# Well-posedness for the KdV hierarchy

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# Korteweg-de Vries (KdV) equation

Korteweg-de Vries (KdV) equation

$$u_t + u_{xxx} = 6uu_x, \qquad u : \mathbb{R} \times [0, T] \to \mathbb{R}$$

models the propagation of shallow water waves along a canal.

Historically, the ship designer John Scott Russell made a remarkable scientific discovery about soliton in 1834.
After a long debate over whether such solitary waves are consistent with the mathematical theory of a frictionless fluid, Korteweg and de Vries(1895) confirmed that Russel was correct by finding explicit travelling wave solutions for KdV equation.

# KdV equation as dispersive equation

- *Dispersive Equation*
  By looking at linear part: Airy equation $u_t + u_{xxx} = 0$. If consider the plane waves $e^{i(kx-\omega(k)t)}$, $\omega(k) = -k^3$, so group velocity is $\Delta_k \omega(k) = -3\xi^2$, telling us that linear wave propagate to the left and higher frequencies travels faster.
- *Derivative semilinear equation*
  *Loss of 1-derivative* in the nonlinearity $uu_x$ will make the regularity theory difficult. In particular, the classical energy method fails. So the key to low regularity theory is the local smoothing estimate that helps to recover the derivative loss.

# KdV as integrable equation

The existence of Lax pair

$$L = -\partial_x^2 + u; \qquad P = -4\partial_x^3 + 3(\partial_x u + u\partial_x).$$

Then

$$u(t) \text{ solves KdV} \iff \frac{d}{dt}L(t) = [P(t), L(t)]$$

In particular, if $u(t)$ is a Schwartz-space solution to KdV, then the unitary operators $U(t)$ defined via $\frac{d}{dt}U(t) = P(t)U(t)$ with $U(0) = Id$, gives

$$L(t) = U(t)L(0)U(t)^*$$

Thus the KdV flow *preserves all spectral properties of L*.

There are infinite many conserved quantities

1. $I_1(u) = \int_{\mathbb{R}} u\,dx$
2. $I_2(u) = \int_{\mathbb{R}} u^2\,dx$
3. $I_3(u) = H(u) = \int_{\mathbb{R}} (\partial_x u)^2 + 2u^3\,dx$
4. $I_4(u) = \int_{\mathbb{R}} (\partial_{xx} u)^2 + 10u_x^2 u + 5u^4\,dx$

# The low regularity problem

Low regularity problem: What is the optimal range of $s$, such that for initial data $u_0 \in H^s(\mathbb{R})$, the Cauchy problem is local/global wellposed?

This problem has been studied intensively.

- Kenig-Ponce-Vega(91') LWP for $s > \frac{3}{4}$. The use of local smoothing estimate.
- Bourgain (93') LWP for $s = 0$, introduced the $X^{s,b}$ space.
- Kenig-Ponce-Vega(96') LWP for $s > -\frac{3}{4}$, the bilinear estimate
- Colliander-Keel-Staffilani-Takaoka-Tao(03') GWP for $s > -\frac{3}{4}$, the success of almost conservation law (I-method).
- Christ-Colliander-Tao (03') LWP $s = -\frac{3}{4}$, and (weak) illposedness $s < -\frac{3}{4}$ in the sense that uniform dependence on data fails.
- Guo, Kishimoto (09') GWP at $s = -\frac{3}{4}$. (Refined resolution space)

# The low regularity problem

The results in previous slide didn't use the integrability, hence the method applies to much wider class of equations.
However, the integrability structure of KdV, has to enter in order to prove the sharp regularity result.

- Kappeler-Tapalov(06') GWP in $H^{-1}(\mathbb{T})$, in the sense that solution flow is continuous in time.
- Molinet (11') strong illposedness $s < -1$, in the sense that the solution flow is not continuous.
- Low regularity conservation law. Koch-Tataru('18), Killip-Visan-Zhang('18)
- Killip-Visan (19') GWP for $s \geq -1$. (Regularized/approximate flow)

# Transimission Coefficient

Consider the Lax operator $L = -\partial_x^2 + u$ and the eigenvalue problem

$$L\phi = z^2\phi, \qquad Imz > 0$$

The left Jost function $\phi_l$ is solution with the normalization at $-\infty$

$$\lim_{x\to-\infty} e^{izx}\phi_l(x) = 1$$

Then the transmission coefficent is defined on the upper half plane as

$$T(z) = \left(\lim_{x\to\infty} e^{izx}\phi_l(x)\right)^{-1}.$$

Alternatively, we can also define the right Jost function $\phi_r$, with $\lim_{x\to\infty} e^{-izx}\phi_r(x) = 1$. Then

$$T(z) = \frac{2iz}{W(\phi_l, \phi_r)}$$

which is formally conserved along the KdV flow.

# Transmission Coefficient

The KdV Hamiltonians are defined for Schwartz functions $u$ as the coefficients (Faddeev)

$$\log T(z) \sim i \sum_{j=-1}^{\infty} H_j^{KdV} (2z)^{-2j-3}$$

of the formal series.
The first few are of the form

$$\begin{aligned} H_{-1}^{KdV} &= \frac{1}{2} \int u dx, \\ H_0^{KdV} &= \frac{1}{2} \int u^2 dx, \\ H_1^{KdV} &= \frac{1}{2} \int u_x^2 + 2u^3 dx, \\ H_2^{KdV} &= \frac{1}{2} \int u_{xx}^2 + 10 u_x^2 u + 5u^4 dx. \end{aligned}$$

# KdV hierarhcy

We equip the space of smooth functions on the Schwartz space with the Poisson structure

$$\{F, G\} = \int \frac{\delta F}{\delta u} \partial_x \frac{\delta G}{\delta u} dx.$$

With respect to this Possion structure, the Hamiltonian flow with Hamiltonian $H$ is of the form

$$u_t = \partial_x \left( \frac{\delta H}{\delta u} \right)$$

So by taking $H = H_j^{KdV}$, we get the corresponding KdV hierarchy

$$u_t = - u_{xxx} + 6uu_x$$

$$u_t = \left( u_4 - 10uu_2 - 5u_1^2 + 10u^3 \right)_x$$

$$u_t = \left( -u_6 + 14uu_4 + 28u_1 u_3 + 21u_2^2 - 70uu_1^2 - 70u^2 u_2 + 35u^4 \right)_x$$

# KdV hierarchy

The KdV hierarchy is a system of integrable equations, which all have the same scaling as KdV. So the scaling critical space is $\dot{H}^{-\frac{3}{2}}(\mathbb{R})$.

- Grunrock showed j-th KdV is LWP in Fourier Lebesgue space $H^r_s$ equiped with norm $\| \langle \xi \rangle^s \hat{f} \|_{L^{r'}}$ , $r = 1+, s > 2j - 2$
- For general 5th KdV, Kwon(08') showed LWP for $s > \frac{5}{2}$, and he also showed that the general 5th KdV uniform dependence on data fails for any $s > 0$.
- Kenig-Pilod, Guo-Kwak-Kwon('12) GWP for $s = 2$. They did it for general 5th order equation, so not necessarily integrable.
- Periodic case, Kappeler and Molnar(18') showed 5th KdV is GWP for $s = 0$, and strongly illposed for $s < 0$ (solution flow is not continuous)
- Recently Bringmann-Killip-Visan(19') showed 5th KdV is GWP for data in $H^{-1}(\mathbb{R})$.

# Main result

$$\|u\|_{L^2_l H^N} := \sup_{x_0} \left( \|u\|_{L^2(I\times(x_0-1,x_0+1))} + \|u^{(N)}\|_{L^2(I\times(x_0-1,x_0+1))} \right)$$

## Theorem (Klaus-Koch-L. '22)

*Given $u_0 \in L^2$, there is a unique weak solution $u \in C([0,\infty), L^2) \cap L^2_{loc} H^N$ to Nth KdV flow. The map $u_0 \in L^2 \to u(t) \in C([0,\infty), L^2) \cap L^2_{loc} H^N$ extends to a continuous map*

$$u_0 \in H^{-1} \to u \in C([0,\infty), H^{-1}) \cap L^2_{loc} H^{N-1}$$

*which further satisfies the Kato smoothing estimate*

$$\|u(t)\|_{H^{-1}} + \sup_{x_0} \|sech(x-\kappa t - x_0) u^{(N)}\|_{L^2([0,\infty)\times(x_0-1,x_0+1))} \leq C\|u_0\|_{H^{-1}}$$

*for all $\kappa > \kappa_0(\|u_0\|_{H^{-1}})$*

# Main result

- The result for $N = 1, 2$ were already obtained by Killip-Visan, Bringmann-Killip-Visan. Notice 5th kdv is signicantly more complicated. Here we try to offer a unified approach for all higher order equations.
- As by product, we obtain unconditional uniqueness of solutions for data in $L^2$.
- We believe the result is sharp as the illposedness proof of Molinet for KdV should carry over.
- On the torus, KdV hierarchy have increasing sharp regularity. i.e. KdV at $H^{-1}$, 5th KdV at $L^2$, 7th KdV at $H^1$. The key difference of line and torus, is the smoothing effect.
- We mainly follows the novel idea of Killip-Visan (approximate flow). But we do it at Gardner level instead of KdV, which seems to be a bit easier to carry over to higher hierarchy.

# Miura map

Miura map(1968)

$$u = M(v) := v_x + v^2$$

connects defocusing mKdV with real-valued KdV, from which we expect that the regularity result for mKdV is 1-derivative higher than that of KdV.

Defocusing mKdV

$$\partial_t v + \partial_x^3 v = 6v^2 \partial_x v \longrightarrow^{u=M(v)} \partial_t u + \partial_x^3 u = 3\partial_x(u^2)$$

It is defocusing in the sense that the Hamiltonian $H(u) = \int_{\mathbb{R}} (\partial_x u)^2 + u^4 \, dx$ is conserved and positive.
The Miura map is not invertible!

# Miura map

If we can invert the Miura map, then we can reduce the problem to solving the mKdV at $L^2$ (The benefit is conservation of mass!)

$M(v) = v_x + v^2$ maps $H^s$ to $H^{s-1}$ for $s \geq 0$.
For $u$ to be in the range of Miura, i.e. $u = v_x + v^2$, the Schrödinger operator $L = -\partial_x^2 + u = (\partial_x + v)(-\partial_x + v)$ has to be positive semidefinite.

For $u \in H^{-1}(\mathbb{R})$, we can find $\tau_0$ large enough (depending on $\|u\|_{H^{-1}(R)}$, such that $-\partial_x^2 + u + \tau_0^2$ is positive definite. Hence we find

$$M(v) = u + \tau_0^2, \quad v = w + \tau_0$$

i.e. $u = M_{\tau_0}(w) = w_x + w^2 + 2\tau_0 w$ (Modified Miura map). This map is invertible! And $w$ sovles the Gardner equation

$$w_t = Gardner(w) := -w_{xxx} + 6w^2 w_x + 12\tau_0 w w_x$$

$$KdV(w_x + w^2 + 2\tau_0 w) = (\partial_x + 2\tau_0 + 2w)Gardner(w).$$

# Gardner Hierarchy

In the same way, we will define a new hierarchy of equations by
$KdV_j(w_x + w^2 + 2\tau_0 w) = (\partial_x + 2\tau_0 + 2w)Gardner_j(w)$.
Remark: $H_j^{Gardner}(w) = H_{j-1}^{KdV}(w_x + w^2 + 2\tau_0 w)$ up to lower order terms...
So our goal is to solve the Gardner hierarchy in $L^2$

$$w_t = Gardner_j(w)$$

For $s \geq 0$, since the modified Miura map $M_{\tau_0}$ is an analytic diffeomorphism from $H^s$ to an open subset $\Omega \subset H^{s-1}$. The range $\Omega$ consists of all potential $u \in H^{s-1}$ so that the ground state energy is above $-\tau_0^2$.

# Approximate Flow

In the work of Killip-Visan on KdV at $H^{-1}(\mathbb{R})$, they introduced the idea of approximate flow as follows:

$$-logT(ik) = -\frac{1}{2k}H_{-1}^{KdV} + \frac{1}{(2k)^3}H_0^{KdV} - \frac{1}{(2k)^5}H^{KdV} + O(k^{-7})$$

So they consider a new Hamiltonian

$$H_k = (2k)^5\left(-\frac{1}{2k}H_{-1}^{KdV} + \frac{1}{(2k)^3}H_0^{KdV} + logT(ik)\right)$$

Then $H_k$ flow will provide a good approximation for the KdV flow when $k$ is large.

- Show KdV is LWP for Schwartz data
- Show $H_k$ flow is wellposed at $H^{-1}$.
- Show the solutions of KdV and $H_k$ flow converge to each other, since the flow commutes, show the difference flow of the Hamiltonian $H^{KdV} - H_k$ converges to 0 when $k \to +\infty$.

# Approximate flow for Gardner

Now in our case of Gardner equation/ Gardner Hierarchy, we can start with the Gardner Lax operator (By Wadati)

$$L^{Gardner} = i \begin{pmatrix} \partial_x & -w \\ 2\tau_0 + w & -\partial_x \end{pmatrix}.$$

Define the Jost function, and hence the transmission coefficient, and by direct computation we could check

$$T^{Gardner}(w) = T^{KdV}(w_x + w^2 + 2\tau_0 w)$$

We can further deduce a similar expansion

$$-\log T^{Gardner}(ik, w, \tau_0) = \frac{\tau_0}{k}\int w dx + \frac{1}{2k}\int w^2 dx - \frac{1}{4k^3}\tilde{H}_1 + \frac{1}{16k^5}\tilde{H}_2 \pm \dots,$$

where $\tilde{H}_j = H^{KdV}_{j-1}(w_x + 2\tau_0 w + w^2) \approx H^{Gardner_j}$.

# Approximate Flow for Gardner

Denote

$$\alpha(k, w, \tau_0) = -\log T^{Gardner}(ik, w, \tau_0) - \frac{\tau_0}{k}\int w dx, \tag{1}$$

and so we obtain an approximation for the $\tilde{H}_i^k$ by

$$\tilde{H}_1^k = 4k^3\left(\frac{1}{2k}\int w^2 dx - \alpha\right) \to \tilde{H}_1,$$
$$\tilde{H}_2^k = 4k^2(\tilde{H}_1 - \tilde{H}_1^k) \to \tilde{H}_2,$$

and in general recursively by $\tilde{H}_j^k = (-1)^j 4k^2(\tilde{H}_{j-1} - \tilde{H}_{j-1}^k) \to \tilde{H}_j$ as $k \to \infty$.

So the strategy is to prove by induction, if we have wellposedness of the $j-1$ Gardnder ($\tilde{H}_{j-1}$) and the $j-1$ approximate flow $\tilde{H}_{j-1}^k$ . Then the $j$ approximate flow $H_j^k$ is wellposed. So we only need to prove the convergence of approximate flow with the Gardner flow.

# The method of approximate flow

Let us quickly highlight the approximate procedure (using Gardner as example)
Take Schwartz data, $u_{n,0} \in \mathcal{S}(R)$ converges in $L^2$ to $u_0$. Write $J = \partial_x$ and denote $u_n(t) = e^{tJ\nabla H^G} u_{n,0}$ as the solution to jth Gardnder equation, $u_{n,k}(t) = e^{tJ\nabla H_k} u_{n,0}$ as the solution to the approximate flow.

Then first show the solution $u_n(t)$ converges weakly, Take $\phi \in \mathcal{S}(R)$

$$\begin{aligned}
\sup_{|t|\leq 1} |\langle \phi, u_n(t) - u_m(t)\rangle| \leq & \sup_{|t|\leq 1} \left|\left\langle \phi, u_n(t) - u_{n,k}(t)\right\rangle\right| \\
& + \sup_{|t|\leq 1} \left|\left\langle \phi, u_m(t) - u_{m,k}(t)\right\rangle\right| \\
& + \sup_{|t|\leq 1} \left|\left\langle \phi, u_{n,k}(t) - u_{m,k}(t)\right\rangle\right|
\end{aligned}$$

Third line vanishs due to the wellposedness of the approximate flow $H_k$. So the main point is to show the first two converges.

# Difference flow

Since the flow commutes, then we have

$$\begin{aligned} u_n - u_{n,k} =& e^{tJ\nabla H^G} u_{n,0} - e^{tJ\nabla H_k} u_{n,0} \\ =& \left(1 - e^{tJ\nabla(H_k - H^G)}\right) e^{tJ\nabla H^G} u_{n,0} \end{aligned}$$

So we are left to consider the difference flow
$w_t = \partial_x \frac{\delta}{\delta w}(H_k - H^G)$
We will prove stronger statement

$$\sup_{|t|\leq 1, w(0)\in Q} \left| \int_0^1 \langle \phi, w(t)' \rangle \, dt \right| \to 0, \qquad k \to +\infty.$$

where $Q \subset L^2$ is an equicontinuous set $Q = \{e^{tJ\nabla H_k} u_{n,0}, t \in [0,1], \forall n\}$
This reduce to estimate

$$\sup_{|t|\leq 1, w(0)\in Q} \left| \int_0^1 \left\langle \phi', \frac{\delta}{\delta w}(H_k - H^G) \right\rangle dt \right| \to 0, \qquad k \to +\infty.$$

# Compactness argument

To upgrade from weak convergence to strong convergence, we need the following precompact property.

Definition: A bounded set $Q \subset L^2(\mathbb{R})$ is precompact in $L^2(\mathbb{R})$ if and only if it is equicontinuous and tight.
By equicontinuous, we mean

$$\lim_{h \to 0} \sup_{w \in Q} \|w(\cdot + h) - w\|_{L^2} \to 0.$$

By tight, we mean

$$lim_{R \to +\infty} \sup_{w \in Q} \|w\|_{L^2([-R,R]^c)} = 0$$

Argue by contradiction, if $\{u_n(t)\}$ is not Cauchy in $C_t L^2$, then we can find sequence $\|u_{n_p}(t_p) - u_{m_p}(t_p)\|_{L^2} \geq \epsilon_0$, and by precompactness in $L^2$, we may assume they have limit. Hence we can find $\phi \in \mathcal{S}(\mathbb{R})$ such that $\langle \phi, u_{n_p}(t_p) - u_{m_p}(t_p) \rangle \geq \frac{1}{2}\epsilon_0$, contradicts with the weak convergence.

# Summary of the task

Now let us sum up the main job

- Equicontinuity
  A bounded subset $Q \subset L^2(\mathbb{R})$ is equicontinuous if and only if

$$\lim_{\tau\to\infty} \sup_{w\in Q} \int_{\mathbb{R}} \frac{\xi^2|\hat{w}(\xi)|^2}{\xi^2+4\tau^2} d\xi = 0. \tag{2}$$

  This will be resolved by showing that $\alpha$ is conserved along the Gardner flow (and the $H_k$) flow. And $\alpha(\tau) \approx \int_{\mathbb{R}} \frac{\xi^2|\hat{w}(\xi)|^2}{\xi^2+4\tau^2} d\xi$ in the sense that

$$|\alpha(\tau) - \int_{\mathbb{R}} \frac{\xi^2|\hat{w}(\xi)|^2}{\xi^2+4\tau^2} d\xi| \leq \tau^{-1}\|w_0\|_{L^2}^2$$

# Summary of the task

- Tightness
  If $\|w_0\|_2 \lesssim 1$, for big $R$, $\|w\|_{L^2([-R,R]^c)} \leq \epsilon$, then there exists $R_1 > 1$ and such that
  $$\|w(t)\|_{L^2([-R_1,R_1]^c)} \leq \epsilon$$
  uniformly for $|t| \leq 1$.
  In particular, if $Q \subset L^2$ is precompact, then $\{e^{tJ\nabla H^G} w, w \in Q, |t| \leq 1|\}$ is also precompact in $L^2$.

# Local smoothing estimate

Take Gardner equation as example (or the defocusing mKdV)

$$\partial_t \int_{\mathbb{R}} \eta w^2 dx = \int_{\mathbb{R}} \eta_t w^2 dx - 3\int_{\mathbb{R}} \eta_x (w_x^2 + w^4) dx + \int_{\mathbb{R}} \eta_{xxx} w^2 dx - 8\tau_0 \int_{\mathbb{R}} \eta_x w^3$$

Integrating in time, this implies by localizing in space, we gain one more derivative for the solution

$$\int_0^1 \int_{\mathbb{R}} \eta_x(x)(w_x^2 + w^4) dxdt \lesssim \|\eta(0,x)^{\frac{1}{2}} w_0\|_{L^2}$$

The choice of $\eta$, if $\eta = 1 + tanh(\frac{x - x_0}{\tau})$ implies local smoothing.
The choice of $\eta = 1 + tanh(\frac{x - R - \tau^2 t}{\tau^2})$, gives tightness on right hand side, since

$$\int_{\mathbb{R}} \eta(t,x) w^2(t) dx \lesssim \|\eta(0,x)^{\frac{1}{2}} w_0\|_{L^2}$$

# Local smoothing estimate

The choice of $\eta = 1 - tanh(\frac{x+R+\tau^2 t}{\tau^2})$, gives estimate in the wrong sign

$$\int_{\mathbb{R}} \eta(1,x)w^2(1,x)dx = \int_{\mathbb{R}} \eta(0,x)w_0^2(x)dx - \int_0^1 \int_{\mathbb{R}} \eta_x w^2 dxdt$$
$$-3\tau^{-2} \int_0^1 \int_{\mathbb{R}} \eta_x (w_x^2 + w^4) dxdt - \tau^{-6} \int_0^1 \int_{\mathbb{R}} \eta_{xxx} w^2 dxdt$$
$$-8\tau^{-2}\tau_0 \int_0^1 \int_{\mathbb{R}} \eta_x w^3 dxdt.$$

We need to control the red part now (since it has the wrong sign) yet, it is essentially local smoothing for another choice of time indenpendent $\eta$!

Hence we get tightness on the left.

The argument carries with little effort to the whole hierarchy

# Difference Flow

- Convergence of the difference flow. Consider the difference Hamiltonian $H^G - H_k$. The main advantage is the following result from Koch-Tataru

$$-logT^{mKdV}(i\tau, u) = \sum_{j=1} \tilde{T}_{2j}(i\tau, u)$$

where $\tilde{T}_{2j}(i\tau, u)$ is of $2j$ linear in $u$, and we have the estimate for

$$|\tilde{T}_{2j}| \lesssim \tau^{1-2j} \|u\|_{L^2}^{2j}$$

The main point is even though $\alpha$ or $-\log T$ is very complicated, but by performing multilinear expansion, only few terms(finite terms) need to be estimated carefully.
Smoothing estimate for difference flow again, need careful cancellation.

Thank You!