Differential operators with singular potentials

Jussi Behrndt (TU Graz)

based on joint papers with
P. Exner, M. Holzmann, M. Langer, V. Lotoreichik,
T. Ourmières-Bonafos, K. Pankrashkin, and M. Tušek

SITE Research Center, NYU Abu Dhabi

Contents

PART I

Differential operators and perturbations by potentials

T. Kato
Perturbation theory for linear operators
Classics in Mathematics, Springer, 1966

M. Reed and B. Simon
Methods of modern mathematical physics I-IV
Academic Press, 1972, 1975, 1977, 1978

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- Scattering theory: $e^{-iA_V t} f \sim e^{-iA_0 t} f_\pm$ for $t \to \pm\infty$

Schrödinger operators in $L^2(\mathbb{R}^n)$

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The Laplace operator (or free Schrödinger operator)

$$A_0 f = -\Delta f,$$
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$$\sharp\{\sigma(A_V) \cap (-\infty, 0)\} \le C_n \int |V_-(x)|^{n/2}\, dx$$

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First order (formal) differential operator in $L^2(\mathbb{R}^3)^4$:

$$A_V f = \big(\underbrace{-i\alpha\cdot\nabla + m\beta}_{A_0} + V\big) f = -i \underbrace{\sum_{j=1}^{3} \alpha_j \partial_j}_{\alpha\cdot\nabla} f + m\beta f + Vf,$$

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with 4×4 Dirac matrices and 2×2 Pauli spin matrices

$$\alpha_j := \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \quad \text{and} \quad \beta := \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix},$$

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or, in addition, with Lorentz scalar potential,

$$V(x) = v_{el}(x) I_4 + v_{sc}(x)\beta \quad \text{with } v_{el}, v_{sc}:\mathbb{R}^3\to\mathbb{R}$$

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Perturb $A_0 = -\Delta$ and $A_0 = -i\alpha \cdot \nabla + m\beta$ by potentials

$$V(x) \quad \text{and} \quad v_{el}(x) I_4 + v_{sc}(x)\beta$$

with V, v_{el}, v_{sc} **strongly localized** near hypersurface Σ and treat instead (as model) singular potential

$$A_\omega = -\Delta + \omega(x)\delta_\Sigma \;\; \text{and} \;\; A_{\eta,\tau} = -i\alpha \cdot \nabla + m\beta + \big(\eta I_4 + \tau\beta\big)\delta_\Sigma$$

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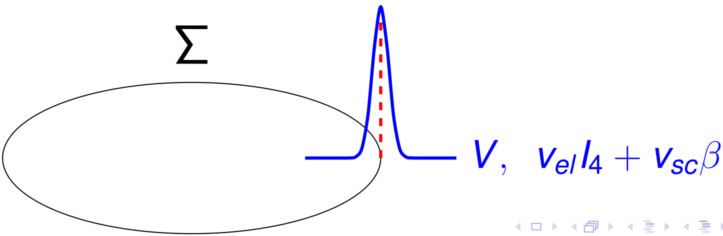

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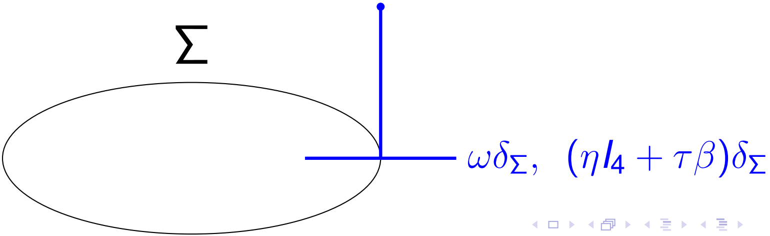

PART II

Schrödinger operators with δ-interactions

JB, M. Langer, V. Lotoreichik
Schrödinger operators with δ and δ'-potentials supported on hypersurfaces
Ann. Henri Poincaré 14 (2013), 385-423

JB, P. Exner, M. Holzmann, V. Lotoreichik
Approximation of Schrödinger operators with δ-interactions supported on hypersurfaces
Math. Nachr. 290 (2017), 1215-1248

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- A_0 unperturbed Laplacian; $\sigma(A_0) = \sigma_{\text{ess}}(A_0) = [0, \infty)$
- A_ω rank one perturbation of A_0:

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For $\varepsilon \to 0$ we obtain the boundary condition:

$$-\psi'(y+) + \psi'(y-) + \omega\psi(y) = 0$$

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- ac-parts of A_ω and A_0 unitarily equivalent

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$\Rightarrow$ Convergence of spectra

Some further references and contributors

- Albeverio, Gesztesy, Hoegh-Krohn, Holden [2nd ed., 2005]
- Albeverio, Kurasov [2000]
- Brasche, Exner, Kuperin, Seba, JMAA 184 (1994), 112-139
- Exner and many collaborators
- Adamyan, Brasche, Derkach, Dijksma, Hassi, Hryniv, Kostenko, Kuperin, Kuzhel, Luger, Malamud, Mikhailets, Minlos, Neidhardt, Neiman-Zade, Pavlov, Popov, Posilicano, Seba, Shkalikov, Shondin, de Snoo,....

PART III

2D and 3D Dirac operators with δ-interactions

JB, M. Holzmann, T. Ourmières-Bonafos, K. Pankrashkin
Two-dimensional Dirac operators with singular interactions supported on closed curves
J. Functional Analysis 279 (2020), 108700, 47pp.

JB, P. Exner, M. Holzmann, V. Lotoreichik
On the spectral properties of Dirac operators with electrostatic δ-shell interactions
J. Mathématiques Pures Appliquées 111 (2018), 47–78.

Free Dirac operator in 2D

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For $m \geq 0$ we consider

$$\begin{aligned} A_0 f &= -i\sigma \cdot \nabla f + m\sigma_3 f = \big(-i(\sigma_1\partial_1 + \sigma_2\partial_2) + m\sigma_3\big) f, \\ \operatorname{dom} A_0 &= H^1(\mathbb{R}^2;\mathbb{C}^2) = H^1(\mathbb{R}^2)^2, \end{aligned}$$

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Proposition

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2D Dirac operator with δ-interaction on C^∞-loop Σ

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For $\eta, \tau \in \mathbb{R}$ consider (formal) operator

$$A_{\eta,\tau} = A_0 + (\eta I_2 + \tau\sigma_3)\delta_\Sigma = -i\sigma \cdot \nabla + m\sigma_3 + \begin{pmatrix} \eta+\tau & 0 \\ 0 & \eta-\tau \end{pmatrix}\delta_\Sigma;$$

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$$-i\,(\sigma \cdot \nu)\big(f_i|_\Sigma - f_e|_\Sigma\big) = \frac{1}{2}(\eta I_2 + \tau\sigma_3)\big(f_i|_\Sigma + f_e|_\Sigma\big) \Bigg\}$$

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Integration by parts shows that δ-interaction $(\eta I_2 + \tau\sigma_3)\delta_\Sigma$ is modeled by the above coupling conditions in dom $A_{\eta,\tau}$

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- More loops $\Rightarrow$ More points in $\sigma_{\text{ess}}(A_{\eta,\tau}) \cap (-m, m)$

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Critical-case $\eta^2 - \tau^2 = 4$: $\sigma_{\mathrm{ess}}(A_{\eta,\tau})$ in $(-m, m)$ may appear!

Some further references

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- Cassano Lotoreichik Mas Tušek 2021
- Benhellal 2021

Thank you for your attention

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