The nonlinear modulational instability of the Stokes waves in 2d water waves

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Outline

- Introduction
- The NLS approximation
- Example 1: the rigorous justification of the Peregrine soliton
- Example 2: the nonlinear modulational instability of the Stokes waves

The water wave equations

- Main assumptions: Inviscid, incompressible, infinite depth, no surface tension
- Water wave equations:

$$\begin{cases} \left.\begin{array}{l} v_t + v \cdot \nabla v = -\nabla P - (0,1) \\ div\ v = 0, \end{array}\right\} & \Omega(t) \\ P \equiv 0 & \Sigma(t) \\ (1, v) \text{ is tangent to the free surface } (t, \Sigma(t)). \end{cases} \tag{1}$$

- $v = (v_1, v_2)$: velocity; P: pressure; $\Omega(t)$: fluid region; $\Sigma(t)$: free interface
- $curl\ v = \partial_y v_2 - \partial_x v_1$: the vorticity. If $curl\ v = 0$: irrotational
- Highly nonlinear, nonlocal, and moving boundary
- In this talk, we focus on irrotational case only

The fundamental questions

- Local wellposednes
- Long time behavior: global wellposedness or finite time singularities? Scattering? ...
- How can the various approximate models be justified?
- Stability of special solutions (solitary waves, travelling waves...)
- ...

The framework

- Lagrangian coordinates: $z_t(\alpha, t) = v(z(\alpha, t), t)$
- The momentum equation on the interface: $z_{tt} - iaz_\alpha = -i$
  a: a real-valued function.
- v incompressible ($div\ v = 0$) and irrotational $\implies$ $\bar{z}_t$ holomorphic
- Equivalent system

$$\begin{cases} z_{tt} - iaz_\alpha = -i, \\ \bar{z}_t \text{ is holomorphic.} \end{cases} \tag{2}$$

- The pressure is recovered by solving an elliptic equation.

Rayleigh-Taylor instability

- If $-\frac{\partial P}{\partial n} < 0$, then the linearized problem admits solutions with exponential growth in time. This is the so-called Rayleigh-Taylor instability: an instability of an interface between two fluids of different densities which occurs when the lighter fluid is pushing the heavier fluid
- The water wave equation is illposed in Sobolev spaces if the Taylor sign fails.

Local wellposedness

Assume irrotational.

- Small data:
  - Nalimov (*Dynamika Splosh. Sredy*, 1974), infinite depth
  - Yoshihara (*RIMS Kyoto*, 1982), with a finite flat bottom
  - Craig (*CPDE*, 1985), infinite depth

  proved LWP in Sobolev spaces. For small data, the Taylor sign condition ($-\frac{\partial P}{\partial n} \geq \delta > 0$) holds automatically.

- Beale-Hou-Lowengrub (*CPAM*, 1993): Assuming the strong Taylor sign condition, they showed that the water waves is stable near arbitrary large solution.

- Wu(*Invent.*, 1997): infinite depth, irrotatoinal, ($C^{1,\alpha}$) and non self-intersecting free surface $\Longrightarrow$ strong Taylor sign condition $(-\frac{\partial P}{\partial n} \geq \delta > 0)$ holds automatically.

- Wu (*JAMS*, 1999): 3d.

- Many results after Wu's works:
  - (irrotational) Alazard, Burq, Zuily, Ambrose, Masmoudi, Coutland, Shkoller, Lannes, Shatah, Zeng, Ai....;
  - (rotational): Iguchi,Tanaka, Tani, Ogawa, Chritodoulou, Lindblad ,Zhang- Zhang...
  - (non-smooth interface) Wu (Invent 2019); Ming-Wang; Cordoba etc...

Long time wellposedness

For how long does the solution exist?

- LWP: at least $O(\epsilon^{-1})$, if initial data size $O(\epsilon)$. A lifespan of order bigger than ϵ^{-1} is called long time existence.
- **Long time: Irrotational, small, smooth, localized**
  - Wu (*Invent.*, 2009), 2d almost global
  - Wu (*JAMS*, 2011) , 3d global
  - Germain-Masmoudi-Shatah (*Ann. Math.*, 2012), 3d global
  - Ionescu-Pusateri (*Invent.*, 2012) ;
    Alazard-Delort (*Ann. Sci. Éc. Norm. Supér*, 2015) , 2d global
- **(Other results)** Ifrim-Hunter-Tataru; Ai-Ifrim-Tataru; Deng-Ionescu-Pausader-Pusateri; Xuecheng Wang; Fan Zheng...

The long time behavior?

- The aforementioned results do not say much about its precise long time behavior.
- In particular, the instability/stability of special solution is not discussed.
- Two examples: the Peregrine soliton and the Stokes waves. Both can be studied by the rigorous justification of the NLS in the full water waves, and both are related to the modulational instability

The NLS approximation

- Ansatz: $\zeta(\alpha, t) = \alpha + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + ...$, with

$$\zeta^{(1)} = B(\epsilon(\alpha - \lambda t), \epsilon^2 t)e^{i(k\alpha+\gamma t)}.$$

- One can choose B such that B solves the 1d focusing cubic NLS

$$iu_t + \frac{1}{2}u_{xx} = -|u|^2 u.$$

- Zakharov (*J. Appl. Mech. Tech. Phys.*, 1968) infinite depth case
  Hasimoto and Ono (*J. Phys. Soc. Jpn.*, 1972) finite depth case.

- The 1d cubic NLS is integrable, there are a lot of interesting solitons! Formally, if we approximate $\zeta(\alpha, t)$ by

$$\zeta_{approx} := \alpha + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)}(+\epsilon^3\zeta^{(3)} + ...),$$

  then we obtain an approximation described by the well-understood 1d cubic NLS.

The rigorous justification

- The above observations are formal. To rigorously approximate the water waves by the NLS, we need to show that the error term $r := \zeta - \zeta_{approx}$ remains small (as compared to ζ_{approx}) for sufficiently long times.

- **To observe such NLS like water waves:** The water waves need to exist on time scale $O(\epsilon^{-2})$ (long time existence).

- Infinite depth: Totz and Wu (*CMP*, '12)

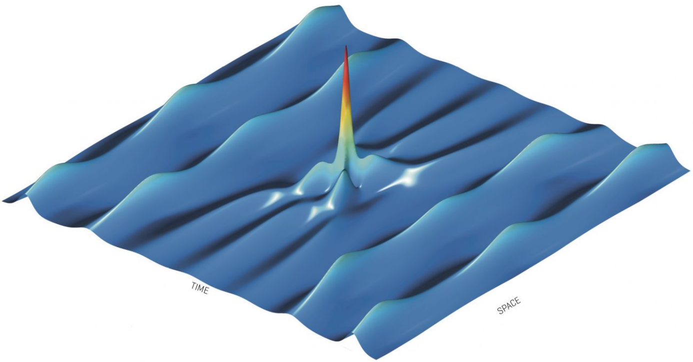
- A canal of finite depth: Düll, Schneider and Wayne (*ARMA*, '16)

- All require $B \in H^s(\mathbb{R})$, cannot cover many important physical phenomena. Example: rogue wave.

Rogue wave

Rogue wave (Crazy dog wave): One type of dangerous and mysterious ocean waves. Mechanism of formation and evolution is unknown.

Figure: Rogue waves. (Source: New Zealand Geographic)



Peregrine soliton

The Peregrine soliton $Q(\xi, \tau) = e^{i\tau}(1 - \frac{4(1+2i\tau)}{1+4\xi^2+4\tau^2})$ solves NLS $iu_\tau + \frac{1}{2}u_{\xi\xi} = -|u|^2 u$.

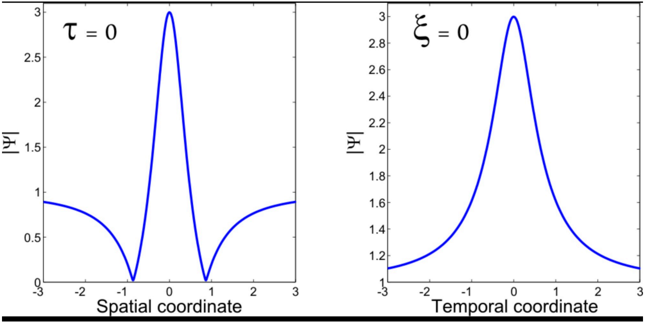


Figure: The modulus of the Peregrine soliton. (Source: wiki)

Rigorous justification of the Peregrine soliton

- The Peregrine soliton is conjectured to be one of the mechanisms for the formation of rogue waves by the ocean waves community.
- Eventually observed in nonlinear optics. Experiments also showed the existence of water waves which look like the Peregrine soliton.

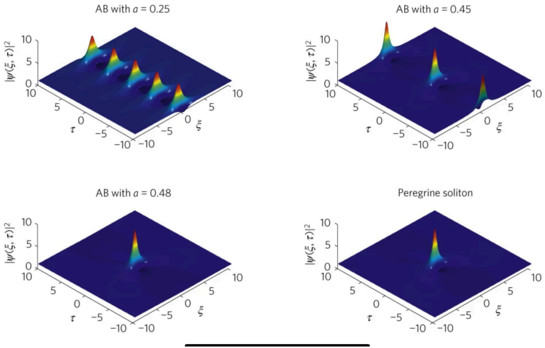


Figure: Peregrine soliton in nonlinear optics (source: nature physics 2010)

- The framework of Totz and Wu does not apply to the Peregrine soliton: $Q \notin H^s(\mathbb{R})$ for any s.

- Working in spaces of the form $H^s(\mathbb{R}) + H^{s'}(\mathbb{T})$, we rigorously justified the Peregrine soliton from the 2d water waves. The key is to rewrite the water wave equation for the error term r as

$$(D_t^2 - iA\partial_\alpha)r = cubic.$$

  Then use the energy method to show that $r(\cdot, t)$ remains the same order as $r(\cdot, 0)$ for $0 \leq t \lesssim \epsilon^{-2}$.

Theorem (Su, 2020, ARMA)

There exists a class of water waves which looks like the Peregrine soliton for sufficiently long times.

Further remarks

- The Peregrine soliton justified above indeed suggest the modulational instability. Recall that $Q(\xi, \tau) = e^{i\tau}(1 - \frac{4(1+2i\tau)}{1+4\xi^2+4\tau^2})$. Suppose initially there is a plane wave and perturb it by a localized perturbation. Then after a long time one can see the breakup of the waveform of the original water waves.

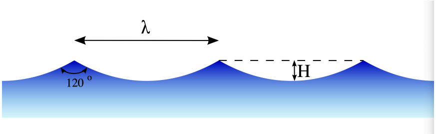
- However, in order that we observe the actually breakup of the waveform, we need to solve the water waves at a timescale of order at least $O(\epsilon^{-2}\delta^{-1})$, where δ is the size of the perturbation of the Peregrine soliton.

- Such a lifespan is too long for us to justify

- If there is some water waves whose instability occurs at the timescale $O(\epsilon^{-2}\log\delta^{-1})$, then we should be able to rigorously justify it. Such an idea was implemented in the proof of nonlinear instability of the Stokes waves.

The Stokes wave

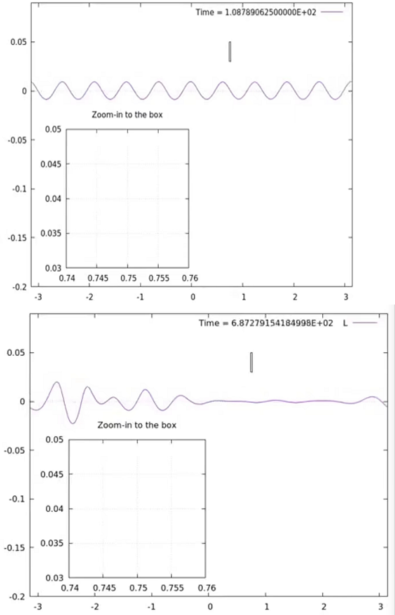
- Stokes wave: periodic symmetric steady water waves travelling at a constant speed.
- A fundamental object of nonlinear waves. Google scholar: Stokes waves, more than 1000,0000 results
- Stokes (1847): constructed formally; (1880) predicted the Stokes wave of greatest height, with its well-known contained angle $2\pi/3$



- Existence (rigorous mathematical proof)
  - Small amplitude: 1920s Levi-Civita, Nekrasov, Struik
  - Large amplitude: 1960s Krasovskii
- The rigorous mathematical investigation of the extreme Stokes: Toland(78), Amick-Fraenkel-Toland (82), Plotnikov (82), Plotnikov-Toland (04), Varvaruca-Weiss (11), Constantin (12), etc.

Modulational instability

- For a long time, people believed that the Stokes are stable
- It was surprised in the 1960s Benjamin and Feir found that the Stokes wave is subject to the Modulational instability, provided that $kh > 1.36...$. k: wave number; h: depth of the unperturbed water wave.
- Modulational instability: A phenonmenon whereby deviations from a periodic waveform are reinforced by the nonlinearity, leading to the generation of specteral-sidebands and the eventual breakup of the waveform into a train of pulses.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Modulational instability of the Stokes waves

- Predicted by Benjamin-Feir (1967) and Whitham (1967). Also discovered in other dispersive PDEs almost simultaneously: Benny and Newell, Ostrovskii, Zakharov, Lightwill...

- Modulational instability (MI) is one of the most commonly seen instability in nature. Google scholar gives almost 4,0000 results on it.

- The fundamental role of Stokes waves + the importance of MI: rigorous mathematical proof of the MI of Stokes waves is desirable

- The problem is difficult: the water wave system is highly nonlinear, nonlocal, and with free boundaries.

Linear MI

- Linear MI of the Stokes waves: If the linearized water wave system around a given Stokes wave has unstable eigenvalues, then it is said to be linear modulational unstable.
- (ARMA '95) Bridges and Mielke: Finite depth, linear modulational instability
- (CPAM '22) H. Nguyen and W. Strauss: arbitrary depth, linear modulational instability
- (Preprint '20) Hur and Yang: finite depth, linear modulational instability
- (Invent. '22) Berti, Maspero, and Ventura, linear modulational instability, full description of all unstable modes

Nonlinear MI

- The proof of the nonlinear modulational instability was missing for a long time (1967-2020)
- The linear MI does not immediately imply the nonlinear MI.
- J.T. Stuart: Nonlinear instability gives us some understanding of nonlinear processes in fluid mechanics, perhaps with reference to the early, relatively simple states of the evolution of laminar flow to turbulence. Even then, the mathematical problems posed are challenging enough
- There are some general frameworks in fluids that linear instability implies nonlinear instability, for example Yan Guo etc; Susan Friedlander etc. All these abstract framework requires the a priori control of higher energy norm, which is not given for free in the water wave system.

Main result

Joint with Gong Chen, using the NLS approximation, we prove the nonlinear modulational instability of small amplitude Stokes waves.

Theorem (Chen, Su, submitted)

A Stokes wave of sufficiently small amplitude is nonlinear modulational unstable under long wave perturbations.

- Using the NLS approximation, we are able to rigorously describe the precise nonlinear process of the MI.
- We don't need to use the linear modulational instability of the Stokes waves proved by others. The instability mechanism is the nonlinear modulational instability of the constant solution e^{it} of the NLS
- This is the first rigorously mathematical proof for the nonlinear MI of the Stokes waves in water waves

Essential ingredients of the proof

1. Nonlinear modulational instability of the 1d focusing cubic NLS. One can directly prove the nonlinear modulational instability of the long wave perturbation of the constant solution e^{it} of $iu_t + u_{xx} = -|u|^2 u$.

2. Formally derive the NLS from the full water waves and construct an approximate solution to the water waves whose modulus of the leading order is the NLS.

3. Rigorous justify the NLS approximation which allows the Stokes wave for sufficiently long times.

Wu's modified Lagrangian formulation

Wu ('09) considered a new labelling $\zeta(\alpha, t)$ of the interface such that

$$\begin{cases} (D_t^2\zeta - iA\partial_\alpha)\zeta = -i \\ D_t\bar{\zeta}, \quad \bar{\zeta} - \alpha \quad \textit{holomorphic}. \end{cases} \tag{3}$$

Here, $D_t = \partial_t + b\partial_\alpha$, with b and A real-valued and given by

$$(I - \mathcal{H}_\zeta)b = -[D_t\zeta, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}, \tag{4}$$

$$(I - \mathcal{H}_\zeta)(A - 1) = i[D_t\zeta, \mathcal{H}_\zeta]\frac{\partial_\alpha D_t\bar{\zeta}}{\zeta_\alpha} + i[D_t^2\zeta, \mathcal{H}_\zeta]\frac{\bar{\zeta}_\alpha - 1}{\zeta_\alpha}. \tag{5}$$

Here, $\mathcal{H}_\zeta$ is the (periodic) Hilbert transform associated with ζ.

$$\mathcal{H}_\zeta f(\alpha) := \frac{1}{2q\pi i}\text{p.v.}\int_{-q\pi}^{q\pi} \zeta_\beta(\beta)\cot\left(\frac{\zeta(\alpha) - \zeta(\beta)}{2q}\right)f(\beta)\, d\beta. \tag{6}$$

$\zeta(\alpha) - \alpha$ and f are $2q\pi$ periodic.

Description of the Stokes wave in Wu's coordinates

- Fix the period of the Stokes waves to be 2π.
- For each ϵ with $0 < |\epsilon| \ll 1$, there is a Stokes wave $(\omega, Z_{ST}, D_t^{ST} Z_{ST})$.
- It has the following asymptotic expansions

$$Z_{ST}(\epsilon; \alpha, t) = \alpha + i\epsilon e^{i\alpha + i\omega t} + i\epsilon^2 + \frac{i}{2}\epsilon^3 e^{-i\alpha - i\omega t} + O(\epsilon^4). \tag{7}$$

and

$$\omega(\epsilon) = 1 + \epsilon^2/2 + \mathcal{O}(\epsilon^3). \tag{8}$$

- In Zakharov-Craig-Sulum formulation, the surface elevation of a Stokes wave of amplitude ϵ, has asymptotic expansion

$$\begin{aligned}\eta(x,t) =& \epsilon\cos(x+\omega t) + \frac{1}{2}\epsilon^2\cos(2(x+\omega t)) + \epsilon^3\Big\{\frac{1}{8}\cos(x+\omega t) \\ &+ \frac{3}{8}\cos(3(x+\omega t))\Big\} + O(\epsilon^4)\end{aligned} \quad (9)$$

- We should immediately notice that in this formulation, up to order $\mathcal{O}(\epsilon^4)$, there are three nontrivial frequencies. But, in Wu's coordinates, up to an error of $O(\epsilon^4)$, (7) has only one nonzero fundamental frequency.

The derivation of the NLS

- Assume the perturbed interface can be written as

$$\zeta(\alpha, t) = \alpha + \epsilon\zeta^{(1)} + \epsilon^2\zeta^{(2)} + \epsilon^3\zeta^{(3)} + \dots \tag{10}$$

- Consider wave packet like water waves: $\zeta^{(1)} = B(X, T)e^{i\alpha + i\omega t}$, with

$$X = \epsilon(\alpha - \frac{1}{2\omega}t), \quad T = \epsilon^2 t.$$

  We seek for B of the form $B = e^{it}(i + perturbation)$ because the Stokes wave admits the expansion

$$\zeta(\alpha, t) = \alpha + i\epsilon e^{i\alpha + i\omega t} + O(\epsilon^2). \tag{11}$$

- We use Multi-scale analysis to find B such that B solves the standard cubic NLS $iB_T + B_{XX} = -|B|^2 B$. Moreover, we show that B is nonlinear modulational unstable near the trivial sotlution e^{it}.

The approximation and the error

- We denote

$$\zeta_{approx} := \zeta_{ST} + (\tilde{\zeta} - \tilde{\zeta}_{ST}), \qquad r := \zeta - \zeta_{approx}. \tag{12}$$

Here,

$$\tilde{\zeta} = \alpha + \epsilon B(X,T) e^{i\alpha + i\omega t} + \epsilon^2 \zeta^{(2)} + \epsilon^3 \zeta^{(3)}, \quad \tilde{\zeta}_{ST} = \alpha + i\epsilon e^{i\alpha + i\omega t} + \epsilon^2 i + \frac{i\epsilon^3}{2} e^{-i\alpha - i\omega t}. \tag{13}$$

This approximation is carefully constructed. Such a choice is crucial for us to prove a lifespan of order $O(\epsilon^{-2}) \log \delta^{-1}$.

- To see the instability, since $B(\alpha, t)$ develops into instability at the time $O(\log \delta^{-1})$, we need to justify the approximation remains valid on times $[0, O(\epsilon^{-2}) \log \delta^{-1}]$.
- So the proof of nonlinear modulational instability is reduced to showing that for $t \leq O(\epsilon^{-2}) \log \delta^{-1}$,

$$\|D_t r\|_{H^{s+1/2}(q\mathbb{T})} + \|\partial_\alpha r\|_{H^s(q\mathbb{T})} \leq C\epsilon^{3/2} \delta e^{\epsilon^2 t}. \tag{14}$$

A model

- Consider a model equation

$$\begin{cases} u_{tt} + |\partial_\alpha| u = u_t^3 \\ u_t(0) = u_0, \quad |\partial_\alpha|^{1/2} u(0) = u_1. \end{cases} \tag{15}$$

Assume that (15) admits a global solution U with

$$\sup_{t\in\mathbb{R}} (\|U_t\|_{H^s(\mathbb{T})} + \left\| |\partial_\alpha|^{1/2} U \right\|_{H^s(\mathbb{T})}) \leq \epsilon \ll 1. \tag{16}$$

- Perturb U by long wave perturbations: $u = U + w$. So w solves

$$\begin{cases} w_{tt} + |\partial_\alpha| w = w_t^3 + 3w_t^2 U_t + 3w_t U_t^2 \\ w_t(0) = w_0, \quad |\partial_\alpha|^{1/2} w(0) = w_1. \end{cases} \tag{17}$$

Long time existence of the toy model

- Let $q \gtrsim 1$ For arbitrary given $0 < \delta \ll 1$, assume that

$$\|w_0\|_{H^s(q\mathbb{T})} + \|w_1\|_{H^s(q\mathbb{T})} \leq \epsilon^{3/2}\delta, \tag{18}$$

  then (17) admits a solution $(w_t, |\partial_\alpha|^{1/2}w) \in C([0, \epsilon^{-2}\log\delta^{-1}]; H^s(q\mathbb{T}) \times H^s(q\mathbb{T}))$, and

$$\sup_{t\in[0,\epsilon^{-2}\log\delta^{-1}]} (\|w_t\|_{H^s(q\mathbb{T})} + \left\| |\partial_\alpha|^{1/2} w \right\|_{H^s}) \lesssim \epsilon^{3/2}. \tag{19}$$

Water waves: governing equation for the error term

- To justify the error bound (14), we need to find an governing equation for r for which we are able to solve r for sufficiently long times.
- Instead of solving r directly, we consider a quantity

$$\rho := (I - \mathcal{H}_\zeta)\Big[\theta - \theta_{ST} - (\tilde{\theta} - \tilde{\theta}_{ST})\Big]$$

which is equivalent to r. Here,

$$\theta := (I - \mathcal{H}_\zeta)(\zeta - \bar{\zeta}),\ \theta_{ST} = (I - \mathcal{H}_{\zeta_{ST}})(\zeta_{ST} - \bar{\zeta}_{ST})$$

and

$$\tilde{\theta} := (I - \mathcal{H}_{\tilde{\zeta}})(\tilde{\zeta} - \bar{\tilde{\zeta}}),\ \tilde{\theta}_{ST} := (I - \mathcal{H}_{\tilde{\zeta}_{ST}})(\tilde{\zeta}_{ST} - \bar{\tilde{\zeta}}_{ST}).$$

- Moreover,

$$(D_t^2 - iA\partial_\alpha)\theta = G_1 + G_2, \qquad ((D_t^{ST}) - iA_{ST}\partial_\alpha)\theta_{ST} = G_1^{ST} + G_2^{ST},$$
$$(\tilde{D}_t^2 - i\tilde{A}\partial_\alpha)\tilde{\theta} = \tilde{G}_1 + \tilde{G}_2, \qquad ((\tilde{D}_t^{ST}) - i\tilde{A}_{ST}\partial_\alpha)\tilde{\theta}_{ST} = \tilde{G}_1^{ST} + \tilde{G}_2^{ST},$$

with $G_j, G_j^{ST}, \tilde{G}_j, \tilde{G}_j^{ST}$ cubic and are of the same structure.

The structure of $(D_t^2 - iA\partial_\alpha)\rho \quad \approx (\partial_t^2 + |\partial_\alpha|)\rho$

- We need to show that the nonlinearities of $(D_t^2 - iA\partial_\alpha)\rho$ can be written as

$$O(\epsilon^2)O(r) + \epsilon^{7/2} O(\|B(\alpha,t) - i\|_{H^{s'}(q_1\mathbb{T})}), \tag{20}$$

- To illustrate this, consider a trilinear map $S(f,g,h)$. Consider for example the quantity $\sum_{j=1}^4 (-1)^{j-1} S(f_j, g_j, h_j)$, consists of terms of the form

$$S(\sum_{j=1}^4 (-1)^{j-1} f_j, g_1, h_1) + S(f_1 - f_3, g_2 - g_4, h_1) + \text{similar terms}$$

In our applications, $(-1)^{j-1} f_j \approx r +$ error terms,
$f_1 - f_3 \approx \epsilon^{1/2} \|B - i\|_{H^{s'}(q_1\mathbb{T})}$, $\|g_2 - g_4\|_{W^{s',\infty}(q\mathbb{T})} \approx \epsilon^2$,
$\|h_1\|_{W^{s-1,\infty}} = O(\epsilon)$. So we obtain
$\left\| S(\sum_{j=1}^4 (-1)^{j-1} f_j, g_1, h_1) \right\|_{H^s(q\mathbb{T})} \approx \epsilon^2 \|r\|_{H^s(q\mathbb{T})}$, and

$$\|S(f_1 - f_3, g_2 - g_4, h_1)\| \approx \epsilon^{7/2} \|B - i\|_{H^{s'}(q_1\mathbb{T})}. \tag{21}$$

Approximate the water waves near the Stokes waves

We obtain

Theorem (Chen, Su, preprint)

Let $s \geq 4$ be given and $s' = s + 7$. Let ζ_{ST} be a Stokes wave of period 2π and amplitude ϵ. Let $0 < \delta \ll 1$ be an arbitrarily small but fixed number. For any given $q \in \mathbb{Q}_+$ with $q \geq \frac{1}{\epsilon}$, and any solution B to the NLS satisfying

$$\|B(\alpha, 0) - i\|_{H^{s'}(q_1\mathbb{T})} \leq \delta, \qquad q_1 = \epsilon q \tag{22}$$

there exist ζ_0 and v_0 such that $(\zeta_0 - \alpha, v_0) \in H^{s+1}(q\mathbb{T}) \times H^{s+1/2}(q\mathbb{T})$ and they satisfy the estimate

$$\left\|(\zeta_0, v_0) - \epsilon(\zeta^{(1)}(\cdot, 0), \partial_t \zeta^{(1)}(\cdot, 0))\right\|_{H^{s+1}(q\mathbb{T}) \times H^{s+1/2}(q\mathbb{T})} \leq \epsilon^{3/2}\delta. \tag{23}$$

(continued)

For all such data (ζ_0, v_0), the water wave system admits a unique solution $\zeta(\alpha, t)$ on $[0, O(\epsilon^{-2})\log\frac{1}{\delta}]$ with

$$(\zeta_\alpha - 1, D_t\zeta) \in C([0, O(\epsilon^{-2})\log\frac{1}{\delta}]; H^s(q\mathbb{T}) \times H^{s+1/2}(q\mathbb{T}))$$

satisfying the following estimate: for all $t \in [0,\ O(\epsilon^{-2})\log\frac{1}{\delta}]$,

$$\left\|\Big(\partial_\alpha\zeta(\alpha, t) - 1, D_t\zeta\Big) - \epsilon\Big(\partial_\alpha\zeta^{(1)}, \partial_t\zeta^{(1)}\Big)\right\|_{H^s(q\mathbb{T})\times H^{s+1/2}(q\mathbb{T})} \leq C\epsilon^{3/2}\delta e^{\epsilon^2 t}.$$

With the theorem, the nonlinear modulational instability of the Stokes waves is proved.

Future work

- High-frequency instability: if the perturbation is not long wave, can we prove the instability of the Stokes waves?
- Stability/instability of the large or even extreme Stokes waves.
- Stability/instability of the solitary waves
- What happens after the instability? Recurrence? Blowup?

Thank you for your attention !