Uniform regularity in the low Mach number and the inviscid limits for the full compressible Navier-Stokes system in a domain with boundaries

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Non-isentropic compressible Navier-Stokes equation

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \tilde{\mu}\operatorname{div}\mathcal{L}u + \nabla P = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega, \\ \partial_t(\rho e) + \operatorname{div}(\rho u e) + P\operatorname{div} u = \tilde{\kappa}\Delta\mathcal{T} + \tilde{\mu}\mathcal{L}u \cdot \mathbb{S}u, \end{cases}$$

- $\rho > 0, u$ are the density and the velocity of the fluids.
- e represents the internal energy, proportional to the temperature:

$$e = C_v \mathcal{T}, \ (C_v > 0 \quad \text{Boltzmann constant})$$

- P denotes the pressure, depending on the density and the temperature $\mathcal{T}$:

$$P = R\rho\mathcal{T}, \ (R > 0 \text{ generic gas constant})$$

- The viscous stress tensor takes the form:

$$\mathcal{L}u = 2\lambda_1 \mathbb{S}u + \lambda_2 \operatorname{div} u \, \mathrm{Id}, \quad \mathbb{S}u = \frac{1}{2}(\nabla u + \nabla^t u), \quad \lambda_1 > 0, \ 2\lambda_1 + 3\lambda_2 > 0.$$

Boundary conditions

- For the energy/temperature: Neumann boundary condition:

$$\partial_{\mathrm{n}} e = 0, \ (\partial_{\mathrm{n}} \mathcal{T} = 0) \qquad \text{on } \partial\Omega$$

- For the velocity: Naviler-Slip boundary condition:

$$u \cdot \mathrm{n} = 0, \quad \Pi(\mathbb{S}u\mathrm{n}) + a\Pi u = 0 \quad \text{on } \partial\Omega \tag{1.1}$$

where $\Pi f = f - (f \cdot \mathrm{n}) \cdot \mathrm{n}$.

Remark

The slip boundary condition is equivalent to the following condition

$$u \cdot \mathrm{n} = 0, \quad \operatorname{curl} u \times \mathrm{n} = 2\Pi(-au + D\mathrm{n} \cdot u) \quad \text{on } \partial\Omega. \tag{1.2}$$

when a is chosen as $a = -D\mathrm{n}$, the second condition reduces to vorticity-slip boundary condition:

$$\operatorname{curl} u \times \mathrm{n} = 0.$$

Scaled system

Mach number:

$$Ma = \frac{u}{c} = \varepsilon << 1.$$

Taking into account the smallness of the Mach number, we perform the following scaling:

$$t \to \varepsilon t,\ x \to x,\ \rho \to \rho^\varepsilon,\ u \to \varepsilon u^\varepsilon,\ e \to e^\varepsilon,\ \tilde{\mu} \to \varepsilon\mu, \tilde{\kappa} \to \varepsilon\kappa$$

and find the system satisfied by $(\rho^\varepsilon, u^\varepsilon, e^\varepsilon)$:

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} - \mu\,\mathrm{div}\,\mathcal{L}u^\varepsilon = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ \partial_t(\rho^\varepsilon e^\varepsilon) + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon e^\varepsilon) + P^\varepsilon \mathrm{div}\, u^\varepsilon = \kappa \Delta \mathcal{T}^\varepsilon + \mu\varepsilon^2 \mathcal{L}u^\varepsilon \cdot \mathbb{S}u^\varepsilon. \end{cases} \tag{1.3}$$

- $\varepsilon \in (0,1], \quad \mu = \frac{1}{Re} \in (0,1], \quad \kappa = \frac{1}{Pe} \in (0,1].$

Low Mach number limit

We consider (1.3) as the equations of the pressure P^ε, the velocity u^ε and the temperature $\mathcal{T}^\varepsilon$:

$$
\begin{cases}
(\partial_t + u^\varepsilon \cdot \nabla) P^\varepsilon + \gamma P^\varepsilon \mathrm{div}\, u^\varepsilon - (\gamma - 1)\kappa \Delta \mathcal{T}^\varepsilon = (\gamma - 1)\mu\varepsilon^2 \mathcal{L} u^\varepsilon \cdot \mathbb{S} u^\varepsilon, \\
\rho^\varepsilon (\partial_t + u^\varepsilon \cdot \nabla) u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} - \mu\, \mathrm{div}\, \mathcal{L} u^\varepsilon = 0, \qquad (t, x) \in \mathbb{R}_+ \times \Omega, \\
\rho^\varepsilon C_v (\partial_t + u^\varepsilon \cdot \nabla) \mathcal{T}^\varepsilon + P^\varepsilon \mathrm{div}\, u^\varepsilon = \kappa \Delta \mathcal{T}^\varepsilon + \mu \varepsilon^2 \mathcal{L} u^\varepsilon \cdot \mathbb{S} u^\varepsilon,
\end{cases}
\tag{1.4}
$$

where $\gamma = 1 + C_v > 1$.

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$$\begin{cases} (\partial_t + u^\varepsilon \cdot \nabla)P^\varepsilon + \gamma P^\varepsilon \mathrm{div}\, u^\varepsilon - (\gamma-1)\kappa\Delta\mathcal{T}^\varepsilon = (\gamma-1)\mu\varepsilon^2 \mathcal{L}u^\varepsilon \cdot \mathbb{S}u^\varepsilon, \\ \rho^\varepsilon(\partial_t + u^\varepsilon \cdot \nabla)u^\varepsilon + \dfrac{\nabla P^\varepsilon}{\varepsilon^2} - \mu\, \mathrm{div}\, \mathcal{L}u^\varepsilon = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega, \\ \rho^\varepsilon C_v(\partial_t + u^\varepsilon \cdot \nabla)\mathcal{T}^\varepsilon + P^\varepsilon \mathrm{div}\, u^\varepsilon = \kappa\Delta\mathcal{T}^\varepsilon + \mu\varepsilon^2 \mathcal{L}u^\varepsilon \cdot \mathbb{S}u^\varepsilon, \end{cases} \tag{1.4}$$

where $\gamma = 1 + C_v > 1$. Sending $\varepsilon \to 0$, one gets formally

$$\begin{cases} \gamma\overline{P}\, \mathrm{div}\, u^0 = (\gamma-1)\kappa\Delta\mathcal{T}^0, \\ \rho^0(\partial_t + u^0 \cdot \nabla)u^0 + \nabla\pi^0 - \mu \mathrm{div}\, \mathcal{L}u^0 = 0, \\ C_v\gamma\rho^0(\partial_t + u^0 \cdot \nabla)\mathcal{T}^0 = \kappa\Delta\mathcal{T}^0, \end{cases} \tag{1.5}$$

where $\rho^0 = \overline{P}/R\mathcal{T}^0$.

Contexts of the low Mach number limit problem

Different contexts:

- Type of solutions: strong solutions, weak solutions (for viscous fluids).
- Assumptions imposed on the initial data:
  - well-prepared: $(\nabla P_0^\varepsilon/\varepsilon, \mathrm{div}\, u_0^\varepsilon) = \mathcal{O}(\varepsilon)$ or
  - ill-prepared $(\nabla P_0^\varepsilon/\varepsilon, \mathrm{div}\, u_0^\varepsilon) = \mathcal{O}(1) \to (\partial_t P^\varepsilon/\varepsilon, \partial_t u^\varepsilon)|_{t=0} = \mathcal{O}(\frac{1}{\varepsilon})$,
- Type of the fluid domain:
  - without boundary (whole space, torus)
  - with boundary: with fixed boundaries or free boundaries.

References for the low Mach number limit problem

Strong solution (without boundaries: $\mathbb{R}^d, \mathbb{T}^d$)

- Ebin; Klainerman-Majda: strong solution, well-prepared data, isentropic fluids
- Ukai, Gallagher, Danchin: strong solution, ill-prepared data, isentropic fluids
- Dutrifoy-Hmidi: convergence of solution in critical space, 2d Euler equation
- Metivier-Schochet, Alazard: non-isentropic Euler or Navier-Stokes, ill-prepared data,

Weak solution for Navier-Stokes:

- Lions-Masmoudi; Desjardins-Grenier-Bresch, Feireisl-Novotny

More references: the survey papers written by
Alazard, Danchin, Feireisl, Gallagher, Jiang-Masmoudi, Schochet...

References

- Non-isentropic Navier-Stkokes equations (strong solutions):
  - In the whole space $\mathbb{R}^3$:
    - Alazard 06', with ill-prepared data, uniform estimates in Mach number (ε) & Reynolds number ($1/\mu$) & Péclet number ($1/\kappa$).
  - In a domain with boundaries,
    - Jiang-Ou 11', with Dirichlet boundary condition & well-prepared data & vanishing thermal viscosity ($\kappa = 0$) (limit system being the non-homogeneous incompressible NS)
    - Dou-Jiang-Ou 15', Ju-Ou 22', with vorticity-slip boundary condition & well-prepared data

Our question:

Motivated by the low Mach number limit problem for the **strong solutions** to the non-isentropic NS in a domain with boundaries:

We aim to establish high regularity estimates which are

- Uniform in the Mach number by allowing
  - **ill-prepared data:** $(\nabla P_0^\varepsilon/\varepsilon, \mathrm{div}\, u_0^\varepsilon) = \mathcal{O}(1)$,
  - large temperature variation: $\mathcal{T}_0^\varepsilon = \mathcal{O}(1)$,
  - general Navier-Slip boundary condition: $\mathrm{curl}\, u^\varepsilon \times \mathrm{n} = 2\Pi(-au^\varepsilon + D\mathrm{n} \cdot u^\varepsilon)$ on $\partial\Omega$.
- Uniform also in the Reynolds number $(1/\mu)$ and Péclet number $(1/\kappa)$?

Uniform regularity in the Mach number for the isentropic system

The isentropic compressible Navier-Stokes system

$$\begin{cases} \partial_t \rho^\varepsilon + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \mathrm{div}\,(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \mathrm{div}\,\mathcal{L}u^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^\varepsilon|_{t=0} = u_0^\varepsilon,\ \rho^\varepsilon|_{t=0} = \rho_0^\varepsilon, \end{cases}$$

Phenomena

Under the ill-prepared assumption & Navier-Slip boundary condition, two phenomena play important roles:

- Fast oscillations of the acoustic waves
- boundary layer effects.

Phenomena 1: fast oscillations

Denote

$$\sigma^{\varepsilon} =: \frac{\ln P^{\varepsilon} - \ln \overline{P}}{\varepsilon}$$

and ignore the equation for the temperature, we are led to consider the following system:

$$\begin{cases} g_1(\varepsilon\sigma)(\partial_t \sigma^{\varepsilon} + u^{\varepsilon} \cdot \nabla \sigma^{\varepsilon}) + \frac{\mathrm{div}\, u^{\varepsilon}}{\varepsilon} = 0, \\ g_2(\varepsilon\sigma)(\partial_t u^{\varepsilon} + u^{\varepsilon} \cdot \nabla u^{\varepsilon}) - \mathrm{div}\, \mathcal{L} u^{\varepsilon} + \frac{\nabla \sigma^{\varepsilon}}{\varepsilon} = 0, \qquad (t,x) \in \mathbb{R}_+ \times \Omega \\ u^{\varepsilon}|_{t=0} = u_0^{\varepsilon}, \sigma^{\varepsilon}|_{t=0} = \sigma_0^{\varepsilon}, \end{cases}$$

where the scalar functions g_1, g_2 are defined by

$$g_2(s) = \rho(\bar{P} e^s), \quad g_1(s) = (\ln g_2)'(s).$$

Uniform regularity in the Mach number for the isentropic system

Phenomena 2: boundary layer effects.

For the simplest case, $\Omega = \mathbb{R}^3_+$ and for the following linearized system:

$$\partial_t U^\varepsilon + \frac{1}{\varepsilon} L U^\varepsilon - \begin{pmatrix} 0 \\ \mathrm{div}\, \mathcal{L} u^\varepsilon \end{pmatrix} = 0, \qquad L = \begin{pmatrix} 0 & \mathrm{div} \\ \nabla & 0 \end{pmatrix}, \quad U = (\sigma^\varepsilon, u^\varepsilon) \in \mathbb{R} \times \mathbb{R}^3.$$

Uniform regularity in the Mach number for the isentropic system

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We expect the following expansions involving boundary layers:

$$\begin{cases} \sigma^\varepsilon(t,x) = \sigma_0^I(\frac{t}{\varepsilon}, t, x) + \varepsilon^{\frac{3}{2}} \sigma^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots, \\ u^\varepsilon(t,x) = u_0^I(\frac{t}{\varepsilon}, t, x) + \sqrt{\varepsilon} \begin{pmatrix} u_{1,\tau}^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) \\ 0 \end{pmatrix} + \varepsilon u_2^B(\frac{t}{\varepsilon}, t, x, \frac{z}{\sqrt{\varepsilon}}) + \cdots \end{cases}$$

where $x = (y, z)$, $z < 0$, which suggests that u_τ is only uniform in $L^2_t H^1$.

Functional setting

Inspired by the previous work of Masmoudi-Rousset[1] (see also [2]) on the inviscid limit problem we use the so-called conormal space: when $\Omega = \mathbb{R}^3_+$, we define

$$L^p_t H^m_{co} = \{f \in L^p\big([0,t], L^2(\Omega)\big), Z^\alpha f \in L^p\big([0,t], L^2(\Omega)\big), |\alpha| \le m\},$$

where $Z^\alpha = Z_0^{\alpha_0} Z_1^{\alpha_1} Z_2^{\alpha_2} Z_3^{\alpha_3}$ with

$$Z_0 = \varepsilon\partial_t,\ Z_1 = \partial_{y_1},\ Z_2 = \partial_{y_2},\ \ Z_3 = \frac{z}{1+z}\partial_z$$

[1] Masmoudi-Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition.

[2] Metivier, Lecture notes on 'Stability of Noncharacteristic Viscous Boundary Layers'

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For general fixed domain, we can work in the local coordinates:

$$\Omega \subset \Omega_0 \cup_{i=1}^N \Omega_i, \quad \Omega_0 \Subset \Omega,$$

$\Omega_i \cap \Omega$ is the graph of a smooth function $z = \varphi_i(x_1, x_2)$.

$$Z^i_k = \partial_{y^k} = \partial_k + \partial_k\varphi_i\partial_3, \quad k = 1, 2 \qquad Z^i_3 = \phi(z)(\partial_1\varphi_1\partial_1 + \partial_2\varphi_1\partial_2 - \partial_3),$$

[1] Masmoudi-Rousset. Uniform regularity for the Navier-Stokes equation with Navier boundary condition.
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Compatibility condition

Definition (Compatibility condition)

We say that $(\sigma_0^\varepsilon, u_0^\varepsilon, \theta_0^\varepsilon)$ satisfy the compatibility conditions up to order m if for any $j = 0, 1 \cdots m-1$,

$$(\varepsilon\partial_t)^j (u^\varepsilon \cdot \mathrm{n})\big|_{t=0} = (\varepsilon\partial_t)^j (\partial_{\mathrm{n}}\theta^\varepsilon)|_{t=0} = 0, \quad \Pi\left[\mathbb{S}\big((\varepsilon\partial_t)^j u^\varepsilon|_{t=0}\big)\mathrm{n}\right] = -a\Pi\left[(\varepsilon\partial_t)^j u^\varepsilon|_{t=0}\right] \text{ on } \partial\Omega.$$

Again, the restriction of the time derivatives of the solution at the initial time can be expressed inductively by using the equations. For example, we have

$$(\varepsilon\partial_t u^\varepsilon)(0) = R\beta(\theta_0^\varepsilon)\big(\varepsilon\mu\Gamma(\varepsilon\sigma_0^\varepsilon)\mathrm{div}\,\mathcal{L}u_0^\varepsilon - \nabla\sigma_0^\varepsilon\big) - \varepsilon u_0^\varepsilon \cdot \nabla u_0^\varepsilon.$$

Main results for fixed domain

Theorem (Masmoudi-Rousset-Sun, JMPA, 2022)

Define

$$\mathcal{N}^{\varepsilon}_{m,T} \approx \|\nabla(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L^{\infty}_T H^{m-2}_{co} \cap L^2_T H^{m-1}_{co} \cap L^{\infty}_T L^{\infty}} + \varepsilon \|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L^{\infty}_T H^m_{co}} + \varepsilon \|\nabla u^{\varepsilon}\|_{L^2_T H^m_{co}}$$

Given an integer $m \geq 6$ and a C^{m+2} smooth domain Ω with boundaries. Consider a family of initial data such that $(\sigma_0^{\varepsilon}, u_0^{\varepsilon})$ satisfy compatibility conditions up to order m and $\sup_{\varepsilon\in(0,1]} \mathcal{N}^{\varepsilon}_{m,0} < +\infty$. Then, there exist $\varepsilon_0 \in (0,1]$ and $T_0 > 0$, such that, for any $0 < \varepsilon \leq \varepsilon_0$, the system $(CNS)_{\varepsilon}$ has a unique solution $(\sigma^{\varepsilon}, u^{\varepsilon})$ which satisfies:

$$\sup_{\varepsilon\in(0,\varepsilon_0]} \mathcal{N}^{\varepsilon}_{m,T_0} < +\infty.$$

Main results for fixed domain

Theorem (Masmoudi-Rousset-Sun, JMPA, 2022)

Define

$$\mathcal{N}^{\varepsilon}_{m,T} \approx \|\nabla(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L^{\infty}_{T}H^{m-2}_{co}\cap L^{2}_{T}H^{m-1}_{co}\cap L^{\infty}_{T}L^{\infty}} + \varepsilon\|(\sigma^{\varepsilon}, u^{\varepsilon})\|_{L^{\infty}_{T}H^{m}_{co}} + \varepsilon\|\nabla u^{\varepsilon}\|_{L^{2}_{T}H^{m}_{co}}$$

Given an integer $m \geq 6$ *and a* C^{m+2} *smooth domain* Ω *with boundaries. Consider a family of initial data such that* $(\sigma^{\varepsilon}_0, u^{\varepsilon}_0)$ *satisfy compatibility conditions up to order* m *and* $\sup_{\varepsilon\in(0,1]}\mathcal{N}^{\varepsilon}_{m,0} < +\infty$. *Then, there exist* $\varepsilon_0 \in (0,1]$ *and* $T_0 > 0$, *such that, for any* $0 < \varepsilon \leq \varepsilon_0$, *the system* $(CNS)_{\varepsilon}$ *has a unique solution* $(\sigma^{\varepsilon}, u^{\varepsilon})$ *which satisfies:*

$$\sup_{\varepsilon\in(0,\varepsilon_0]}\mathcal{N}^{\varepsilon}_{m,T_0} < +\infty.$$

Moreover, $\rho^{\varepsilon} = \rho(\varepsilon\sigma^{\varepsilon})$ *converges to* $\bar{\rho}$ *in* $C([0,T_0], L^2(\Omega))$, *and* u^{ε} *converges in* $L^2([0,T_0], L^2(\Omega))$ *(if* Ω *is the exterior domain) to* u^0 *which is the weak solution to the incompressible NS system with Navier boundary condition.*

Difficulties

The most involved part lies in the control of the gradient of the solution:

- The vorticity $\omega^\varepsilon = \mathrm{curl}\, u^\varepsilon$ reads:

$$\partial_t \omega^\varepsilon + u^\varepsilon \cdot \nabla \omega^\varepsilon - \mu \Delta \omega^\varepsilon = \omega^\varepsilon \cdot \nabla u^\varepsilon - \omega^\varepsilon \mathrm{div}\, u^\varepsilon + \cdots$$

Unlikely to be controlled by direct energy estimates.

- Lack of the uniform control of the tangential spatial derivatives:

$$\nabla u^\varepsilon \approx (\partial_y u^\varepsilon + \mathrm{div}\, u^\varepsilon + \omega^\varepsilon \times \mathsf{n}), \quad \omega^\varepsilon \times \mathsf{n}|_{\partial\Omega} \approx u^\varepsilon|_{\partial\Omega}.$$

there is no uniform control of the spatial tangential derivatives since the tangential vector fields $\partial_{y^1}, \partial_{y^2}$ do not commute with the penalized operator $(\nabla, \mathrm{div}\,)/\varepsilon$.

$(\partial_{y^k} = \partial_k + \partial_k \varphi_i \partial_3, \quad k = 1, 2.)$

Strategies

• To get the missing information, we would use the Leray-Helmholtz projection to split the velocity u^ε into a compressible part $\nabla\Psi^\varepsilon$ and incompressible part v^ε. More precisely, Ψ^ε is defined as the unique solution of

$$\begin{cases} \Delta\Psi^\varepsilon = \mathrm{div}\, u^\varepsilon & \text{in} \quad \Omega, \\ \partial_{\mathrm{n}}\Psi^\varepsilon = u^\varepsilon \cdot \mathrm{n} & \text{on} \quad \partial\Omega, \\ \int_\Omega \Psi^\varepsilon dx = 0. \end{cases}$$

• Control $\nabla^2\Psi^\varepsilon$ by $\mathrm{div}\, u^\varepsilon$ which is derived by using the equation and a-priori control of the weighted time derivatives.

• Control the incompressible part v^ε by direct energy estimates.

• To finish the estimate of the gradient, we need to control $\omega^\varepsilon \times \mathrm{n}$ where we use a lifting of the boundary and control the remainder by energy estimates.

Remark

Part of these strategies are also employed to justify the incompressible limit for the free surface Navier-Stokes system, where the uniform control of the regularity of the surface is an additional (crucial) obstacle.

$$\begin{cases} \partial_t \rho^\varepsilon + \operatorname{div}(\rho^\varepsilon u^\varepsilon) = 0, \\ \partial_t(\rho^\varepsilon u^\varepsilon) + \operatorname{div}(\rho^\varepsilon u^\varepsilon \otimes u^\varepsilon) - \operatorname{div} \mathcal{L} u^\varepsilon + \dfrac{\nabla P(\rho^\varepsilon)}{\varepsilon^2} = 0, \end{cases} \qquad (t,x) \in \mathbb{R}_+ \times \Omega_t^\varepsilon$$

where the (moving) domain:

$$\Omega_t^\varepsilon = \{x = (y,z) |\ y \in \mathbb{R}^2, z < h^\varepsilon(t,y)\},$$

and the surface is subjected to:

$$\partial_t h^\varepsilon - u^\varepsilon(t, y, h^\varepsilon(t,y)) \cdot \mathbf{N}^\varepsilon = 0, \quad y \in \mathbb{R}^2$$

[3] Masmoudi-Rousset-Sun, Incompressible limit for the free surface Navier-Stokes system, arxiv.

The following study:

- more general **non-isentropic** system.
- uniform regularity estimates also in the Reynolds number and the Péclet number.

Rewritten of the system

Let

$$\sigma^\varepsilon =: \frac{\ln P^\varepsilon - \ln \overline{P}}{\varepsilon}, \quad \theta^\varepsilon =: \ln \mathcal{T}^\varepsilon - \ln \overline{\mathcal{T}},$$

$$\begin{cases} \frac{1}{\gamma}(\partial_t + u^\varepsilon \cdot \nabla)\sigma^\varepsilon + \frac{\mathrm{div}\, u^\varepsilon}{\varepsilon} - \frac{\gamma - 1}{\gamma\varepsilon}\kappa\Delta\theta^\varepsilon = 0, \\ \beta(\theta^\varepsilon)(\partial_t + u^\varepsilon \cdot \nabla)u^\varepsilon + \frac{\nabla\sigma^\varepsilon}{\varepsilon} - \mu\, \mathrm{div}\, \mathcal{L}u^\varepsilon = 0, \qquad (t, x) \in \mathbb{R}_+ \times \Omega, \\ C_v(\partial_t + u^\varepsilon \cdot \nabla)\theta^\varepsilon + \mathrm{div}\, u^\varepsilon - \kappa\Delta\theta^\varepsilon = 0 \end{cases} \tag{1.6}$$

where we denote for simplicity $\beta(\theta^\varepsilon) = \overline{\mathcal{T}}^{-1} e^{-\theta^\varepsilon}$.

New feature: Two kinds of the boundary layers !

- Viscous boundary layer with size of $\sqrt{\varepsilon\mu}$ (and $\sqrt{\mu}$)
- Thermal boundary layer of size $\sqrt{\kappa}$.

Uniform regularity estimates for the non-isentropic NS system

Theorem (Uniform estimates in ε, μ, κ, Sun, arxiv, 22')

Given an integer $m \geq 7$ and a C^{m+2} smooth domain Ω. Let $A \subset (0,1]^2$ denote the set:

$$A =: \left\{ (\mu, \kappa) \in (0,1]^2 \middle| \left|\mu - \frac{\kappa}{C_v \gamma \lambda_1}\right| \lesssim \mu \kappa^{\frac{1}{2}} \right\}$$

Define the quantity:

$$\mathcal{N}_{m,T}^{\mu,\kappa} \approx \|(\nabla(\sigma, u, \theta), \partial_t \theta)\|_{L_T^\infty H_{co}^{m-2} \cap L_T^\infty L^\infty} + \|\nabla(\kappa^{\frac{1}{2}}(\sigma, \theta), \mu^{\frac{1}{2}} u)\|_{L_T^2 H_{co}^{m-1}} + \mu^{\frac{1}{2}} \|(\varepsilon\sigma, \varepsilon u, \theta)\|_{L_T^\infty H_{co}^m}.$$

Assume that the initial data satisfy compatibility conditions up to order m and be such that $\sup_{(\varepsilon,\mu,\kappa) \in (0,1] \times A} \mathcal{N}_{m,0}^{\mu,\kappa} < +\infty$.

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Assume that the initial data satisfy compatibility conditions up to order m and be such that $\sup_{(\varepsilon,\mu,\kappa)\in(0,1]\times A} \mathcal{N}_{m,0}^{\mu,\kappa} < +\infty$. *Then there exist $\varepsilon_0 \in (0,1]$ and $T_0 > 0$, such that, for any $0 < \varepsilon \leq \varepsilon_0$, any $(\mu, \kappa) \in A$, the system (1.6) has a unique solution $(\sigma^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ which satisfies:*

$$\sup_{(\mu,\kappa)\in A, \varepsilon\in(0,1]} \mathcal{N}_{m,T_0}^{\mu,\kappa}(\sigma^\varepsilon, u^\varepsilon, \theta^\varepsilon) < +\infty.$$

Remarks

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$$\|(\nabla(\sigma,u,\theta),\partial_t\theta)\|_{L^\infty_T H^{m-2}_{co}\cap L^\infty_T L^\infty} + \mu^{\frac{1}{2}}\|\nabla(\sigma,u,\theta)\|_{L^2_T H^{m-1}_{co}} < +\infty.$$

Remarks

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Remark

We are considering ill-prepared data *and allowing* large *initial temperature variation.*

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$$\|(\nabla(\sigma, u, \theta), \partial_t\theta)\|_{L^\infty_T H^{m-2}_{co} \cap L^\infty_T L^\infty} + \mu^{\frac{1}{2}} \|\nabla(\sigma, u, \theta)\|_{L^2_T H^{m-1}_{co}} < +\infty.$$

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We are considering ill-prepared data *and allowing* large *initial temperature variation.*

Remark

The relation $\left|\mu - \frac{\kappa}{C_v \gamma \lambda_1}\right| \lesssim \mu \kappa^{\frac{1}{2}}$ *between* μ *and* κ *is assumed due to the interactions of two kinds of boundary layers. However, the Reynolds number and the low Mach number can be completely independent.*

Main differences compared to the isentropic system

New issues due to the **non-constant temperature**:

- The uniform estimate of the incompressible part of the velocity:

$$\beta(\theta)(\partial_t + u \cdot \nabla)(\mathbb{P}u) + \nabla q - \mu\Delta(\mathbb{P}u) = -[\mathbb{P}, \beta]\partial_t u + \cdots$$

and also the vorticity:

$$\beta(\partial_t + u \cdot \nabla)\omega - \mu\Delta\omega = -\nabla\beta \times \partial_t u + \cdots .$$

Classical trick: Define the modified velocity $\tilde{u} = \beta u$ which satisfies:

$$(\partial_t + u \cdot \nabla)\tilde{u} + \frac{\nabla\sigma}{\varepsilon} - \mu \mathrm{div}\, \mathcal{L}u = -\beta' u(\partial_t + u \cdot \nabla)\theta$$

and consider its incompressible part and vorticity:

$$\tilde{v} = \mathbb{P}\tilde{u}, \quad \tilde{\omega} = \mathrm{curl}\, \tilde{u}$$

New issues for the estimates uniformly in the Reynolds number .

The new issues mainly arise from the control of the incompressible part and its gradient.

- Problem: Let $\tilde{v} = \mathbb{P}(\beta u)$ which is governed by:

$$(\partial_t + u \cdot \nabla)\tilde{v} + \nabla q - \mu \Delta \tilde{v} = \beta' u \underbrace{(\partial_t + u \cdot \nabla)\theta}_{\approx \operatorname{div} u + \cdots} + \cdots$$

The control of $\|\tilde{v}\|_{L^\infty_t H^{m-1}_{co}}$ requires the boundedness of $\|\operatorname{div} u\|_{L^2_t H^{m-1}_{co}}$ which is not uniform in μ, κ.

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- Strategy: introduce another weight equation:

$$r_2 = \beta(\theta)\exp(\varepsilon\sigma/C_v\gamma) = \beta(0)\exp(-\tilde{\theta}/C_v), \quad \tilde{\theta} = C_v\theta - \frac{\varepsilon\sigma}{\gamma},$$

and study its incompressible part:

$$(\partial_t + u \cdot \nabla)\big(\mathbb{P}(r_2 u)\big) + \nabla q - \mu\Delta\big(\mathbb{P}(r_2 u)\big) = -C_v r_2 u \underbrace{(\partial_t + u \cdot \nabla)\tilde{\theta}}_{\approx \kappa\Delta\theta + \cdots} + \cdots .$$

New issues for the estimates uniformly in the Reynolds number.

- Problem: estimate of the modified vorticity $\mathrm{curl}(r_2 u) \times \mathsf{n} =: \omega_{r_2}^{\mathsf{n}}$ which satisfies a transport-diffusion equation:

$$\left(\partial_t + u \cdot \nabla - \mu r_2^{-1} \Delta\right) \omega_{r_2}^{\mathsf{n}} = \mathrm{curl}\, G_{r_2} \times \mathsf{n} + \cdots$$

with Dirichlet boundary condition:

$$\omega_{r_2}^{\mathsf{n}}\big|_{\partial\Omega} = \frac{\varepsilon}{C_v \gamma} r_2 \partial_{\mathsf{n}} \sigma \Pi u + 2 r_2 \Pi\big(- a u + (D\mathsf{n}) u\big)\big|_{\partial\Omega}.$$

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We split it into two parts (after the change of the coordinates):

$$\begin{cases} (\partial_t - \mu(\frac{1}{r_2})^{\Psi}\big|_{z=0} \partial_z^2)\zeta_1 = 0 \\ \zeta_1|_{t=0} = 0, \quad \zeta_1|_{z=0} = \omega_{r_2}^{\mathrm{n}}|_{\partial\Omega} \circ \Psi, \end{cases} \quad \begin{cases} (\partial_t + u \cdot \nabla)\zeta_2 - \mu(\frac{1}{r_2})^{\Psi} \Delta^{\Psi} \zeta_2 = (u \cdot \nabla)\zeta_1 + \cdots, \\ \zeta_2|_{t=0} = \omega_{r_2}^{\mathrm{n}}|_{t=0} \circ \Psi, \quad \zeta_2|_{z=0} = 0, \end{cases}$$

We need to control $\|\zeta_1\|_{L_t^2 H_{co}^{m-1}}$, which requires at least $|\partial_{\mathrm{n}} \sigma|_{L_t^2 H^{m-3/2}}$ that cannot be controlled uniformly in κ.

New issues for the estimates uniformly in the Reynolds number.

- Strategy: do further correction to the weight function r_2.

$$r_0 = \beta(\theta)\exp(\varepsilon\tilde{\sigma}/C_v\gamma) \quad \text{with } \tilde{\sigma} = \sigma - \varepsilon\mu(2\lambda_1 + \lambda_2)\Gamma \mathrm{div}\, u$$

and switch to study $\omega^{\mathrm{n}}_{r_0} =: \mathrm{curl}(r_0 u) \times \mathrm{n}$, whose boundary condition is changed into

$$\omega^{\mathrm{n}}_{r_0}\big|_{\partial\Omega} = \frac{\varepsilon}{C_v\gamma} r_0 \partial_{\mathrm{n}}\tilde{\sigma}\Pi u + 2r_0\Pi\big(-au + (D\mathrm{n})u\big)\big|_{\partial\Omega}. \tag{1.7}$$

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By using the equation of the velocity,

$$\partial_{\mathrm{n}}\tilde{\sigma} \approx -\big(\beta(\varepsilon\partial_t + \varepsilon u\cdot\nabla)u + \varepsilon\mu\lambda_1 \,\mathrm{curl}\,\mathrm{curl}\, u\big)\cdot \mathrm{n} \quad \text{on} \quad \partial\Omega,$$

can be controlled uniformly in $L_t^2 H^{m-3/2}$.

Comments on the relation between μ and κ

$$\begin{cases} (\partial_t + \tilde{u}\cdot\nabla)\zeta_2 - \mu\lambda_1\overline{\Gamma}(\frac{1}{r_0})^{\Psi}(\partial_z^2+\Delta_g)\zeta_2 = \big(\mathrm{curl}\, G_{r_2}\times \mathrm{n}\big)^{\Psi} + \cdots, \quad (t,x)\in\mathbb{R}_+\times\mathbb{R}^3_+, \\ \zeta_2|_{z=0}=0, \quad \zeta_2|_{t=0}=\zeta|_{t=0}, \end{cases}$$

where $G_{r_0} =: u(\partial_t + u\cdot\nabla)r_2 + \mu\lambda_1\overline{\Gamma} r_2^{-1}[\mathrm{curl}\,\mathrm{curl}, r_2]u.$

$$\|\zeta_2(t)\|^2_{H^{m-2}_{co}} + \mu\iota\|\nabla\zeta_2\|^2_{L^2_t H^{m-2}_{co}} \le \|\zeta_2(0)\|^2_{H^{m-2}_{co}}$$

$$+\sum_{|I|\le m-2}\Big|\iint Z^I(\mathrm{curl}\, G_{r_0}\times \mathrm{n})^{\Psi}\cdot Z^I\zeta_2\,\mathrm{d}x\mathrm{d}s\Big| + \Big|\mu\iint Z^I\big((\frac{1}{r_0})^{\Psi}\big)\partial_z\zeta_2\cdot\partial_z Z^I\zeta_2\mathrm{d}x\mathrm{d}s\Big| + \cdots$$

- The second integral: Control of

$$\mu^{\frac{1}{2}}\|\big(Z^I(\frac{1}{r_0})^{\Psi}\big)\partial_z\zeta_2\|_{L^2_tL^2(\mathbb{R}^3_+)}, \quad \text{when } |I| = m-2.$$

Problem: $\mu^{\frac{1}{2}}\partial_z\zeta_2$ only bounded in $L^2_tL^2$ and $\|Z^I(\frac{1}{r_0})^{\Psi})\|_{L^\infty_{t,x}} \lesssim \|\nabla\theta\|_{L^\infty_t H^{m-1}_{co}} + \cdots$ is not uniformly bounded in κ.

Main steps

Step 1: Uniform high-order $\varepsilon\partial_t$ derivatives up to order $m-1$: In this step, we aim to control $\|(\varepsilon\partial_t)^j(u,\sigma)\|_{L^\infty L^2}, (j \le m-1)$:

$$\begin{pmatrix} 0 & \mathrm{div} & \frac{\gamma-1}{\gamma}\kappa\overline{\Gamma}\mathrm{div}\,(\beta(0)\nabla\cdot) \\ \nabla & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not skew-symmetric, we need to work on the unknown $w = u - \kappa\frac{\gamma-1}{\gamma}\Gamma(0)\beta(\theta)\nabla\theta$ and perform energy estimates for $(\sigma, w, \kappa\nabla\theta)$.

Step 2: Uniform estimates for θ. Here we want to prove the boundedness of

$$\|\theta\|_{L_T^\infty H_{co}^{m-1}} + \sqrt{\kappa}\|\nabla\theta\|_{L_T^\infty H_{co}^{m-1}} + \kappa\|\nabla^2\theta\|_{L_T^\infty H_{co}^{m-2}\cap L_T^2 H_{co}^{m-1}} + \kappa^{\frac{3}{2}}\|\nabla^3\theta^\varepsilon\|_{L_T^2 H_{co}^{m-2}},$$

Main steps

Step 3: Uniform estimates for the compressible part of the system.

Control of $\kappa^{\frac{1}{2}} \|(\nabla\sigma, \mathrm{div}\, u)\|_{L_t^\infty H_{co}^{m-2} \cap L_t^2 H_{co}^{m-1}}$.

$$\begin{aligned} -\mathrm{div}\, u &\approx \frac{1}{\gamma}(\varepsilon\partial_t + \varepsilon u \cdot \nabla)\sigma + \kappa\frac{\gamma-1}{\gamma}\Delta\theta, \\ -\nabla\sigma &\approx \theta\varepsilon\partial_t u + \varepsilon(u\cdot\nabla u - \mu \mathrm{div}\, \mathcal{L}u). \end{aligned}$$

Step 4: Uniform estimates for the incompressible part of the velocity. Define

$$r_0 = \beta(\theta)\exp(\varepsilon R\tilde{\sigma}/C_v\gamma) \quad \text{with } \tilde{\sigma} = \sigma - \varepsilon\mu(2\lambda_1 + \lambda_2)\mathrm{div}\, u$$

and denote

$$v = \mathbb{P}(r_0 u), \quad \nabla\Psi = \mathbb{Q}(r_0 u)$$

the incompressible and compressible part of the modified velocity respectively.

The incompressible part v solves:

$$\begin{cases} (\partial_t + u\cdot\nabla)v - \lambda_1\mu\Delta v + \nabla q = G_{r_0} - [\mathbb{P}, u\cdot\nabla](r_0 u) \quad \text{in} \quad \Omega \\ v\cdot \mathrm{n} = 0, \quad \Pi(\partial_{\mathrm{n}} v) = \Pi\big(-2ar_0 u + (D\mathrm{n})(\nabla\Psi + r_0 u)\big) + \frac{R\varepsilon}{C_v\gamma} r_0 \partial_{\mathrm{n}}\tilde{\sigma}\Pi u \quad \text{on} \quad \partial\Omega \end{cases}$$

where

$$\nabla q = -\mathbb{Q}(G_{r_0}) \approx -\mathbb{Q}\Big(u(\partial_t + u\cdot\nabla)r_0 + \mu\lambda_1\overline{\Gamma} r_0^{-1}[\mathrm{curl}\,\mathrm{curl}, r_0]u\Big)$$

Step 4.1: Control of $\|v\|_{L_t^\infty H_{co}^{m-1}}$ and $\mu^{\frac{1}{2}}\|\nabla v\|_{L_t^2 H_{co}^{m-1}}$.
Step 4.2: Control of $\|\nabla v\|_{L_t^\infty H_{co}^{m-2}}$.

We reduce the matter to the estimate of $\mathrm{curl}(r_0 u)\times \mathrm{n}$ where we use the lift the boundary and control the reminder.

Related study

Inviscid incompressible limit for CNS with Dirichlet boundary condition?

Thanks for your attention!