Nonnegative martingale solutions to the stochastic thin-film equation with nonlinear gradient noise

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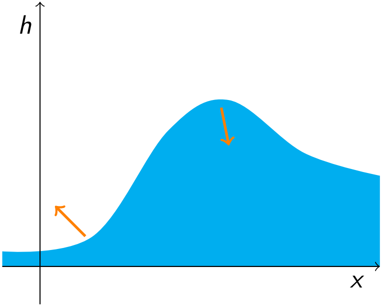
Stochastic Thin-film equation (STFE)

Stochastic
nonlinear
degenerate-parabolic
fourth-order
evolution equation for the
film height $h(t,x) \geq 0$:

$$\eta \, \mathrm{d}h = -\gamma \lambda^{3-n} \partial_x \left(h^n \partial_x^3 h \right) \mathrm{d}t + T \partial_x (h^{\frac{n}{2}} \circ \mathrm{d}W)$$

in $\{h > 0\}$, with mobility exponent $n \in [1, 3]$, $\lambda > 0$ slip length.



2D droplet spreading

Dynamics of thin films in a lubrication approximation

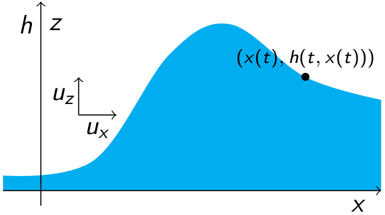
- driven by
  - surface tension γ and
  - thermal fluctuations (temperature T),
- limited by viscosity η.

Modelling:

- Davidovitch & Moro & Stone PRL '05
- Grün & Mecke & Rauscher JSP '06

Lubrication approximation

Lubrication approximation assumes/uses viscous and thin films:



- Transport equation for the film height: particle trajectory $x(t)$,
  $u_z(t, x(t))$
  $= \frac{\mathrm{d}}{\mathrm{d}t} h(t, x(t))$
  $= \partial_t h + \dot{x}(t) \circ \partial_x h(t, x(t))$.
  $\Rightarrow \ \partial_t h = u_z - u_x \circ \partial_x h$.

- Reduced (Navier-)Stokes system (small h and Reynolds number):
  - bulk equations: $0 = -\partial_x p + \eta \partial_z^2 u_x + \partial_z S_{zx}$ and $0 = \partial_z p$ (S denoting stochastic thermal stresses and p the pressure),
  - upper boundary: $p = -\gamma \partial_x^2 h$ (Laplace's law) and $\partial_z u_x + S_{zx} = 0$,
  - lower boundary: $u_z = 0$ and $u_x = \lambda^{3-n} h^{n-1} \partial_z u_x$ (Navier slip).
- Integrating equations using incompressibility leads to

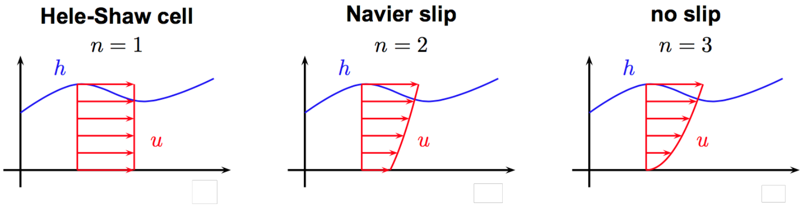
$$\mathrm{d}h = -\frac{\gamma}{3\eta} \partial_x \left((h^3 + \lambda^{3-n} h^n) \partial_x^3 h \right) \mathrm{d}t + \partial_x \left(\sqrt{h^3 + \lambda^{3-n} h^n} \circ \mathrm{d}W \right)$$

  with $\mathbb{E} W = 0$ and $\mathbb{E}\left[W(t, x) W(t', x') \right] = 2T \delta(t - t') \delta(x - x')$.
- Deterministic rigorous lubrication approximation:
  - Giacomelli & Otto IFB '02 ($n = 1$),
  - Knüpfer & Masmoudi CMP '13, ARMA '15 ($n = 1$).
  - $h \geq \mathrm{const.} > 0$: Matioc & Prokert IFB '12, Günther & Prokert JDE '08 ($n = 1, 3$)

Slippage models: Slip length λ

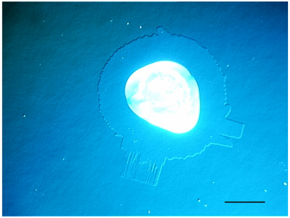
$$\mathrm{d}h = -\partial_x \left((h^3 + \lambda^{3-n} h^n) \partial_x^3 h \right) \mathrm{d}t + \partial_x \left(\sqrt{h^3 + \lambda^{3-n} h^n} \circ \mathrm{d}W \right)$$

$\lambda > 0 \quad \Leftrightarrow \quad$ various slip conditions depending on $n \in [1, 3)$



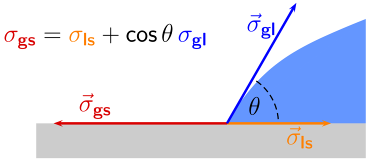
- No-slip paradox: Huh & Scriven (JCIS 1971)
- Navier slip: Jäger & Mikelic JDE '01
- alternative: precursor film

http://www.nano-lane.com/applications /materials/thin-film/



Contact angle

- $\partial_x h = \tan\theta$ on $\partial\{h>0\}$: Young's law
  - $\theta = 0 \Leftrightarrow$ complete wetting,
  - $\theta > 0 \Leftrightarrow$ partial wetting



(microscopic contact-angle)

- Microscopic versus dynamic apparent contact angle (Voinov JAMTP '77, Tanner JPD '79, Cox JFM '86, Hocking JFM '92): Giacomelli & G. & Otto NL '16, G. & Wisse NL '22, Giacomelli & Otto '22+

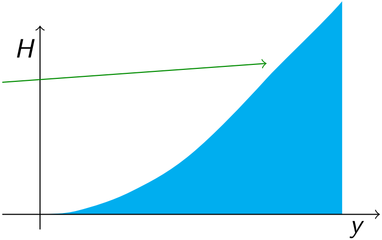
$$H(y) = h(t,x) \text{ with } y = x + Vt$$

$$\left(\frac{\mathrm{d}H}{\mathrm{d}y}\right)^3 = 3V \ln\left(B(3V)^{\frac{1}{3}}\lambda^{-1}y\right)$$

$$\text{for } y \gg V^{-\frac{1}{3}}\lambda$$

$$B = B(n,\theta) \text{ is } C^1$$

$$\frac{\mathrm{d}H}{\mathrm{d}y}\Big|_{y=0} = \theta$$

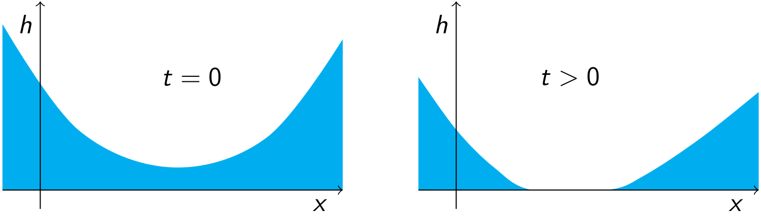


Deterministic Thin-Film Equation (TFE)

- weak solutions to (TFE): Bernis & Friedman JDE '90; Beretta & Bertsch & Dal Passo ARMA '95; Bertozzi & Pugh CPAM '96: weak (entropy-weak) solutions
- Many follow-up works covering global existence in any dimension and qualitative properties.
- Degenerate parabolic: Also true for the porous-medium equation:

$$\text{(PME)} \quad \rho_t - (\rho^m)_{xx} = 0 \quad \text{on } \{\rho > 0\}.$$

  Uniqueness for (PME): Bénilan, Crandall, Pierre IUMJ '82.
- but: in general no uniqueness of weak solutions to (TFE),
- deterministic (TFE) is higher order: no comparison principle



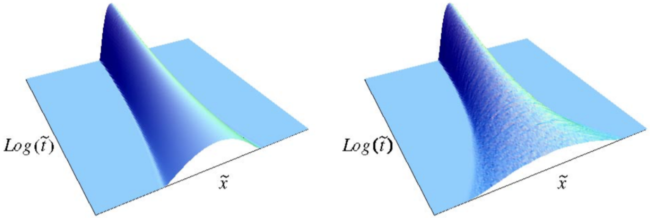
- Uniqueness and classical solutions for deterministic (TFE): starting with Bringmann & Giacomelli & Knüpfer & Otto '08, '16, ...

Stochastic thin-film equation

Thermal noise is important for very thin films: infinite-dimensional Langevin equation

$$\mathrm{d}h = -\partial_x\left(|h|^n\partial_x^3 h\right)\mathrm{d}t + \underbrace{\partial_x\left(|h|^{\frac{n}{2}}\circ \mathrm{d}W\right)}_{\text{thermal fluctuations}} \quad \text{on} \quad ([0,\infty)\times\mathbb{T})\cap\{h>0\}$$

- Davidovitch & Moro & Stone PRL '05 with Stratonovich noise



taken from Davidovitch & Moro & Stone PRL '05

spreading $\sim t^{\frac{1}{7}}$ (deterministic) versus $\sim t^{\frac{1}{4}}$ (stochastic), where $n = 3$.

- Grün & Mecke & Rauscher JSP '06, Fischer & Grün SIMA '18 ($n = 2$), Cornalba '18 with Itô noise and interface potential Φ:

$$\mathrm{d}h = -\partial_x\left(h^n\partial_x\left(\partial_x^2 h - \Phi'(h)\right)\right)\mathrm{d}t + \partial_x\left(h^{\frac{n}{2}}\circ \mathrm{d}W\right) \quad \text{on} \quad \{h>0\},$$

where e.g. $\Phi(h)\sim h^{-p}$ (or attractive-repulsive: Lennard-Jones).

Stochastic thin-film equation II

- (STFE) $\quad \mathrm{d}h = -\partial_x\left(|h|^n\partial_x^3 h\right)\mathrm{d}t + \partial_x\left(|h|^{\frac{n}{2}}\circ \mathrm{d}W\right)$ on $\{h>0\}$.
- colored noise $W=\sum_{k\in\mathbb{Z}}\sigma_k\beta^k$ with
  - mutually independent standard real-valued Wiener processes $(\beta^k)_{k\in\mathbb{Z}}$,
  - $\sum_{k\in\mathbb{Z}}\|\sigma_k\|^2_{W^{2,\infty}(\mathbb{T})}<\infty$.
- in Itô calculus: $\partial_x(|h|^{n/2}\circ \mathrm{d}W)=\sum_{k\in\mathbb{Z}}\partial_x(|h|^{n/2}\sigma_k)\circ \mathrm{d}\beta^k$

$$\begin{aligned}\mathrm{d}h \quad = \quad & -\partial_x\left(|h|^n\partial_x^3 h\right)\mathrm{d}t\\ & +\frac{1}{2}\sum_{k\in\mathbb{Z}}\partial_x\left(\sigma_k(|h|^{n/2})'\partial_x(\sigma_k|h|^{n/2})\right)\mathrm{d}t+\sum_{k\in\mathbb{Z}}\partial_x(\sigma_k|h|^{n/2})\mathrm{d}\beta^k\end{aligned}$$

- initial data h_0: $\|h_0\|_{H^1(\mathbb{T})}<\infty$ (finite surface energy), $h_0\geq 0$.
- Entropy $G_0(h):=\frac{h^{2-n}}{(2-n)(1-n)}$ for $h>0$ and $G_0(h):=\infty$ for $h\leq 0$.

Theorem (Gess & G. SPA '20)

For $n=2$ $\exists$ stochastic basis ("ensemble") and "weak" solution h to (STFE) *s.t. $h\geq 0$ a.s.*

Theorem (Dareiotis & Gess & G. & Grün ARMA '21)

For $n\in[8/3,4)$, $\mathbb{E}\left\|G_0(u^{(0)})\right\|^p_{L^1(\mathbb{T})}<\infty$ with $p>n+2$, $\exists$ stochastic basis and "weak" solution h to (STFE) *s.t. $h\geq 0$ a.s.*

Proof strategy

- For GG '20: On time interval $[0, T)$, take $N \in \mathbb{N}_0$, define $\delta := \frac{T}{N+1}$, and solve the Trotter scheme:
  - deterministic dynamics: $\mathrm{d}v_N = -\partial_x(v_N^2 \partial_x^3 v_N)\mathrm{d}t$ for $(t, x) \in [(j-1)\delta, j\delta) \times \mathbb{T}$ using Beretta, Bertsch, Dal Passo '95,
  - stochastic dynamics: $\mathrm{d}w_N = \partial_x(w_N \circ \mathrm{d}W)$ for $(t, x) \in [(j-1)\delta, j\delta) \times \mathbb{T}$ using variational (monotone operator) approach,
  - initial/jump conditions:
    - $v_N(0, x) := h_0(x)$,
    - $w_N((j-1)\delta, x) = \lim_{t \nearrow j\delta} v_N(t, x)$, and
    - $v_N(j\delta, x) = \lim_{t \nearrow j\delta} w_N(t, x)$.
  - concatenation:

$$h_N(t, x) := \begin{cases} v_N(2t - (j-1)\delta, x) & \text{for } t \in [(j-1)\delta, (j-\frac{1}{2})\delta), \\ w_N(2t - j\delta, x) & \text{for } t \in [(j-\frac{1}{2})\delta, j\delta). \end{cases}$$

  - Compactness argument as $N \to \infty$ using the energy estimate.
- For DG[3] '21: three-step approximation:
  - Galerkin approximation + regularization of the mobility $h^n \mapsto (h^2 + \varepsilon^2)^{\frac{n}{2}}$ + cut off of $\|h\|_{L^\infty(\mathbb{T})}$.
  - Compactness in the Galerkin scheme using the energy estimate.
  - Compactness to remove the regularization $(\varepsilon \searrow 0)$ and cut off $(R \to \infty)$ using the energy and entropy estimate.

A-priori estimates

- SPDE at the ε-R-level:

$$\begin{aligned} \mathrm{d}h &= \partial_x \left(-F_\varepsilon^2(h)\partial_x^3 h\right) \mathrm{d}t \\ &\quad + \frac{1}{2}\sum_k \gamma_R \partial_x \left(\sigma_k F_\varepsilon'(h)\partial_x(\sigma_k F_\varepsilon(h))\right) \mathrm{d}t \\ &\quad + \sum_k \gamma_R \partial_x \left(\sigma_k F_\varepsilon(h)\right) \mathrm{d}\beta^k, \end{aligned}$$

  where $F_\varepsilon(h) := (h^2+\varepsilon^2)^{n/4}$, $\gamma_R := g_R\left(\|h\|_{L^\infty(\mathbb{T})}\right)$, $g_R(s) := g(s/R)$ with g smooth such that $g|_{[0,1]} = 1$ and $g|_{[2,\infty)} = 0$.
- Regularized entropy: $G_\varepsilon(h) := \int_h^\infty \int_{r_2}^\infty \frac{\mathrm{d}r_2\,\mathrm{d}r_2}{F_\varepsilon^2(r_2)}$.
- Entropy estimate: For any $T \in (0,\infty)$, $p \geq 1$, ε-R-independent estimate

$$\begin{aligned} &\mathbb{E}\left(\sup_{t\in[0,T]} \|G_\varepsilon(h(t))\|_{L^1(\mathbb{T})}^p + \left\|\partial_x^2 h\right\|_{L^2([0,T]\times\mathbb{T})}^{2p}\right) \\ &\lesssim \mathbb{E}\left(1 + \left|\overline{h^{(0)}}\right|^{2p} + \left\|G_\varepsilon(h^{(0)})\right\|_{L^1(\mathbb{T})}^p\right). \end{aligned}$$

A-priori estimates II: Proof sketch of entropy estimate

- Apply Itô formula (Pardoux 1975):

$$\int_{\mathbb{T}} G_\varepsilon(h(t))\,\mathrm{d}x = \int_{\mathbb{T}} G_\varepsilon(h^{(0)})\,\mathrm{d}x + \int_0^t \int_{\mathbb{T}} G_\varepsilon''(h) F_\varepsilon^2(h)(\partial_x^3 h)(\partial_x h)\,\mathrm{d}x\,\mathrm{d}t'$$
$$+\frac{1}{2}\sum_k \int_0^t \gamma_R^2 \int_{\mathbb{T}} G_\varepsilon'(h)\partial_x\left(\sigma_k F_\varepsilon'(h)\partial_x(\sigma_k F_\varepsilon(h))\right)\mathrm{d}x\,\mathrm{d}t'$$
$$+\frac{1}{2}\sum_k \int_0^t \gamma_R^2 \int_{\mathbb{T}} G_\varepsilon''(h)\left(\partial_x(\sigma_k F_\varepsilon(h))\right)^2\mathrm{d}x\,\mathrm{d}t'$$
$$+\sum_k \int_0^t \gamma_R \int_{\mathbb{T}} G_\varepsilon'(h)\partial_x(\sigma_k F_\varepsilon(h))\,\mathrm{d}x\,\mathrm{d}\beta^k.$$

- Use $G_\varepsilon''(h) = F_\varepsilon^{-2}(h)$. Integration by parts leads to cancellation of $(\partial_x h)^2$-term ("fluctuation-dissipation theorem"):

$$\int_{\mathbb{T}} G_\varepsilon(h(t))\,\mathrm{d}x = \int_{\mathbb{T}} G_\varepsilon(h^{(0)})\,\mathrm{d}x - \int_0^t \int_{\mathbb{T}} (\partial_x^2 h)^2\,\mathrm{d}x\,\mathrm{d}t' + \frac{1}{2}\sum_k \int_0^t \gamma_R^2 \int_{\mathbb{T}} (\partial_x \sigma_k)^2\,\mathrm{d}x\,\mathrm{d}t'$$
$$-\frac{1}{4}\sum_k \int_0^t \gamma_R^2 \int_{\mathbb{T}} (\partial_x^2 \sigma_k^2)\,\ln F_\varepsilon(h)\,\mathrm{d}x\,\mathrm{d}t' - \sum_k \int_0^t \gamma_R^2 \int_{\mathbb{T}} \sigma_k F_\varepsilon^{-1}(h)\,\partial_x h\,\mathrm{d}x\,\mathrm{d}\beta^k.$$

- Use $\sum_k \left((\partial_x \sigma_k)^2 + \left|(\partial_x^2 \sigma_k)\ln F_\varepsilon(h)\right|\right) \le C_{\delta,\sigma}\left(1 + \|G_\varepsilon(h)\|_{L^1(\mathbb{T})}\right) + \delta\,\|h\|_{L^2(\mathbb{T})}$ and $\|h\|_{L^2(\mathbb{T})} \lesssim \|\partial_x^2 h\|_{L^2(\mathbb{T})} + \left|\overline{h^{(0)}}\right|$ (Poincaré + mass conservation).
- Use Burkholder-Davis-Gundy inequality to estimate the martingale.
- Use Grönwall to conclude.

A-priori estimates III

- Energy estimate: For any $T \in (0,\infty)$, $p \geq 1$, ε-R-uniform estimate for any $q > 1$

$$\mathbb{E}\left(\sup_{t\in[0,T]} \|\partial_x h(t)\|^p_{L^2(\mathbb{T})} + \left\|F_\varepsilon(h)\partial_x^3 h\right\|^p_{L^2([0,T]\times\mathbb{T})}\right)$$

$$\lesssim \mathbb{E}\left(1 + \left|\overline{h^{(0)}}\right|^{\frac{3p}{2}} + \left\|\partial_x h^{(0)}\right\|^p_{L^2(\mathbb{T})} + \sup_{t\in[0,T]} \|G_\varepsilon(h(t))\|^p_{L^1(\mathbb{T})} + \left\|\partial_x^2 h\right\|^{2pq}_{L^2([0,T]\times\mathbb{T})}\right)$$

  Absorb remainder using the entropy estimate.

- Regularity in time: Obtain from the equation for $p > 1$, $q > 1$, $\tilde{p} \in [1, 2p)$,

$$\mathbb{E}\,\|h\|^{\tilde{p}}_{C^{1/4}([0,T];L^2(\mathbb{T}))} \lesssim \left[\mathbb{E}\left(1 + \left|\overline{h^{(0)}}\right|^{2(n+2)pq} + \left\|G_0(h^{(0)})\right\|^{(n+2)pq}_{L^1(\mathbb{T})} + \left\|\partial_x h^{(0)}\right\|^{(n+2)p}_{L^2(\mathbb{T})}\right)\right]^{\frac{\tilde{p}}{2p}}.$$

- Interpolation + Sobolev embedding or Fischer & Grün '18: $\mathbb{E}\,\|h\|^{\tilde{p}}_{C^{1/8,1/2}([0,T]\times\mathbb{T})} < \infty$.

Compactness (the degenerate limit $\varepsilon \searrow 0$)

- Compactness leads to tightness of laws h_ε, $J_\varepsilon := F_\varepsilon(h_\varepsilon)\partial_x^3 h_\varepsilon$ (pseudo flux), and W_ε in $C^{1/8-,1/2-}([0,T]\times\mathbb{T})$, $L^2([0,T]\times\mathbb{T})$ with weak topology, and $C^0\left([0,T];H^2(\mathbb{T})\right)$, respectively, follows by compact embeddings using a-priori estimates.
- Applying Skorokhod-Prokhorov argument (Jakubowski '97 in the non-metric case) gives for

$$(h_\varepsilon, J_\varepsilon, W_\varepsilon) \sim (\tilde{h}_\varepsilon, \tilde{J}_\varepsilon, \tilde{W}_\varepsilon)$$

  $\tilde{\mathbb{P}}$-almost surely as $\varepsilon \searrow 0$

$$\begin{aligned} \tilde{h}_\varepsilon &\to \tilde{h} && \text{in } C^{1/8-,1/2-}([0,T]\times\mathbb{T}),\\ \tilde{J}_\varepsilon &\rightharpoonup \tilde{J} && \text{in } L^2([0,T]\times\mathbb{T}),\\ \tilde{W}_\varepsilon &\to \tilde{W} && \text{in } C^0\left([0,T];H^2(\mathbb{T})\right). \end{aligned}$$

- Problem: identify limit $\tilde{J} = \mathbb{1}_{\{\tilde{h}>0\}}\tilde{h}^{n/2}\partial_x^3\tilde{h}$ (some work).
- Passing to the limit in the weak formulation using compactness and $\tilde{\mathbb{P}}$-almost sure convergence as above.

Conclusions/Outlook

- numerical implementation of Trotter scheme,
- solutions with non-full support for nonlinear mobilities (with Dareiotis & Gess & Sauerbrey):
  - use of α-entropies $G_\alpha(u) = \frac{1}{\alpha(\alpha+1)}|u|^{\alpha+1}$ with $\alpha \in (-1, 2-n)$, $n \in (2,3)$ (cf. Beretta & Bertsch & Dal Passo ARMA '95) $\rightsquigarrow$ "very weak" solutions
  - question: finite speed of propagation?
- higher dimensions (see Grün & Metzger '21, Sauerbrey to appear in AIHP B '22+)
- self-similar asymptotics, rougher noise, ...

Outlook II

with Knüpfer & Masmoudi & Sauer: (Navier-)Stokes equations with moving contact point:

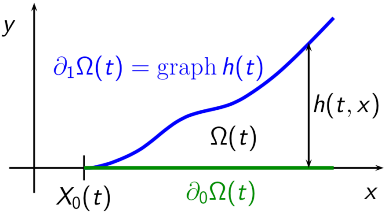
$$\begin{cases} \partial_t U - \nabla \cdot T_{U,P} = 0, \\ \nabla \cdot U = 0 \end{cases} \quad \text{in} \quad \Omega(t),$$

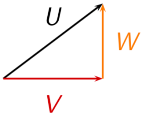
where $T_{U,P} = P\,\mathrm{id} - \nu\left(\nabla U + (\nabla U)^T\right)$ and $U = (V, W)$, subject to

$$W = 0 \quad \text{and} \quad V - \lambda \partial_y V = 0 \quad \text{on} \quad \partial_0\Omega(t),$$

$$\partial_t h = W - V\partial_x h \quad \text{and} \quad T_{U,P} n = -\sigma \partial_x \left(\frac{\partial_x h}{\sqrt{1 + (\partial_x h)^2}} \right) \quad \text{on} \quad \partial_1\Omega(t)$$

Contact point $\partial_0\Omega(t) \cap \partial_1\Omega(t) = \{X_0(t)\}$ meets $\frac{\mathrm{d}}{\mathrm{d}t} X_0(t) = V|_{x = X_0(t)}$.





See also Tice & Zheng SIMA '17, Guo & Tice ARMA '18, Tice & Wu JDE '21, Roodenburg (& Bravin & G.) '22(+)

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