Time-inconsistent mean-field optimal stopping and zero-sum Dynkin games

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Formulation of the problem

- Let $T > 0$ be a finite time horizon. On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ W is a 1-dim Brownian motion and X_0 is r.v. independent of W. $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$, where $\mathcal{F}_t$ is the completion of $\sigma(X_0, W_s, s \leq t)$.

- $\mathcal{T}_t$ is set of $\mathbb{F}$-stopping times with values in $[t, T]$.

- On $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, let X be the mean-field diffusion

$$\begin{cases} X_t = X_0 + \int_0^t b(s, X_s, \mathbb{P}_{X_s})ds + \int_0^t \sigma(s, X_s, \mathbb{P}_{X_s})dW_s, \quad X_0 \in L^2(\mathbb{P}), \\ \mathbb{P}_{X_t} := \text{law}(X_t) \text{ under } \mathbb{P}, \end{cases}$$

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- whose particle approximation is

$$X_t^{i,n} = X_0^i + \int_0^t b(s, X_s^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_s^{j,n}})ds + \int_0^t \sigma(s, X_s^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_s^{j,n}})dW_s^i,$$

where the r.v. X_0^i are i.i.d. with the same law as X_0 and W^i's are independent Brownian processes

We want to solve the following time-inconsistent optimal stopping problems:

- Optimal stopping of a mean-field diffusion:

$$\mathcal{T}_0 \ni \hat{\tau} = \operatorname*{arg\,max}_{\tau \in \mathcal{T}_0} \mathbb{E}\left[h(X_\tau, \mathbb{P}_{X_\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right], \tag{OSPa}$$

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  for a certain $\mathcal{F}_T$-measurable final condition ξ and a payoff function h.
- We consider payoff processes defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ of the form

$$Y_t = \underset{\tau \in \mathcal{T}_t}{\text{ess sup}}\, E\left[h(Y_\tau, \mathbb{P}_{Y_\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}} \,|\mathcal{F}_t\right],$$

  for a certain $\mathcal{F}_T$-measurable final condition ξ and a payoff function h.
  - Optimal stopping of a recursive utility function:

$$\mathcal{T}_0 \ni \tau^* = \underset{\tau \in \mathcal{T}_0}{\arg\max}\, \mathbb{E}\left[h(Y_\tau, \mathbb{P}_{Y_\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right] = Y_0, \tag{OSPb}$$

  for a certain $\mathcal{F}_T$-measurable final condition ξ and a payoff function h.

Later on I will introduce the formulation of the MF-zero-sum Dynkin game.

Motivating example: Optimal stopping of the variance

Find $\hat{\tau} \in \mathcal{T}_0$ such that

$$\hat{\tau} = \underset{\tau \in \mathcal{T}_0}{\arg\max}\, \mathbb{E}\left[\left(X_\tau - \mathbb{E}[X_\tau]\right)^2\right]?$$

- The value-process

$$Y_t := \underset{\tau \in \mathcal{T}_0}{\text{ess sup}}\, \mathbb{E}\left[\left(X_\tau - \mathbb{E}[X_\tau]\right)^2 \mid \mathcal{F}_t\right]$$

  is no longer a supermartingale (time-inconsistent) due to the term $\mathbb{E}[X_t]$.
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  is no longer a supermartingale (time-inconsistent) due to the term $\mathbb{E}[X_t]$.
- We cannot directly apply the techniques based on the Snell envelope of processes to solve the problem.
- Pedersen (2011) solved the OSP of the variance of a Markov diffusion X by embedding it into a standard OSP: Let $X_0 = x$. If $c_*(x)$ is a constant s.t.

$$\begin{cases} \tau^{c_*(x)} = \underset{\tau}{\arg\max}\, \mathbb{E}_x[(X_\tau - c_*(x))^2], \\ c_*(x) = \mathbb{E}_x[X_{\tau^{c_*(x)}}] \quad \text{(Matching condition)}, \end{cases}$$

  then $\tau^{c_*(x)} = \hat{\tau}$.

But there is a large class of MF-OSP whose value processes are supermartingales (i.e. for which the DPP holds (in the Markovian case))

- D, Elie and Hamadène (2019), D, Dumitrescu and Zeng (2021):

$$Y_0 = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[h(Y_\tau, \mathbb{P}_{Y_s}|_{s=\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right],$$

(we ignore the integral term). The mean-field coupling $\mathbb{P}_{Y_s}|_{s=\tau}$ is not the law of the random variable Y_τ. The value process solves a MF-reflected BSDE.

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- Talbi, Touzi and Zhang (2021) (a weak formulation of the OSP): Given $\mu \in \mathcal{P}_2(\mathbb{R}^d)$,

$$Y_0 = \sup_{\mathbb{P}} \int_0^T F(s, \mathbb{P}_{(X_s, I_s)})ds + g(\mathbb{P}_{(X_T, I_T)})$$

  where under $\mathbb{P}$, the 'coordinate process' (X_s, I_s) satisfies

$$X_t = X_0 + \int_0^t b(s, X_s, \mathbb{P}_{(X_s,I_s)})I_s ds + \int_0^t \sigma(s, X_s, \mathbb{P}_{(X_s,I_s)})I_s dW_s^{\mathbb{P}}, \quad I_t = I_{0-} \mathbb{1}_{\{t<\tau\}}$$

  under the constraint $\mathbb{P}_{X_0} = \mu, \ \mathbb{P}(I_{0-} = 1) = 1$.

Optimal stopping via the Snell envelope approach

Consider the 'standard' Optimal stopping problem (OSP)

$$Y_0 = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[L_\tau\right]$$

where the reward process $(L_t)_{t\in[0,T]}$ is adapted, right continuous with left limits (càdlàg) and 'uniformly integrable' (i.e. of class D or DL).

- The Snell envelope of the process L is defined by

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- Y is the smallest supermartingale that dominates L.

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- The Snell envelope of the process L is defined by

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[L_\tau \mid \mathcal{F}_t\right].$$

- Y is the smallest supermartingale that dominates L.
- If L has only nonnegative jumps, then Y is continuous and it is optimal to stop when Y hits L i.e. the hitting time

$$\tau_t^* = \inf\{s \geq t,\ Y_s = L_s\} \wedge T \qquad (1)$$

is optimal after t. In particular, $\tau^* := \tau_0^*$ is optimal for Y_0.

(Bismut and Skalli (1977) and El Karoui (Saint-Flour 1979)).

Optimal stopping for mean-field diffusions: The Main Result

Consider the OSP

$$Y_0 = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[h(X_\tau, \mathbb{P}_{X_\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right],$$

whose value-process is

$$Y_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t} \mathbb{E}\left[h(X_\tau, \mathbb{P}_{X_\tau})\mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}} \,|\mathcal{F}_t\right].$$

Assumption A1

The coefficients ξ and h satisfy

(i) $\xi \in L^2(\mathcal{F}_T)$

(ii) $h\colon \Omega \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ is Lipschitz continuous w.r.t. (y, μ), i.e. there exist two positive constants γ_1 and γ_2 such that

$$|h(\omega, y_1, \mu_1) - h(\omega, y_2, \mu_2)| \leq \gamma_1 |y_1 - y_2| + \gamma_2 \mathcal{W}_2(\mu_1, \mu_2),$$

for any $\omega \in \Omega$, any $y_1, y_2 \in \mathbb{R}^d$ and any $\mu_1, \mu_2 \in \mathcal{P}_2(\mathbb{R})$.

(This condition is merely needed for the OSP of recursive utilities. Weaker conditions suffice.)

Despite time-inconsistency, under these conditions on ξ and h we can prove that

Theorem

the hitting time

$$\tau_t^* = \inf\{s \geq t,\ \ Y_s = h(X_s, \mathbb{P}_{X_s})\} \wedge T$$

is optimal after t. In particular,

$$\tau_0^* = \inf\{s \geq 0,\ \ Y_s = h(X_s, \mathbb{P}_{X_s})\} \wedge T$$

is optimal for our OSP i.e.

$$\tau_0^* = \arg\max_{\tau \in \mathcal{T}_0} \mathbb{E}\left[h(X_\tau, \mathbb{P}_{X_\tau})1\!\!1_{\{\tau < T\}} + \xi 1\!\!1_{\{\tau = T\}}\right].$$

Comparison with randomized optimal stopping times

By embedding this class of OSPs into the ones w.r.t. the set of randomized stopping times which is compact in the Baxter-Chacon topology (see Baxter and Chacon (1977)), following many papers including Edgar, Millet and Sucheston (1981), Arenas (1990), El Karoui, Lepeltier and Millet (1991) and Touzi and Vieille (2002), one can show that there exists an optimal randomized stopping time for Y_0, without further characterization compared to the explicit optimal stopping times τ_0 or τ_0^*.

Outline of the proof: A limit approach

We consider the system of n weakly interacting diffusions defined by

$$X_t^{i,n} = X_0^i + \int_0^t b(s, X_s^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_s^{j,n}})ds + \int_0^t \sigma(s, X_s^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_s^{j,n}})dW_s^i,$$

to which we associate the family of value-processes of 'standard' OSPs

$$Y_t^{i,n} = \operatorname*{ess\,sup}_{\tau \in \mathcal{T}_t^i} \mathbb{E}\left[h(X_\tau^{i,n}, \frac{1}{n}\sum_{j=1}^n \delta_{X_\tau^{j,n}})\mathbb{1}_{\{\tau<T\}} + \xi^i \mathbb{1}_{\{\tau=T\}} \,|\mathcal{F}_t^i \right], \quad i = 1, 2, \ldots, n,$$

- $(X_0^1, W^1) = (X_0, W)$ and $\{(X_0^i, W^i)\}_{i\geq 1}$ are i.i.d.
- $\mathbb{F}^i := \{\mathcal{F}_t^i\}$ is the completion of the filtration generated by (X_0^i, W^i).
- $\mathcal{T}_t^i$ is the set of $\mathbb{F}^i$ stopping times with values in $[t, T]$.
- $\xi^1 = \xi$ and $\{\xi^i\}_{1\leq i\leq n}$ are i.i.d. and each $\xi^i \in L^2(\mathcal{F}_T^i)$.

Provided that b and σ satisfy standard linear growth and Lipschitz continuity conditions we have (see e.g. Theorem 3 in Jourdain *et al.* (2008)),

- For each $i \geq 1$, $\sup_{t\in[0,T]} |X^i|$ is square-integrable,
- For each $i = 1, \ldots, n$, $\sup_{t\in[0,T]} |X^{i,n}|$ is square-integrable,
- $\lim_{n\to\infty} \sup_{i\leq n} \mathbb{E}\left[\sup_{t\in[0,T]} |X_t^{i,n} - X_t^i|^2\right] = 0,$
- By the standard 'Snell envelope approach', for each $i = 1, \ldots, n$, the hitting time

$$\hat{\tau}_t^{i,n} = \inf\{s \geq t;\ Y_s^{i,n} = \mathbb{E}[h(X_s^{i,n}, \frac{1}{n}\sum_{j=1}^{n} \delta_{X_s^{j,n}}) \,|\mathcal{F}_s^i]\} \wedge T$$

  is optimal for $Y_t^{i,n}$.

We need to show that

- $\hat{\tau}^{1,n}$ converges to τ_0^* as $n \to \infty$ in some sense, where

$$\tau_0^* := \inf\{s \geq 0,\ Y_s = h(X_s, \mathbb{P}_{X_s})\} \wedge T$$

- the optimality of each $\hat{\tau}^{1,n}$ implies optimality of τ_0^*.

We have

Proposition

- $$\lim_{n\to\infty}\sup_{i\leq n}\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t^{i,n}-Y_t^i|^2\right]=0.$$

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- $$\lim_{n\to\infty} \sup_{i\leq n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2\right] = 0.$$
- $\hat{\tau}^{1,n}$ *converges to* τ_0^* *in probability, as* n *goes to infinity.*

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Proposition

- $$\lim_{n\to\infty} \sup_{i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - Y_t^i|^2\right] = 0.$$
- *$\hat{\tau}^{1,n}$ converges to τ_0^* in probability, as n goes to infinity.*
- *We have*
$$\lim_{n\to\infty} \mathbb{E}[Y_{\hat{\tau}^{1,n}}] = \mathbb{E}[Y_{\tau_0^*}].$$
- *Up to a subsequence, it holds that*
$$\lim_{n\to\infty} \mathbb{E}\left[h(X^{1,n}_{\hat{\tau}^{1,n}}, \tfrac{1}{n}\textstyle\sum_{j=1}^n \delta_{X^{j,n}_{\hat{\tau}^{j,n}}})\mathbb{1}_{\{\hat{\tau}^{1,n}<T\}} + \xi\mathbb{1}_{\{\hat{\tau}^{1,n}=T\}}\right] = \mathbb{E}\left[h(X_{\tau_0^*}, \mathbb{P}_{X_{\tau_0^*}}])\mathbb{1}_{\{\tau_0^*<T\}} + \xi\mathbb{1}_{\{\tau_0^*=T\}}\right].$$

These results yield optimality of the hitting time τ_0^*.

The same approach goes through for the recursive utility problem:

$$\mathcal{T}_0 \ni \tau^* = \underset{\tau \in \mathcal{T}_0}{\arg\max}\, \mathbb{E}\left[h(Y_\tau, \mathbb{P}_{Y_\tau})\mathbb{1}_{\{\tau < T\}} + \xi \mathbb{1}_{\{\tau = T\}}\right] \qquad \text{(OSPb)}$$

provided that

$$\gamma_1^2 + \gamma_2^2 < \frac{1}{16}.$$

This condition is needed in order to apply a fixed point argument.

Optimal stopping of the variance of a mean-field diffusion

For $h(\omega, x, \mu) := (x - \int_{\mathbb{R}^d} y\mu(dy))^2$ and $\xi \geq 0$, we obtain an OSP of the variance:

$$Y_0 = \sup_{\tau \in \mathcal{T}_0} \mathbb{E}\left[(X_\tau - \mathbb{E}[X_\tau])^2 \mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right].$$

Although h is not Lipschitz continuous we can still claim that the hitting time

$$\tau^* = \inf\{s \geq 0;\ \ Y_s = (X_s - \mathbb{E}[X_s])^2\} \wedge T$$

satisfies

$$\tau^* = \arg\max_{\tau \in \mathcal{T}_0} \mathbb{E}\left[(X_\tau - \mathbb{E}[X_\tau])^2 \mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}}\right].$$

We only need to show that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{1,n} - Y_t|\right] = 0$$

where (note that $\mathcal{F}_t^1 = \mathcal{F}_t$)

$$Y_t^{1,n} = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t^1} \mathbb{E}\left[(X_\tau^{1,n} - \tfrac{1}{n}\textstyle\sum_{j=1}^n X_\tau^{j,n})^2 \mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}} \,|\, \mathcal{F}_t\right],$$

$$Y_t = \operatorname*{ess\,sup}_{\tau\in\mathcal{T}_t} \mathbb{E}\left[(X_\tau - \mathbb{E}[X_\tau])^2 \mathbb{1}_{\{\tau<T\}} + \xi \mathbb{1}_{\{\tau=T\}} \,|\, \mathcal{F}_t\right],$$

which follows from the fact that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\in[0,T]} \left|\mathbb{E}[(X_t^{1,n} - \frac{1}{n}\sum_{j=1}^{n} X_t^{j,n})^2 \,|\, \mathcal{F}_t] - (X_t - \mathbb{E}[X_t])^2\right|\right] = 0.$$

whose derivation uses the strong law of large numbers for i.i.d. $C([0,T];\mathbb{R})$-valued random variables with finite second moments (i.e. in $\mathcal{S}^2$).

MF-Dynkin zero-sum games for recursive utilities

The payoff process associated with the game is

$$\mathcal{J}(\tau,\sigma,V) := h_1(X_\tau,\mathbb{P}_{X_\tau},V_\tau,\mathbb{P}_{V_\tau})\mathbb{1}_{\{\tau\le\sigma<T\}} \\ +h_2(X_\sigma,\mathbb{P}_{X_\sigma},V_\sigma,\mathbb{P}_{V_\sigma})\mathbb{1}_{\{\sigma<\tau\}}+\xi\mathbb{1}_{\{\tau\wedge\sigma=T\}}.$$

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Assumption A2

(i) $h_1(y,\mu)\le h_2(t,y,\mu)$ for all $(y,\mu)\in(\mathbb{R}\times\mathcal{P}_2(\mathbb{R}))^{\otimes 2}$.

(ii) $\xi\in L^2(\mathcal{F}_T)$ and

$$h_1(X_T,\mathbb{P}_{X_T},\xi,\mathbb{P}_\xi)\le\xi\le h_2(X_T,\mathbb{P}_{X_T},\xi,\mathbb{P}_\xi)\quad \mathbb{P}\text{-a.s.}$$

(iii) For each $i=1,2$, $h_i\colon(\mathbb{R}\times\mathcal{P}_2(\mathbb{R}))^{\otimes 2}\to\mathbb{R}$ is Lipschitz continuous w.r.t. $(y:=(x_1,x_2),\mu:=(\nu_1,\nu_2))$, i.e. there exist two positive constants $(\alpha_i,\beta_i,\gamma_i$ and κ_i such that

$$|h_i(y,\mu)-h_i(\overline{y},\overline{\mu})|\le\alpha_i|x_1-\overline{x}_1|+\gamma_i|x_2-\overline{x}_2| +\beta_i\mathcal{W}_2(\nu_1,\overline{\nu}_1)+\kappa_i\mathcal{W}_2(\nu_2,\overline{\nu}_2),$$

for any $y,\overline{y}\in\mathbb{R}^2$ and any $\mu,\overline{\mu}\in(\mathcal{P}_2(\mathbb{R}))^{\otimes 2}$.

(iv) $\gamma_1^2+\gamma_2^2+\kappa_1^2+\kappa_2^2<2^{-2}$.

The upper and lower values of the game

The upper and lower values of the game (if they exist) are given by the following recursive utilities.

$$\overline{V}_t := \operatorname*{ess\,inf}_{\tau \geq t} \operatorname*{ess\,sup}_{\sigma \geq t} \mathbb{E}[\mathcal{J}(\tau, \sigma, \overline{V}) | \mathcal{F}_t], \quad \underline{V}_t := \operatorname*{ess\,sup}_{\sigma \geq t} \operatorname*{ess\,inf}_{\tau \geq t} \mathbb{E}[\mathcal{J}(\tau, \sigma, \underline{V}) | \mathcal{F}_t].$$

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Proposition

Under Assumption 2, there is a unique $\mathcal{S}_c^2$-valued upper value $\overline{V}_.$ and a unique $\mathcal{S}_c^2$-valued lower value $\underline{V}_.$ of the game.

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Definition (*Value of the game and the notion of S-saddle point*)

The MF-zero-sum Dynkin game has a value if

$$\underline{V}_. = \overline{V}_. \qquad \mathbb{P}\text{-a.s.}$$

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Proposition

Under Assumption 2, there is a unique $\mathcal{S}^2_c$-valued upper value $\overline{V}_.$ and a unique $\mathcal{S}^2_c$-valued lower value $\underline{V}_.$ of the game.

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Let $S \in \mathcal{T}_0$. A pair $(\tau^\star, \sigma^\star) \in (\mathcal{T}_S)^2$ is called an S-saddle point for the *mean-field Dynkin game problem* if for each $(\tau, \sigma) \in (\mathcal{T}_S)^2$ we have

$$\mathbb{E}[\mathcal{J}(\tau^\star, \sigma, V)|\mathcal{F}_S] \leq \mathbb{E}[\mathcal{J}(\tau^\star, \sigma^\star, V)|\mathcal{F}_S] \leq \mathbb{E}[\mathcal{J}(\tau, \sigma^\star, V)|\mathcal{F}_S].$$

The MF-Zero-sum Dynkin games has a value and an S-saddle-point

Let $L_n[\mathbf{x}] := \frac{1}{n}\sum_{i=1}^n \delta_{x_i}$ denote the empirical measure associated with the vector $\mathbf{x} := (x_1, x_2, \ldots, x_n)$ in $\mathbb{R}^n$.

Assumption A3

- For each $n \geq 1$, $\xi^n := (\xi^{1,n}, \xi^{2,n}, \ldots, \xi^{n,n}) \in L^2(\mathcal{F}^n_T)$.
- For each $n \geq 1$ and $i = 1, \ldots, n$,

$$h_1(X^{i,n}_T, L_n[\mathbf{X}^n_T], \xi^{i,n}, L_n[\xi^n]) \leq \xi^{i,n} \leq h_2(X^{i,n}_T, L_n[\mathbf{X}^n_T], \xi^{i,n}, L_n[\xi^n]) \quad \mathbb{P}\text{-a.s.}$$

- Mokobodzki's condition: For each n, there exits an $\mathbb{F}^n$-adapted quasimartingale $S^{i,n} \in \mathcal{S}^2_c$ such that

$$h_1(X^{i,n}_t, L_n[\mathbf{X}^n_t], y, \mu) \leq S^{i,n}_t \leq h_2(X^{i,n}_t, L_n[\mathbf{X}^n_t], y, \mu), \quad \mathbb{P}\text{-a.s.}$$

  for any $(t, y, \mu) \in [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R})$.

  (Any Mokobodzki's type condition which yields the last inequality is ok)

Assumption A4

- $\lim_{n\to\infty} \sup_{i\le n} \mathbb{E}\left[|\xi^{i,n} - \xi^i|^2\right] = 0$, where $\xi^1 = \xi$ and $\{\xi^i\}_{1\le i\le n}$ are i.i.d., each $\xi^i \in L^2(\mathcal{F}^i_T)$.

Theorem (*Existence of a value and a saddle-point of the game*)

Suppose that Assumptions A2, A3 and A4 is in force. Then,

(i) *the time inconsistent mean-field Dynkin game admits a value* $V = \overline{V} = \underline{V} \in \mathcal{S}^2_c$,

(ii) *For each* $S \in \mathcal{T}_0$, *the pair of stopping times* $(\tau^\star_S, \sigma^\star_S)$ *defined by*

$$\tau^\star_S := \inf\{t \ge S : \ V_t = h_1(X_t, \mathbb{P}_{X_t}, V_t, \mathbb{P}_{V_t})\} \wedge T,$$
$$\sigma^\star_S := \inf\{t \ge S : \ V_t = h_2(X_t, \mathbb{P}_{X_t}, V_t, \mathbb{P}_{V_t})\} \wedge T,$$

is an S*-saddle point for the game.*

Method of the proof: A limit approach

To a process $\mathbf{Y}^n := (Y^{1,n}, Y^{2,n}, \dots, Y^{n,n})$ and the $\mathcal{F}_T^n$-measurable r.v. $\xi^n := (\xi^{1,n}, \xi^{2,n}, \dots, \xi^{n,n})$, we associate the payoff process

$$\begin{aligned}\mathcal{J}^{i,n}(\tau,\sigma,\mathbf{Y}^n) \quad &:= h_1(X_\tau^{i,n}, L_n[\mathbf{X}_\tau^n], Y_\tau^{i,n}, L_n[\mathbf{Y}_\tau^n])\mathbb{1}_{\{\tau\le\sigma<T\}} \\ &\quad + h_2(X_\sigma^{i,n}, L_n[\mathbf{X}_\sigma^n], Y_\sigma^{i,n}, L_n[\mathbf{Y}_\sigma^n])\mathbb{1}_{\{\sigma<\tau\}} \\ &\quad + \xi^{i,n}\mathbb{1}_{\{\tau\wedge\sigma=T\}}.\end{aligned}$$

The upper and lower values of the interacting Dynkin zero-sum game (if they exist) are given by the following recursive utilities.

$$\overline{V}_t^{i,n} := \operatorname*{ess\,inf}_{\tau\ge t}\operatorname*{ess\,sup}_{\sigma\ge t}\mathbb{E}[\mathcal{J}^{i,n}(\tau,\sigma,\overline{V}^n)|\mathcal{F}_t^i],$$

$$\underline{V}_t^{i,n} := \operatorname*{ess\,sup}_{\sigma\ge t}\operatorname*{ess\,inf}_{\tau\ge t}\mathbb{E}[\mathcal{J}^{i,n}(\tau,\sigma,\underline{V}^n)|\mathcal{F}_t^i].$$

A system of weakly interacting doubly reflected BSDEs

Consider the system ($t \in [0, T]$)

$$\begin{cases} Y_t^{i,n} = \mathbb{E}[\xi^{i,n} \,|\, \mathcal{F}_T^i] + K_T^{1,i,n} - K_t^{1,i,n} + K_t^{2,i,n} - K_T^{2,i,n} - \int_t^T Z_s^{i,n} dW_s^i, \\ \mathbb{E}[h_2(X_t^{i,n}, L_n[\mathbf{X}_t^n], Y_t^{i,n}, L_n[\mathbf{Y}_t^n]) \,|\mathcal{F}_t^i] \geq Y_t^{i,n} \geq \mathbb{E}[h_1(X_t^{i,n}, L_n[\mathbf{X}_t^n], Y_t^{i,n}, L_n[\mathbf{Y}_t^n]) \,|\mathcal{F}_t^i], \\ \int_0^T (Y_t^{i,n} - \mathbb{E}[h_1(X_t^{i,n}, L_n[\mathbf{X}_t^n], Y_t^{i,n}, L_n[\mathbf{Y}_t^n]) \,|\mathcal{F}_t^i]) dK_t^{1,i,n} = 0, \\ \int_0^T (Y_t^{i,n} - \mathbb{E}[h_2(X_t^{i,n}, L_n[\mathbf{X}_t^n], Y_t^{i,n}, L_n[\mathbf{Y}_t^n] \,|\mathcal{F}_t^i]) dK_t^{2,i,n} = 0, \\ dK_t^{1,i,n} \perp dK_t^{2,i,n}, \end{cases}$$

where the last condition $dK_t^{1,i,n} \perp dK_t^{2,i,n}$ (mutual singularity) means that there exists $D \in \mathcal{P}$ such that

$$\mathbb{E}\left[\int_0^T 1_{D^c} dK_t^{1,i,n}\right] = \mathbb{E}\left[\int_0^T 1_D dK_t^{2,i,n}\right] = 0.$$

This condition is imposed in order to ensure the uniqueness of the solution.

By Theorem 3.1 in D. and Dumitrescu (2022) we have

Theorem (*Existence of a value and link with interacting system of doubly reflected BSDEs*)

Suppose that Assumptions A2 and A3 are in force. Then,

(i) *the interacting system of doubly reflected BSDEs has an unique solution* $(\mathbf{Y}^n, \mathbf{Z}^n, \mathbf{K}^{1,n}, \mathbf{K}^{2,n})$ *in* $\mathcal{S}^{2,\otimes n} \otimes \mathcal{H}^{2,n\otimes n} \otimes \mathcal{S}_{c,i}^{2,\otimes n} \otimes \mathcal{S}_{c,i}^{2,\otimes n}$,

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(ii) *We have* $\underline{V}_{\cdot}^{i,n} = Y_{\cdot}^{i,n} = \overline{V}_{\cdot}^{i,n}$, *for all* $1 \leq i \leq n$. *i.e. the interacting Dynkin game admits a value* $\mathbf{V}^n \in \mathcal{S}_c^{2,\otimes n}$

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(iii) *For each* $S \in \mathcal{T}_0^i$, *consider the pair of stopping times* $(\tau_S^{\star,i,n}, \sigma_S^{\star,i,n})$ *defined by*

$$\begin{cases} \tau_S^{\star,i,n} := \inf\{t \geq S : \ V_t^{i,n} = \mathbb{E}[h_1(X_t^{i,n}, L_n[\mathbf{X}_t^n], V_t^{i,n}, L_n[\mathbf{V}_t^n]) \,|\mathcal{F}_t^i]\} \wedge T, \\ \sigma_S^{\star,i,n} := \inf\{t \geq S : \ V_t^{i,n} = \mathbb{E}[h_2(X_t^{i,n}, L_n[\mathbf{X}_t^n], V_t^{i,n}, L_n[\mathbf{V}_t^n] \,|\mathcal{F}_t^i]\} \wedge T. \end{cases}$$

is an S*-saddle point for each of the values* $V^{i,n}$.

Final step of the proof

Proposition

Assume A2, A3 and A4 are in force. We have

- $$\lim_{n\to\infty} \sup_{i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - \overline{V}_t^i|^2\right] = \lim_{n\to\infty} \sup_{i\le n} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{i,n} - \underline{V}_t^i|^2\right] = 0.$$

  i.e. $\overline{V}^i = \underline{V}^i := V^i \quad \mathbb{P}\text{-a.s.}$ *, where* $\{V^i\}_{i\ge 1}$ *are i.i.d. copies of the value* $V = V^1$ *of the game:*

  $$V_t := \operatorname*{ess\,inf}_{\tau\ge t} \operatorname*{ess\,sup}_{\sigma\ge t} \mathbb{E}[\mathcal{J}(\tau,\sigma,V)|\mathcal{F}_t] = \operatorname*{ess\,sup}_{\sigma\ge t} \operatorname*{ess\,inf}_{\tau\ge t} \mathbb{E}[\mathcal{J}(\tau,\sigma,V)|\mathcal{F}_t].$$

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- $(\tau_S^{*,1,n}, \sigma_S^{*,1,n})$ *converges to* (τ_S^*, σ_S^*) *in probability, as n goes to infinity.*

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Assume A2, A3 and A4 are in force. We have

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  i.e. $\overline{V}^i = \underline{V}^i := V^i \quad \mathbb{P}\text{-a.s.}$, *where* $\{V^i\}_{i\ge1}$ *are i.i.d. copies of the value* $V = V^1$ *of the game:*

  $$V_t := \operatorname*{ess\,inf}_{\tau\ge t} \operatorname*{ess\,sup}_{\sigma\ge t} \mathbb{E}[\mathcal{J}(\tau,\sigma,V)|\mathcal{F}_t] = \operatorname*{ess\,sup}_{\sigma\ge t} \operatorname*{ess\,inf}_{\tau\ge t} \mathbb{E}[\mathcal{J}(\tau,\sigma,V)|\mathcal{F}_t].$$

- $(\tau_S^{*,1,n}, \sigma_S^{*,1,n})$ *converges to* (τ_S^*, σ_S^*) *in probability, as n goes to infinity.*

- *We have*

  $$\lim_{n\to\infty} \mathbb{E}[V_{\tau_S^{*,1,n}}] = \mathbb{E}[V_{\tau_S^*}], \qquad \lim_{n\to\infty} \mathbb{E}[V_{\sigma_S^{*,1,n}}] = \mathbb{E}[V_{\sigma_S^*}].$$

Proposition

- *Up to a subsequence, it holds that*

$$\lim_{n\to\infty} \mathbb{E}\left[h_1\big(X^{1,n}_{\tau_S^{*,1,n}}, \tfrac{1}{n}\textstyle\sum_{j=1}^n \delta_{X^{j,n}_{\tau_S^{*,j,n}}}, Y^{1,n}_{\tau_S^{*,1,n}}, \tfrac{1}{n}\textstyle\sum_{j=1}^n \delta_{Y^{j,n}_{\tau_S^{*,j,n}}}\big)\mathbb{1}_{\{\tau_S^{*,1,n}<T\}} + \xi^{1,n}\mathbb{1}_{\{\tau_S^{*,1,n}=T\}} \right]$$

$$= \mathbb{E}\left[h_1(X_{\tau_S^*}, \mathbb{P}_{X_{\tau_S^*}}, V_{\tau_S^*}, \mathbb{P}_{V_{\tau_S^*}})\mathbb{1}_{\{\tau_S^*<T\}} + \xi\mathbb{1}_{\{\tau_S^*=T\}} \right],$$

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$$\lim_{n\to\infty} \mathbb{E}\left[h_1\Big(X^{1,n}_{\tau_S^{*,1,n}}, \tfrac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}_{\tau_S^{*,j,n}}}, Y^{1,n}_{\tau_S^{*,1,n}}, \tfrac{1}{n}\sum_{j=1}^n \delta_{Y^{j,n}_{\tau_S^{*,j,n}}}\Big) \mathbb{1}_{\{\tau_S^{*,1,n}<T\}} + \xi^{1,n}\mathbb{1}_{\{\tau_S^{*,1,n}=T\}} \right]$$

$$= \mathbb{E}\left[h_1(X_{\tau_S^*}, \mathbb{P}_{X_{\tau_S^*}}, V_{\tau_S^*}, \mathbb{P}_{V_{\tau_S^*}}) \mathbb{1}_{\{\tau_S^*<T\}} + \xi \mathbb{1}_{\{\tau_S^*=T\}} \right],$$

and

$$\lim_{n\to\infty} \mathbb{E}\left[h_2\Big(X^{1,n}_{\sigma_S^{*,1,n}}, \tfrac{1}{n}\sum_{j=1}^n \delta_{X^{j,n}_{\sigma_S^{*,j,n}}}, Y^{1,n}_{\sigma_S^{*,1,n}}, \tfrac{1}{n}\sum_{j=1}^n \delta_{Y^{j,n}_{\sigma_S^{*,j,n}}}\Big) \mathbb{1}_{\{\sigma_S^{*,1,n}<T\}} + \xi^{1,n}\mathbb{1}_{\{\sigma_S^{*,1,n}=T\}} \right]$$

$$= \mathbb{E}\left[h_2(X_{\sigma_S^*}, \mathbb{P}_{X_{\sigma_S^*}}, V_{\sigma_S^*}, \mathbb{P}_{V_{\sigma_S^*}}) \mathbb{1}_{\{\sigma_S^*<T\}} + \xi \mathbb{1}_{\{\sigma_S^*=T\}} \right].$$

From these two limits we derive the saddle-point property of (τ_S^*, σ_S^*).

References

• B. Djehiche, R. Dumitrescu, and J. Zeng, A propagation of chaos result for weakly interacting nonlinear snell envelopes, arXiv, 2021.

• B. Djehiche and R. Dumitrescu, Zero-sum mean-field Dynkin games: characterization and convergence, arXiv, 2022.

• B. Djehiche, R. Elie, and S. Hamadéne, Mean-field reflected backward stochastic differential equations, arXiv, 2019.

• B. Jourdain, S. Méléard, and W. Woyczynski, Nonlinear sdes driven by lévy processes and related pdes, Alea 4 (2008), 1–29.

• J. L. Pedersen, Explicit solutions to some optimal variance stopping problems, Stochastics 83 (2011), no. 4- 6, 505–518.

• M. Talbi, N. Touzi, and J. Zhang, Dynamic programming equation for the mean field optimal stopping problem, arXiv, 2021.