol{f}, \boldsymbol{v} >_{\mathrm{H}^{-1}, \mathrm{H}_0^1} \qquad \forall \boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d.$$

Staggered discretizations for incompressible flows

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Staggered discretizations for incompressible flows

$$\operatorname{div} \boldsymbol{u} = 0,$$

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- Step 1 : Eliminate the pressure :

  Solve for $\boldsymbol{u} \in V = \{\boldsymbol{v} \in \mathrm{H}_0^1(\Omega)^d, \operatorname{div} \boldsymbol{v} = 0\}$:

$$\mu \int_\Omega \boldsymbol{\nabla} \boldsymbol{u} : \boldsymbol{\nabla} \boldsymbol{v} =< \boldsymbol{f}, \boldsymbol{v} >_{\mathrm{H}^{-1},\mathrm{H}_0^1} \qquad \forall \boldsymbol{v} \in V.$$

- Step 2: Recover the pressure:

  Solve for $p \in \mathrm{L}_0^2(\Omega) = \{q \in \mathrm{L}^2(\Omega) \ \int_\Omega q = 0\}$:

$$\boldsymbol{\Delta} p = \operatorname{div} \boldsymbol{f}.$$

  Is this laplacian operator invertible ?

Staggered discretizations for incompressible flows

Solve for $p \in \mathrm{L}^2_0(\Omega) = \{q \in \mathrm{L}^2(\Omega)\ \int_\Omega q = 0\}$:

$$\boldsymbol{\nabla} p = \mu \boldsymbol{\Delta u} + \boldsymbol{f} \in V^\perp.$$

Is there a continuous inverse $\boldsymbol{\nabla}^{-1} : V^\perp \longrightarrow \mathrm{L}^2_0(\Omega)$?

Theorem : (inf-sup property) *There exists* $\beta_\Omega > 0$ *s.t.*

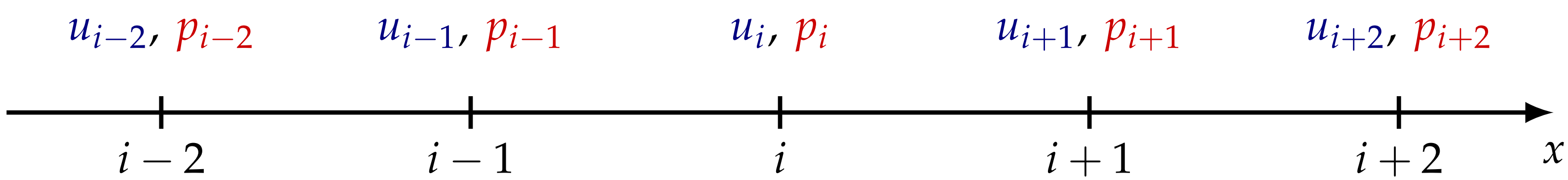
$$\inf_{p \in \mathrm{L}^2_0} \sup_{\boldsymbol{v} \in \mathrm{H}^1_0} \frac{\displaystyle\int_\Omega p \operatorname{div} \boldsymbol{v}}{\|p\|_{\mathrm{L}^2} \|\boldsymbol{v}\|_{\mathrm{H}^1_0}} \geqslant \beta_\Omega.$$

In other words, for all $p \in \mathrm{L}^2_0(\Omega)$ there exists $\boldsymbol{v} \in \mathrm{H}^1_0$ such that :

$$\frac{\displaystyle\int_\Omega p \operatorname{div} \boldsymbol{v}}{\|\boldsymbol{v}\|_{\mathrm{H}^1_0}} \geqslant \beta_\Omega \|p\|_{\mathrm{L}^2}.$$

Therefore, if $\boldsymbol{\nabla} p \in \mathrm{H}^{-1}$ then $\|p\|_{\mathrm{L}^2} \leqslant \frac{1}{\beta_\Omega} \|\boldsymbol{\nabla} p\|_{\mathrm{H}^{-1}}$.

Co-localized schemes : odd-even decoupling



Discrete momentum equation:

$$-\mu \frac{u_{i+1} - 2u_i + u_{i-1}}{\delta x^2} + \frac{p_{i+1} - p_{i-1}}{2\delta x} = f_i, \qquad \forall i.$$

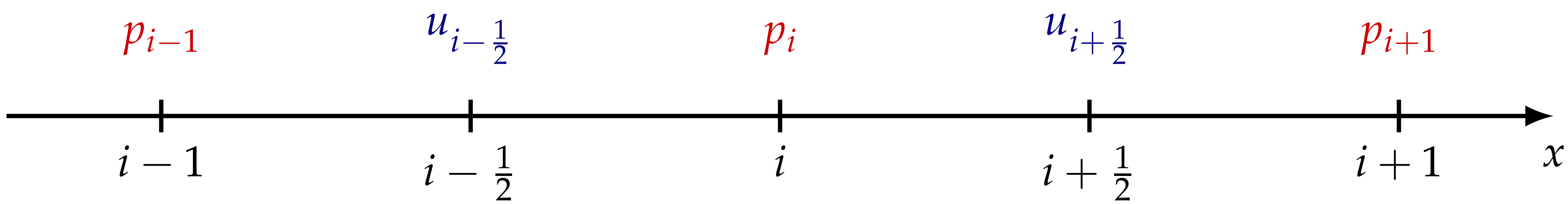
Take the divergence $(\partial_x X)_i = \big(X_{i+1} - X_{i-1}\big) / \delta x$

$$\frac{p_{i+2} - 2p_i + p_{i-2}}{4\delta x^2} = \frac{f_{i+1} - f_{i-1}}{2\delta x}$$

This discrete laplacian operator (and the discrete gradient operator) is not invertible

($\ker \boldsymbol{\Delta} \neq \{0\}$!)

Staggered discretization



Discrete divergence on the mesh $\mathbb{Z}$:

$$\frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{\delta x} = 0, \qquad \forall i$$

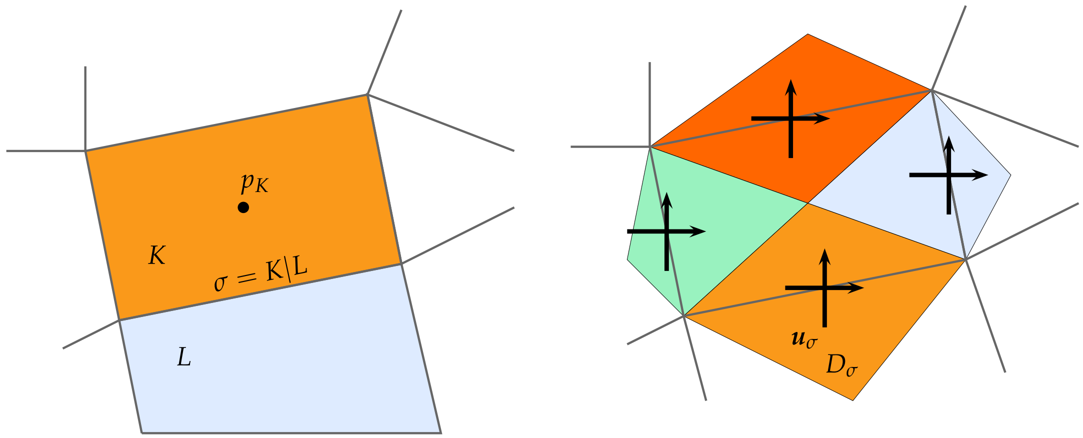
Discrete momentum equation on the mesh $\mathbb{Z} + \frac{1}{2}$:

$$-\mu \frac{u_{i+\frac{3}{2}} - 2u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}}{\delta x^2} + \frac{p_{i+1} - p_{i-1}}{2\delta x} = f_{i+\frac{1}{2}}, \qquad \forall i.$$

Take the divergence $(\partial_x X)_i = \left(X_{i+\frac{1}{2}} - X_{i-\frac{1}{2}}\right)/\delta x$

$$\frac{p_{i+1} - 2p_i + p_{i-1}}{\delta x^2} = \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{\delta x}$$

Staggered discretization



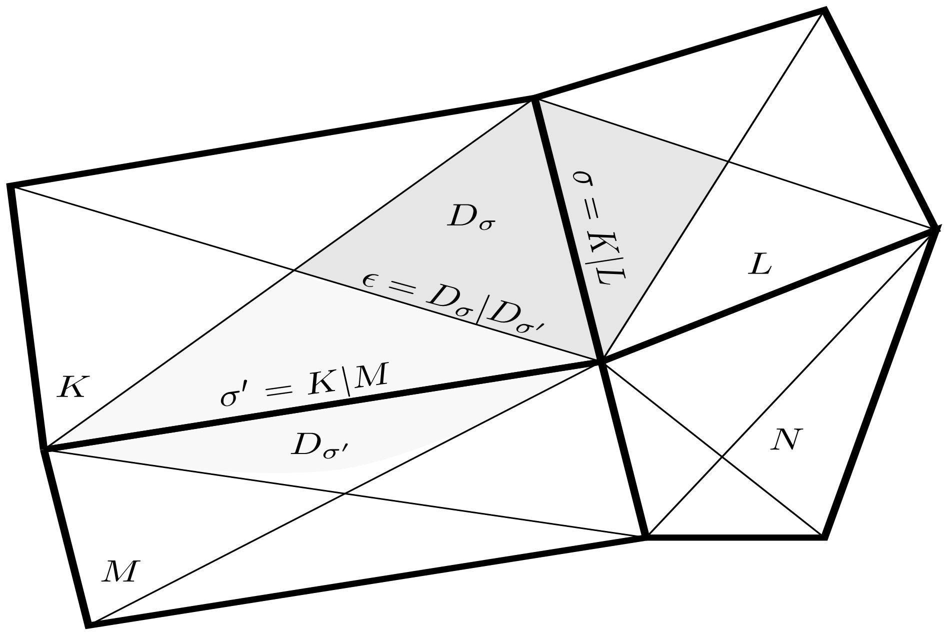
- Primal mesh : $\mathcal{M}$ = { convexe polygones K}.
- Pressure unknowns at the cells centers of the primal mesh: $(p_K)_{K\in\mathcal{M}}$. Discrete pressure space :

$$\mathrm{L}_{\mathcal{M}} = \{p = \sum_{K\in\mathcal{M}} p_K \mathbb{1}_K\}$$

- Velocity unknowns at the faces: $(\boldsymbol{u}_\sigma)_{\sigma\in\mathcal{E}}$. Dual mesh: $\mathcal{D} = \{$ dual cells D_σ, $\sigma \in \mathcal{E}$ $\}$. Discrete velocity space:

$$\mathrm{H}_{\mathcal{E}} = \{\boldsymbol{u} = \sum_{\sigma\in\mathcal{E}} \boldsymbol{u}_\sigma \mathbb{1}_{D_\sigma}\}$$

Staggered discretization



Solve for $(\boldsymbol{u}, p) \in \mathrm{H}_{\mathcal{E}} \times \mathrm{L}_{\mathcal{M},0}$:

$$\begin{aligned} (\operatorname{div} \boldsymbol{u})_K = 0, \qquad & K \in \mathcal{M}, \\ -\mu\,(\boldsymbol{\Delta u})_\sigma + (\boldsymbol{\nabla} p)_\sigma = 0, \quad & \sigma \in \mathcal{E}_{\text{int}}, \end{aligned}$$

with $(\operatorname{div} \boldsymbol{u})_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}, \qquad (\boldsymbol{\nabla} p)_\sigma = \frac{|\sigma|}{|D_\sigma|}(p_L - p_K)\boldsymbol{n}_{K,\sigma}.$

Accurate schemes for all Mach number regimes

Usual strategy: Stabilize the numerical methods designed for compressible flows (often co-localized) in the low Mach number regime:

- Approximate Godunov solvers with modified numerical fluxes.
- Implicitation of acoustic stiff terms.

Our strategy: Extend to compressible models the existing numerical methods designed for incompressible flows.

- Staggered space discretization.

  $$\text{Discrete inf-sup condition} \Longrightarrow \text{stabilized pressure}$$

- Time discretization projection method adapted to compressible flows.

Compressible isentropic Navier-Stokes equations

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\,\boldsymbol{\tau} + \boldsymbol{\nabla} \wp(\rho) = 0.$$

- Boundary condition $\boldsymbol{u} = 0$ on $\partial\Omega$.
- The numerical schemes described in this talk can be extended to other models:
  - Full compressible Navier-Stokes system (with energy equations), Majda-Sethian low Mach number model.
  - Evolution equations for (reactive) chemical species.
  - Baer-Nunziato two-phase flow model.
  - Other boundary condition . . .

An implicit scheme

- **Continuity equation**: (Finite Volumes on the primal mesh)

$$\frac{1}{\delta t}\,(\rho_K^{n+1}-\rho_K^{n})+\mathrm{div}(\rho^{n+1}\boldsymbol{u}^{n+1})_K=0, \qquad K\in\mathcal{M},$$

with $\quad \mathrm{div}(\rho\boldsymbol{u})_K=\dfrac{1}{|K|}\displaystyle\sum_{\sigma\in\mathcal{E}(K)}F_{K,\sigma}$ $\quad$ and $\quad F_{K,\sigma}=|\sigma|\,\rho_\sigma^{\mathrm{up}}\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma},$

$\rho_\sigma^{\mathrm{up}}$: *upwind* approximation of ρ on the faces σ of K.

- **Momentum equation** : (Hybrid VF/ EF method on the dual mesh)

$$\frac{1}{\delta t}\,\Big[\rho_\sigma^{n+1}\,\boldsymbol{u}_\sigma^{n+1}-\rho_\sigma^{n}\boldsymbol{u}_\sigma^{n}\Big]+\mathrm{div}(\rho^{n+1}\boldsymbol{u}^{n+1}\otimes\boldsymbol{u}^{n+1})_\sigma+(\mathrm{div}\,\boldsymbol{\tau})_\sigma^{n+1}+(\boldsymbol{\nabla}p)_\sigma^{n+1}=0, \quad \sigma\in\mathcal{E}_{\mathrm{int}},$$

with $\quad \mathrm{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})_\sigma=\dfrac{1}{|D_\sigma|}\displaystyle\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F_{\sigma,\epsilon}\,\boldsymbol{u}_\epsilon,$

$\boldsymbol{u}_\epsilon$ *upwind or centered* approximation of $\boldsymbol{u}$ on the faces ϵ of the dual cell D_σ.

Zero Mach number limit : the continuous problem

For weak solutions :

Main difficulty in the analysis : passing to the limit in the convective non linear term.

- Lions-Masmoudi '98: $\Omega = \mathbb{T}^d$.
- Desjardins-Grenier '99 : $\Omega = \mathbb{R}^d$.
- Desjardins-Grenier-Lions-Masmoudi '02 : Ω bounded with homogeneous Dirichlet conditions.
- Fereisl-Novotny '09: Navier-Stokes-Fourier with energy and *ad hoc* pressure law.
- Masmoudi-Rousset-Sun '21: (Navier boundary conditions).

For strong solutions :

- Klainerman - Majda.
- Métivier - Schochet.
- Hagstrom-Lorenz.
- Gallagher.
- Danchin.
- Alazard.

Zero Mach number limit : the continuous problem

A priori Estimates:

- Kinetic energy:

$$\boldsymbol{u}^{\varepsilon}\cdot \qquad \partial_t(\rho^{\varepsilon}\boldsymbol{u}^{\varepsilon}) + \operatorname{div}(\rho^{\varepsilon}\boldsymbol{u}^{\varepsilon}\otimes\boldsymbol{u}^{\varepsilon}) - \operatorname{div}(\boldsymbol{\tau}^{\varepsilon}) + \frac{1}{\varepsilon^2}\boldsymbol{\nabla} p^{\varepsilon} = 0,$$

$$\hookrightarrow \qquad \partial_t\Big(\frac{1}{2}\rho^{\varepsilon}|\boldsymbol{u}^{\varepsilon}|^2\Big) + \operatorname{div}\Big(\frac{1}{2}\rho^{\varepsilon}|\boldsymbol{u}^{\varepsilon}|^2\boldsymbol{u}^{\varepsilon}\Big) - \operatorname{div}(\boldsymbol{\tau}^{\varepsilon})\cdot\boldsymbol{u}^{\varepsilon} + \frac{1}{\varepsilon^2}\boldsymbol{\nabla} p^{\varepsilon}\cdot\boldsymbol{u}^{\varepsilon} = 0.$$

- Renormalization identity

$$b'(\rho^{\varepsilon})\times \qquad \partial_t\rho^{\varepsilon} + \operatorname{div}(\rho^{\varepsilon}\boldsymbol{u}^{\varepsilon}) = 0,$$

$$\hookrightarrow \qquad \partial_t b(\rho^{\varepsilon}) + \operatorname{div}(b(\rho^{\varepsilon})\boldsymbol{u}^{\varepsilon}) + (\rho^{\varepsilon}b'(\rho^{\varepsilon}) - b(\rho^{\varepsilon}))\operatorname{div}\boldsymbol{u}^{\varepsilon} = 0.$$

We choose b s.t. $sb'(s) - b(s) = s^{\gamma}$, *i.e.*:

$$b(s) = \frac{s^{\gamma}}{\gamma-1}.$$

Zero Mach number limit : the continuous problem

A priori estimates:

- Kinetic energy and "free energy Π_γ":

$$\partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{1}{(\gamma-1)\varepsilon^2}\partial_t((\rho^\varepsilon)^\gamma) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(f) = 0,$$

$$\hookrightarrow \qquad \partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{1}{(\gamma-1)\varepsilon^2}\partial_t((\rho^\varepsilon)^\gamma - 1) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(f) = 0.$$

Zero Mach number limit : the continuous problem

A priori estimates:

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$$\partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{1}{(\gamma-1)\varepsilon^2}\partial_t((\rho^\varepsilon)^\gamma) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(f) = 0,$$

$$\hookrightarrow \qquad \partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{1}{(\gamma-1)\varepsilon^2}\partial_t((\rho^\varepsilon)^\gamma - 1) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(f) = 0.$$

Mass conservation equation :

$$-\frac{\gamma}{(\gamma-1)\varepsilon^2}\times \quad \partial_t(\rho^\varepsilon - 1) + \mathrm{div}(\rho^\varepsilon\boldsymbol{u}^\varepsilon) = 0.$$

Summing:

$$\partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{C_\gamma}{\varepsilon^2}\partial_t(\Pi_\gamma(\rho^\varepsilon)) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(g) = 0.$$

with

$$\Pi_\gamma(s) = s^\gamma - 1 - \gamma(s-1) \qquad C_\gamma = \frac{1}{\gamma-1}$$

Zero Mach number limit : the continuous problem

A priori estimates:

$$\partial_t(\frac{1}{2}\rho^\varepsilon|\boldsymbol{u}^\varepsilon|^2) + \frac{C_\gamma}{\varepsilon^2}\partial_t(\Pi_\gamma(\rho^\varepsilon)) - \mathrm{div}(\boldsymbol{\tau}^\varepsilon)\cdot\boldsymbol{u}^\varepsilon + \mathrm{div}(g) = 0,$$

The function Π_γ satisfies:

$$\begin{aligned}\Pi_\gamma(\rho) &= \wp(\rho) - \wp(1) - \gamma\wp'(1)(\rho-1)\\ &= |\rho-1|^2\,\gamma\int_0^1 (1+s(\rho-1))^{\gamma-2}(1-s)\mathrm{d}s.\end{aligned}$$

Integrating on $\Omega\times(0,t)$ + BC:

$$\begin{aligned}\int_\Omega \frac{1}{2}\rho^\varepsilon(.,t)|\boldsymbol{u}^\varepsilon(.,t)|^2\,\mathrm{d}\boldsymbol{x} + \frac{C_\gamma}{\varepsilon^2}\int_\Omega \Pi_\gamma(\rho^\varepsilon(.,t))\,\mathrm{d}\boldsymbol{x} + \mu\int_0^t\int_\Omega|\boldsymbol{\nabla}\boldsymbol{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t\\ + (\lambda+\mu)\int_0^t\int_\Omega(\mathrm{div}\boldsymbol{u}^\varepsilon)^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leqslant \int_\Omega\frac{1}{2}\rho_0^\varepsilon|\boldsymbol{u}_0^\varepsilon|^2\,\mathrm{d}\boldsymbol{x} + \frac{C_\gamma}{\varepsilon^2}\int_\Omega\Pi_\gamma(\rho_0^\varepsilon)\,\mathrm{d}\boldsymbol{x}.\end{aligned}$$

Zero Mach number limit : the continuous problem

A priori estimates: what arguments have been used ?

(i) Kinetic energy:

$$\boldsymbol{u} \cdot \Big[\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u})\Big] \quad \geqslant \quad \partial_t(\frac{1}{2}\rho|\boldsymbol{u}|^2) + \mathrm{div}(\frac{1}{2}\rho|\boldsymbol{u}|^2\boldsymbol{u}).$$

(ii) Renormalization identity (for a convex function b)

$$b'(\rho) \times \Big[\partial_t\rho + \mathrm{div}(\rho\boldsymbol{u})\Big] \quad \geqslant \quad \partial_t b(\rho) + \mathrm{div}(b(\rho)\boldsymbol{u}) + (\rho b'(\rho) - b(\rho))\,\mathrm{div}\,\boldsymbol{u}.$$

(iii) Gradient-Divergence duality

$$\boldsymbol{\nabla} p \cdot \boldsymbol{u} + p\mathrm{div}\boldsymbol{u} = \mathrm{div}(p\boldsymbol{u}).$$

(iv) Global conservativity:

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \partial\Omega \quad \Longrightarrow \quad \int_\Omega \mathrm{div}(\phi\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = 0.$$

(v) Coercivity of the diffusion term:

$$-\int_\Omega \mathrm{div}(\boldsymbol{\tau}) \cdot \boldsymbol{u}\,\mathrm{d}\boldsymbol{x} \geqslant C \int_\Omega |\boldsymbol{\nabla}\boldsymbol{u}^\varepsilon|^2\,\mathrm{d}\boldsymbol{x}.$$

An implicit scheme

- **Continuity equation**: (Finite Volumes on the primal mesh)

$$\frac{1}{\delta t}\,(\rho_K^{n+1} - \rho_K^{n}) + \mathrm{div}(\rho^{n+1}\boldsymbol{u}^{n+1})_K = 0, \qquad K \in \mathcal{M},$$

with $\quad \mathrm{div}(\rho \boldsymbol{u})_K = \dfrac{1}{|K|} \displaystyle\sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} \quad$ et $\quad F_{K,\sigma} = |\sigma|\, \rho_\sigma^{\mathrm{up}}\, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma},$

$\rho_\sigma^{\mathrm{up}}$: *upwind* approximation of ρ on the faces σ of K.

- **Momentum equation** : (Hybrid VF/ EF method on the dual mesh)

$$\frac{1}{\delta t}\left[\rho_\sigma^{n+1}\,\boldsymbol{u}_\sigma^{n+1} - \rho_\sigma^{n}\boldsymbol{u}_\sigma^{n}\right] + \mathrm{div}(\rho^{n+1}\boldsymbol{u}^{n+1} \otimes \boldsymbol{u}^{n+1})_\sigma + (\mathrm{div}\,\boldsymbol{\tau})_\sigma^{n+1} + (\boldsymbol{\nabla} p)_\sigma^{n+1} = 0, \quad \sigma \in \mathcal{E}_{\mathrm{int}},$$

with $\quad \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u})_\sigma = \dfrac{1}{|D_\sigma|} \displaystyle\sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}\, \boldsymbol{u}_\epsilon,$

$\boldsymbol{u}_\epsilon$ *upwind or centered* approximation of $\boldsymbol{u}$ on the faces ϵ of the dual cell D_σ.

Discrete renormalization identity

$$\frac{|K|}{\delta t}\,(\rho_K - \rho_K^*) + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\,\rho_\sigma^{\mathrm{up}} = 0.$$

Assume that :

- he densities ρ_K et ρ_K^* are positive
- the function b is convex
- $\rho_\sigma^{\mathrm{up}}$ is the upwind approximation of ρ on σ.

Then : multiplying by $b'(\rho_K)$ yields:

$$\frac{|K|}{\delta t}\,(b(\rho_K) - b(\rho_K^*)) + \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma}\,b(\rho_\sigma^{\mathrm{up}}) \\ + (\rho_K b'(\rho_K) + b(\rho_K)) \sum_{\sigma\in\mathcal{E}(K)} |\sigma|\,\boldsymbol{u}_\sigma\cdot\boldsymbol{n}_{K,\sigma} + R_K = 0.$$

with $R_K \geqslant 0$.

Discrete ∇ – div duality

Discrete divergence:

$$(\operatorname{div} u)_K = \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma}.$$

Discete gradient :

$$\text{pour } \sigma = K|L, \qquad (\nabla p)_\sigma = \frac{|\sigma|}{|D_\sigma|} (p_L - p_K) \boldsymbol{n}_{K,\sigma}.$$

We get, for all discrete fields $(p_K)_{K \in \mathcal{M}}$, $(\boldsymbol{u}_\sigma)_{\sigma \in \sigma}$ (with $\boldsymbol{u}_\sigma = 0$ for $\sigma \subset \partial\Omega$):

$$\begin{aligned}
\sum_{K \in \mathcal{M}} |K| \, p_K \, (\operatorname{div} \boldsymbol{u})_K &= \sum_{K \in \mathcal{M}} |K| \, p_K \, \frac{1}{|K|} \sum_{\sigma \in \mathcal{E}(K)} |\sigma| \, \boldsymbol{u}_\sigma \cdot \boldsymbol{n}_{K,\sigma} \\
&= - \sum_{\sigma \in \mathcal{E}} |\sigma| \boldsymbol{u}_\sigma (p_L - p_K) \boldsymbol{n}_{K,\sigma} \\
&= - \sum_{\sigma \in \mathcal{E}} |D_\sigma| \, \boldsymbol{u}_\sigma \cdot (\boldsymbol{\nabla} p)_\sigma.
\end{aligned}$$

Discrete kinetic energy :

$$\partial_t \rho + \mathrm{div}(\rho \boldsymbol{u}) = 0,$$

$$\partial_t(\rho \boldsymbol{u}) + \mathrm{div}(\rho \boldsymbol{u} \otimes \boldsymbol{u}) - \mathrm{div}\boldsymbol{\tau} + \boldsymbol{\nabla} p = 0.$$

$$\Downarrow$$

$$\rho \left(\partial_t \frac{|\boldsymbol{u}|^2}{2} + \boldsymbol{\nabla} \frac{|\boldsymbol{u}|^2}{2} \right) - \mathrm{div}\boldsymbol{\tau} \cdot \boldsymbol{u} + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0.$$

$$\Downarrow$$

$$\partial_t \frac{\rho|\boldsymbol{u}|^2}{2} + \boldsymbol{\nabla} \frac{\rho|\boldsymbol{u}|^2}{2} - \mathrm{div}\boldsymbol{\tau} \cdot \boldsymbol{u} + \boldsymbol{\nabla} p \cdot \boldsymbol{u} = 0.$$

- The continuity equation has been used twice.
- The two equations are not discretized on the same mesh !

$$\frac{|K|}{\delta t}(\rho_K - \rho_K^\star) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}(\rho_\sigma \boldsymbol{u}_\sigma - \rho_\sigma^\star \boldsymbol{u}_\sigma^\star) + \frac{1}{|D_\sigma|} \sum_{\epsilon \in \tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon} \boldsymbol{u}_\epsilon - (\mathrm{div}\boldsymbol{\tau})_\sigma + (\boldsymbol{\nabla} p)_\sigma = 0, \qquad \sigma \in \mathcal{E}_{\mathrm{int}}.$$

Discrete kinetic energy :

Theeorem (Herbin, Kheriji, Latché) :

- Assume that a mass balance equation is satisfied on the dual mesh cells :

$$\frac{|D_\sigma|}{\delta t}(\rho_\sigma - \rho_\sigma^\star) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0, \qquad \forall \sigma \in \mathcal{E}_{\text{int}}.$$

with $\rho_\sigma > 0$ for all $\sigma \in \mathcal{E}_{\text{int}}$.

- Then a local discrete kinetic energy equation is satisfied on this dual mesh :

$$\frac{1}{2\delta t}\left(\rho_\sigma\,|\boldsymbol{u}_\sigma|^2 - \rho_\sigma^\star\,|\boldsymbol{u}_\sigma^\star|^2\right) + \frac{1}{2|D_\sigma|} \sum_{\substack{\epsilon \in \tilde{\mathcal{E}}(D_\sigma) \\ \epsilon = D_\sigma | D_{\sigma'}}} F_{\sigma,\epsilon}\, \boldsymbol{u}_\sigma \cdot \boldsymbol{u}_{\sigma'} - (\text{div}\,\boldsymbol{\tau})_\sigma \cdot \boldsymbol{u}_\sigma + (\boldsymbol{\nabla} p)_\sigma \cdot \boldsymbol{u}_\sigma \leqslant 0.$$

Discrete kinetic energy :

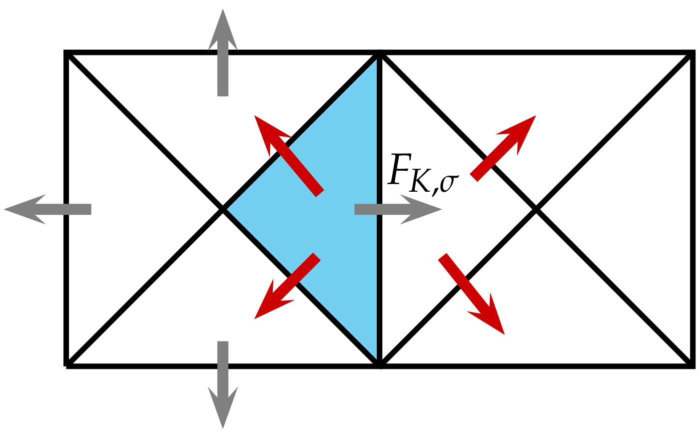
We want :

$$\frac{|D_\sigma|}{\delta t}(\rho_\sigma - \rho_\sigma^\star) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

We have :

$$\frac{|K|}{\delta t}(\rho_K - \rho_K^\star) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0,$$

$$\frac{|L|}{\delta t}(\rho_L - \rho_L^\star) + \sum_{\sigma \in \mathcal{E}(L)} F_{L,\sigma} = 0.$$



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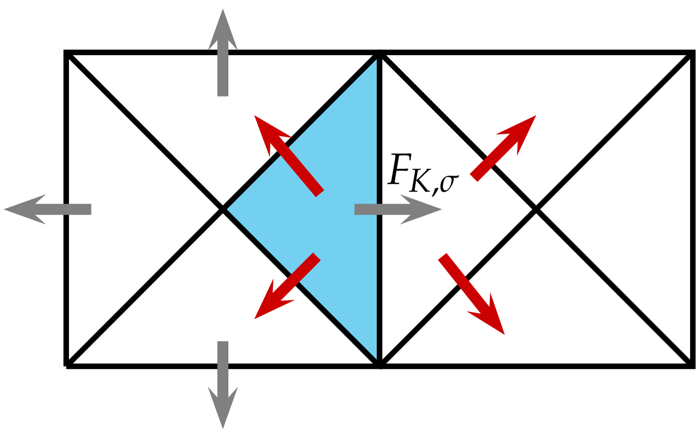

Let $\boldsymbol{w}$ s.t. $\mathrm{div}\boldsymbol{w} = cst$ and for every face σ of K:

$$\int_\sigma \boldsymbol{w} \cdot \boldsymbol{n}_{K,\sigma} = F_{K,\sigma}$$ (lifting of the fluxes).

We then define : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

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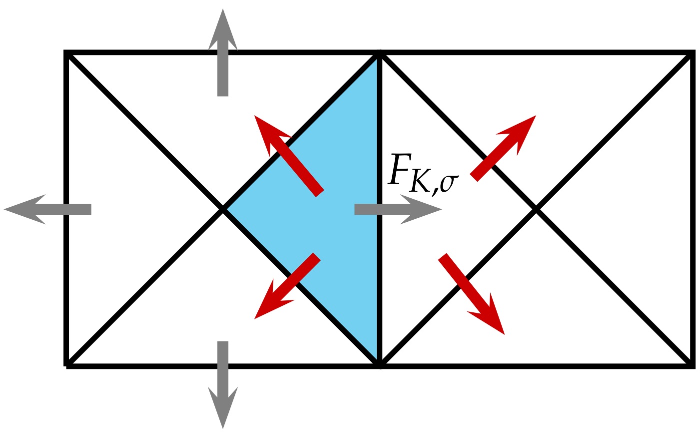

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We then define : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = \int_{\partial D_{K,\sigma}} \boldsymbol{w} \cdot \boldsymbol{n} = \int_{D_{K,\sigma}} \mathrm{div}\boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_K \mathrm{div}\boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|} \int_{\partial K} \boldsymbol{w} \cdot \boldsymbol{n}$$

$$= \frac{|D_{K,\sigma}|}{|K|} \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = -\frac{|D_{K,\sigma}|}{|K|} \frac{|K|}{\delta t}(\rho_K - \rho_K^\star)$$

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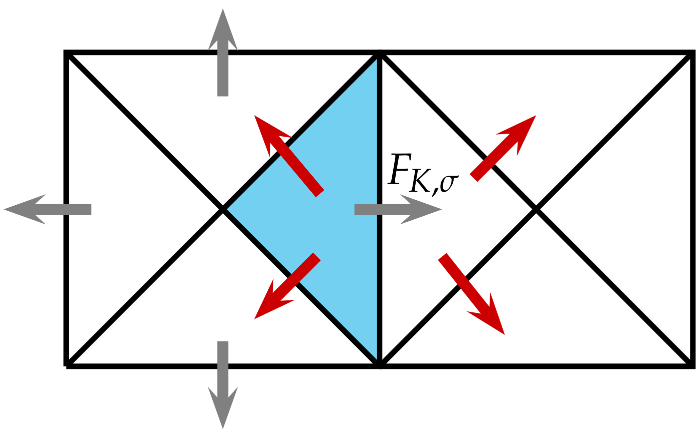

Let $\boldsymbol{w}$ s.t. $\mathrm{div}\boldsymbol{w} = cst$ and for every face σ of K:

$$\int_\sigma \boldsymbol{w} \cdot \boldsymbol{n}_{K,\sigma} = F_{K,\sigma} \qquad \text{(lifting of the fluxes).}$$

We then define : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$\frac{|D_{K,\sigma}|}{\delta t}(\rho_K - \rho_K^\star) + F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = 0,$$

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We want :

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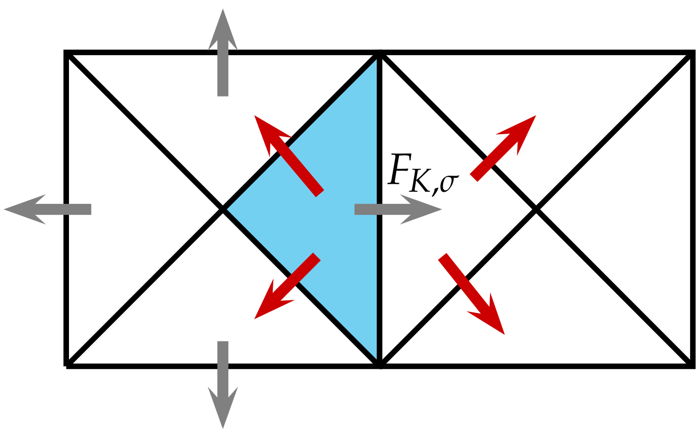

Let $\boldsymbol{w}$ s.t. $\mathrm{div}\boldsymbol{w} = cst$ and for every face σ of K:

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We then define : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

$$\frac{|D_{K,\sigma}|}{\delta t}(\rho_K - \rho_K^\star) + F_{K,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset K}} F_{\sigma,\epsilon} = 0, \quad \text{et} \quad \frac{|D_{L,\sigma}|}{\delta t}(\rho_L - \rho_L^\star) + F_{L,\sigma} + \sum_{\substack{\epsilon \in \mathcal{E}(D_\sigma) \\ \epsilon \subset L}} F_{\sigma,\epsilon} = 0.$$

Discrete kinetic energy :

We want :

$$\frac{|D_\sigma|}{\delta t}(\rho_\sigma - \rho_\sigma^\star) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

We have :

$$\frac{|K|}{\delta t}(\rho_K - \rho_K^\star) + \sum_{\sigma \in \mathcal{E}(K)} F_{K,\sigma} = 0,$$

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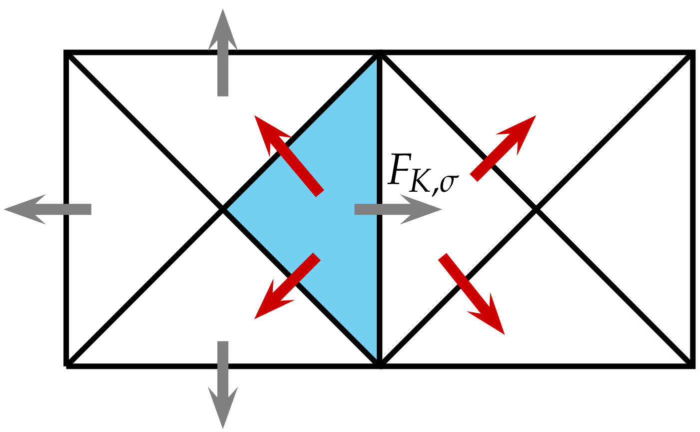

Let $\boldsymbol{w}$ s.t. $\mathrm{div}\boldsymbol{w} = cst$ and for every face σ of K:

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We then define : $F_{\sigma,\epsilon} = \int_\epsilon \boldsymbol{w} \cdot \boldsymbol{n}_{\sigma,\epsilon}$.

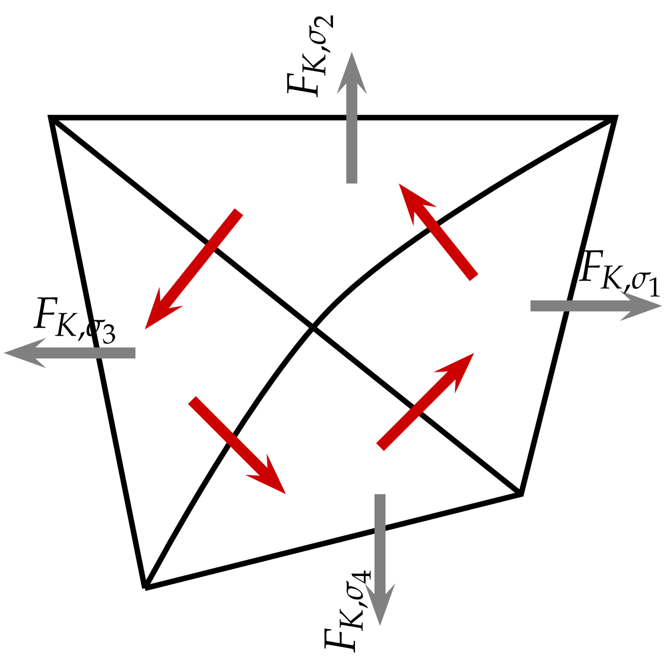
$$\frac{1}{\delta t}\Big(\underbrace{|D_{K,\sigma}|\rho_K + |D_{L,\sigma}|\rho_L}_{=:|D_\sigma|\rho_\sigma} - |D_{K,\sigma}|\rho_K^\star - |D_{L,\sigma}|\rho_L^\star\Big) + \sum_{\epsilon \in \mathcal{E}(D_\sigma)} F_{\sigma,\epsilon} = 0.$$

Discrete kinetic energy :

$$F_{K,\sigma} + \sum_{\substack{\epsilon\in\mathcal{E}(D_\sigma)\\ \epsilon\subset K}} F_{\sigma,\epsilon} = \int_{\partial D_{K,\sigma}} \boldsymbol{w}\cdot\boldsymbol{n} = \int_{D_{K,\sigma}} \mathrm{div}\boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|}\int_K \mathrm{div}\boldsymbol{w} = \frac{|D_{K,\sigma}|}{|K|}\int_{\partial K} \boldsymbol{w}\cdot\boldsymbol{n}$$

$$= \frac{|D_{K,\sigma}|}{|K|}\sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma} = -\frac{|D_{K,\sigma}|}{|K|}\frac{|K|}{\delta t}(\rho_K - \rho_K^\star)$$

with $\xi_K^\sigma = \frac{|D_{K,\sigma}|}{|K|}$ independent of K and σ.



One solves the linear system :

$$F_{K,\sigma} + \sum_{\substack{\epsilon\in\mathcal{E}(D_\sigma)\\ \epsilon\subset K}} F_{\sigma,\epsilon} = \xi_K^\sigma \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}, \qquad \forall\sigma\in\mathcal{E}(K).$$

Other properties of the scheme

- **Coercivity of the diffusion term :**

  Rannacher-Turek Finite Element:

$$-\sum_{\sigma\in\mathcal{E}} |D_\sigma|(\mathrm{div}\boldsymbol{\tau})_\sigma \cdot \boldsymbol{u}_\sigma \geqslant C\|\boldsymbol{u}\|_{1,\mathrm{d}}^2.$$

- **Discrete *inf-sup* condition :**

$$\inf_{p=(p_K)} \sup_{\boldsymbol{v}=(\boldsymbol{u}_\sigma)} \frac{\sum_{K\in\mathcal{M}} |K| p_K \,(\mathrm{div}\boldsymbol{v})_K}{\|p\|_{\mathrm{L}^2(\Omega)} \|\boldsymbol{v}\|_{1,\mathrm{d}}} \geqslant \beta.$$

For all $p = (p_K)$ such that $\sum_{K\in\mathcal{M}} |K| p_K = 0$, there exists $\boldsymbol{v} = (\boldsymbol{v}_\sigma)$ such that $\|\boldsymbol{v}\|_{1,\mathrm{d}} = 1$ and:

$$\sum_{K\in\mathcal{M}} |K| p_K \,(\mathrm{div}\boldsymbol{v})_K \geqslant \beta \|p\|_{\mathrm{L}^2(\Omega)}.$$

Zero Mach number asymptotics

$$\frac{1}{\delta t}\,(\rho_K^{n+1}-\rho_K^n)+|K|^{-1}\sum_{\sigma\in\mathcal{E}(K)}F_{K,\sigma}^{n+1}=0, \qquad K\in\mathcal{M},$$

$$\frac{1}{\delta t}\,\Big[\rho_\sigma^{n+1}\,\boldsymbol{u}_\sigma^{n+1}-\rho_\sigma^n\boldsymbol{u}_\sigma^n\Big]+|D_\sigma|^{-1}\sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)}F_{\sigma,\epsilon}^{n+1}\boldsymbol{u}_\epsilon^{n+1}+(\operatorname{div}\boldsymbol{\tau})_\sigma^{n+1}+\frac{1}{\varepsilon^2}(\boldsymbol{\nabla}p)_\sigma^{n+1}=0, \quad \sigma\in\mathcal{E}_{\rm int}.$$

Theorem : *Assume ill prepared initial conditions. Then, for a fixed mesh, with* $\varepsilon\to 0$*:*

- $\rho_\varepsilon\to 1$ *with an* ε *rate in* $\mathrm{L}^\infty((0,T);\mathrm{L}^q(\Omega))$ *norm for all* $q\in[1,\min(2,\gamma)]$.
- $\sqrt{\rho^\varepsilon}\boldsymbol{u}^\varepsilon$ *is bounded in* $\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d)$ *and* $\boldsymbol{u}^\varepsilon$ *is bounded in* $\mathrm{L}^2((0,T);\mathrm{H}^1_d(\Omega)^d)$.
- $\|\delta p^\varepsilon\|_{\mathrm{L}^2}:=\|\frac{1}{\varepsilon^2}\left(p^\varepsilon-|\Omega|^{-1}\int_\Omega p^\varepsilon\,\mathrm{d}\boldsymbol{x}\right)\|_{\mathrm{L}^2}\leqslant C/\delta t$.

Zero Mach number asymptotics

$$\frac{1}{\delta t}\,(\rho_K^{n+1}-\rho_K^{n}) + |K|^{-1} \sum_{\sigma\in\mathcal{E}(K)} F_{K,\sigma}^{n+1} = 0, \qquad K\in\mathcal{M},$$

$$\frac{1}{\delta t}\,\Big[\rho_\sigma^{n+1}\,\boldsymbol{u}_\sigma^{n+1} - \rho_\sigma^{n}\boldsymbol{u}_\sigma^{n}\Big] + |D_\sigma|^{-1} \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{n+1}\boldsymbol{u}_\epsilon^{n+1} + (\operatorname{div}\boldsymbol{\tau})_\sigma^{n+1} + \frac{1}{\varepsilon^2}(\boldsymbol{\nabla} p)_\sigma^{n+1} = 0, \quad \sigma\in\mathcal{E}_{\rm int}.$$

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- $\|\delta p^\varepsilon\|_{\mathrm{L}^2} := \|\frac{1}{\varepsilon^2}\big(p^\varepsilon - |\Omega|^{-1}\int_\Omega p^\varepsilon\,\mathrm{d}\boldsymbol{x}\big)\|_{\mathrm{L}^2} \leqslant C/\delta t$.

For every converging subsequence of $(\boldsymbol{u}^\varepsilon,\delta p^\varepsilon)$ *when* $\varepsilon\to 0$*, the limit* $(\boldsymbol{u},\pi)$ *is a solution to a stable implicit staggered scheme for the incompressible Navier-Stokes equations:*

$$|K|^{-1} \sum_{\sigma\in\mathcal{E}(K)} \boldsymbol{u}_\sigma^{n+1}\cdot\boldsymbol{n}_{K,\sigma} = 0, \qquad K\in\mathcal{M},$$

$$\frac{1}{\delta t}\,(\boldsymbol{u}_\sigma^{n+1}-\boldsymbol{u}_\sigma^{n}) + |D_\sigma|^{-1} \sum_{\epsilon\in\tilde{\mathcal{E}}(D_\sigma)} F_{\sigma,\epsilon}^{n+1}\boldsymbol{u}_\epsilon^{n+1} + (\operatorname{div}\boldsymbol{\tau})_\sigma^{n+1} + (\boldsymbol{\nabla}\pi)_\sigma^{n+1} = 0, \quad \sigma\in\mathcal{E}_{\rm int}.$$

Zero Mach number asymptotics

Discrete a priori estimates :

$$\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\,\rho_\sigma^n|\boldsymbol{u}_\sigma^n|^2+\mu\sum_{k=1}^{n}\delta t\|\boldsymbol{u}^k\|_{1,\rm d}^2+\frac{C_\gamma}{\varepsilon^2}\sum_{K\in\mathcal{M}}|K|\,\Pi_\gamma(\rho_K^n)$$
$$\leqslant\frac{1}{2}\sum_{\sigma\in\mathcal{E}_{\rm int}} |D_\sigma|\,\rho_\sigma^0|\boldsymbol{u}_\sigma^0|^2+\frac{C_\gamma}{\varepsilon^2}\sum_{K\in\mathcal{M}}|K|\,\Pi_\gamma(\rho_K^0).$$

Ill prepared initial conditions $\Longrightarrow \quad \dfrac{C_\gamma}{\varepsilon^2}\displaystyle\sum_{K\in\mathcal{M}}|K|\,\Pi_\gamma(\rho_K^0)\leqslant C.$

We deduce that:

- $\rho_\varepsilon\to 1$ with an ε rate in $\mathrm{L}^\infty((0,T);\mathrm{L}^q(\Omega))$ norm for all $q\in[1,\min(2,\gamma)]$.
- $\sqrt{\rho^\varepsilon}\boldsymbol{u}^\varepsilon$ is bounded in $\mathrm{L}^\infty((0,T);\mathrm{L}^2(\Omega)^d)$ and $\boldsymbol{u}^\varepsilon$ is bounded in $\mathrm{L}^2((0,T);\mathrm{H}^1_d(\Omega)^d)$.
- From the momentum equation, we obtain :

$$\|\boldsymbol{\nabla}(\delta p^\varepsilon)\|_{\mathrm{H}^{-1}}\leqslant C/\delta t \quad \text{with} \quad \delta p^\varepsilon:=\frac{1}{\varepsilon^2}\left(p^\varepsilon-|\Omega|^{-1}\int_\Omega p^\varepsilon\,\mathrm{d}\boldsymbol{x}\right).$$

- By the discrete *inf-sup* condition, we get $\|\delta p^\varepsilon\|_{\mathrm{L}^2}\leqslant C/\delta t$.

Zero Mach number asymptotics

- By the Bolzano-Weierstrass theorem (fixed mesh $\Rightarrow$ finite dimension): there exists a subsequence of $(\boldsymbol{u}^\varepsilon, \delta p^\varepsilon)$ converging towards $(\boldsymbol{u}, \pi)$. Passing to the limit in the scheme yields:

$$\mathrm{div}\,(u^{n+1})_K = 0, \qquad K \in \mathcal{M},$$

$$\frac{1}{\delta t}\,(\boldsymbol{u}_\sigma^{n+1} - \boldsymbol{u}_\sigma^{n}) + \mathrm{div}\,(\boldsymbol{u}^{n+1} \otimes \boldsymbol{u}^{n+1})_\sigma + (\mathrm{div}\,\boldsymbol{\tau})_\sigma^{n+1} + (\boldsymbol{\nabla}\pi)_\sigma^{n+1} = 0, \quad \sigma \in \mathcal{E}_{\mathrm{int}}.$$

- If the solution of this system is unique, the whole sequence converges.

A pressure correction scheme

- **Prediction step :** Solve for $\tilde{\boldsymbol{u}}^{n+1}$:

$$\frac{1}{\delta t}\left[\rho_\sigma^n\,\tilde{\boldsymbol{u}}_\sigma^{n+1} - \rho_\sigma^{n-1}\boldsymbol{u}_\sigma^n\right] + \mathrm{div}(\rho^n\boldsymbol{u}^n\otimes\tilde{\boldsymbol{u}}^{n+1})_\sigma + (\mathrm{div}\,\boldsymbol{\tau}(\tilde{\boldsymbol{u}}^{n+1}))_\sigma + (\boldsymbol{\nabla}p)_\sigma^n = 0, \quad \sigma\in\mathcal{E}_{\mathrm{int}},$$

- **Correction step**: Solve for p^{n+1}, ρ^{n+1} and $\boldsymbol{u}^{n+1}$:

$$\frac{1}{\delta t}\,\rho_\sigma^n\,(\boldsymbol{u}_\sigma^{n+1} - \tilde{\boldsymbol{u}}_\sigma^{n+1}) + (\boldsymbol{\nabla}p)_\sigma^{n+1} - (\boldsymbol{\nabla}p)_\sigma^n = 0, \quad \sigma\in\mathcal{E}_{\mathrm{int}},$$

$$\frac{1}{\delta t}\,(\rho_K^{n+1} - \rho_K^n) + \mathrm{div}(\rho^{n+1}\boldsymbol{u}^{n+1})_K = 0, \quad K\in\mathcal{M},$$

$$p_K^{n+1} = (\rho_K^{n+1})^\gamma, \quad K\in\mathcal{M}.$$

- Unconditionally stable w.t. δt and low Mach asymptotics analysis similar to the implicit scheme.
- Difficulty: proving the discrete kinetic energy equation.

Conclusion and perspectives

Conclusion : (comparison with co-located methods)

- The proof is still valid if $\mu = 0 \quad \Longrightarrow \quad$ AP scheme for Euler equations.

$$\frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_\sigma| \, \rho_\sigma^n |\boldsymbol{u}_\sigma^n|^2 + \mu \sum_{k=1}^{n} \delta t \|\boldsymbol{u}^k\|_{1,\text{d}}^2 + \frac{C_\gamma}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \, \Pi_\gamma(\rho_K^n) \leqslant \frac{1}{2} \sum_{\sigma \in \mathcal{E}_{\text{int}}} |D_\sigma| \, \rho_\sigma^0 |\boldsymbol{u}_\sigma^0|^2 + \frac{C_\gamma}{\varepsilon^2} \sum_{K \in \mathcal{M}} |K| \, \Pi_\gamma(\rho_K^0).$$

- Time stabilization when $\varepsilon \to 0$:
  - Implicitation of the acoustic terms : Liou '98 (AUSM), Degond '11, Chalons '14 (Splitting), Zakerzadeh '15, Dimarco...
  - Pressure correction method: staggered schemes *but also* co-located schemes (Chady Zaza's thesis).

Perspectives :

- Other boundary conditions ?
- Numerical scheme which preserves the diffusive limit for a radiative transfer model (with M. Ghattassi and N. Masmoudi).

Thank you !