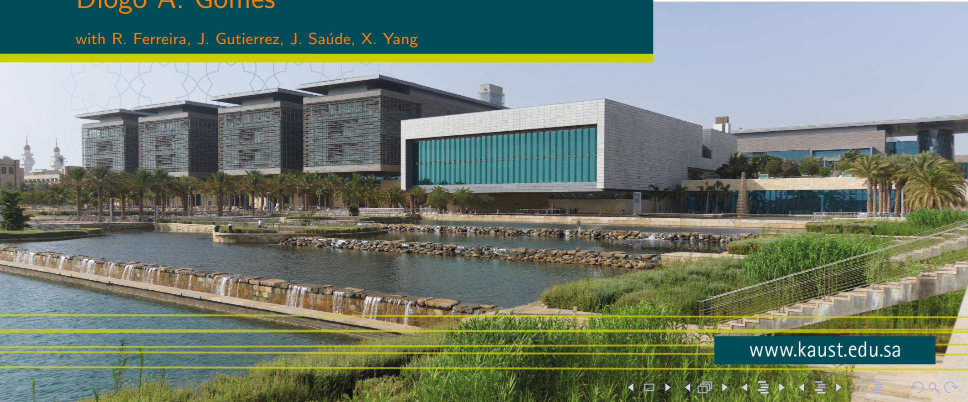
جامعة الملك عبدالله
للعلوم والتقنية
King Abdullah University of
Science and Technology

Hessian Riemannian flows in MFGs

Diogo A. Gomes

with R. Ferreira, J. Gutierrez, J. Saúde, X. Yang



Optimal control and Hamilton-Jacobi equations

- We fix $T > 0$ and consider an agent with state $\mathbf{x}(t) \in \mathbb{R}^d$ for $0 \leq t \leq T$.
- Agents change their state by choosing a control in $\mathbf{v} \in \mathcal{W} = L^\infty([t, T], \mathbb{R}^d)$.
- The state of an agent evolves according to

$$\dot{\mathbf{x}}(t) = \mathbf{v}(t).$$

- We fix a Lagrangian $\tilde{L} : \mathbb{R}^d \times \mathbb{R}^d \times [0, T] \to \mathbb{R}$, with $v \mapsto L(x, v, t)$ uniformly convex.
- Agents preferences are encoded by the functional,

$$J(\mathbf{v}; x, t) = \int_t^T \tilde{L}(\mathbf{x}(s), \mathbf{v}(s), s)ds + u_T(\mathbf{x}(T)),$$

  where $\dot{\mathbf{x}} = \mathbf{v}$ with $\mathbf{x}(t) = x$.
- Each agent seeks to minimize J in $\mathcal{W}$. The value function is

$$u(x, t) = \inf_{\mathbf{v} \in \mathcal{W}} J(\mathbf{v}; x, t).$$

The Hamiltonian, $\tilde{H}$, is Legendre transform of $\tilde{L}$

$$\tilde{H}(x,p,t) = \sup_{v\in\mathbb{R}^d} \left[-p\cdot v - \tilde{L}(x,v,t) \right].$$

By uniform convexity, the maximum is achieved at a unique point

$$v^* = -D_p\tilde{H}(x,p,t).$$

Theorem (Verification Theorem)

- *Let* $\tilde{u} \in C^1(\mathbb{R}^d \times [t_0, T])$ *solve the Hamilton–Jacobi equation with the terminal condition* $u_T(x)$.
- *Let*
$$\mathbf{v}^*(t) = -D_p\tilde{H}(\mathbf{x}^*(t), D_x\tilde{u}(\mathbf{x}^*(t), t), t)$$
and $\mathbf{x}^*(t)$ *be the corresponding trajectory.*

Then,

- $\mathbf{v}^*(t)$ *is an optimal control*
- $\tilde{u}(x, t)$ *is the value function,* u.

Transport equation

Let $b : \mathbb{R}^d \times [0, T] \to \mathbb{R}^d$ be a Lipschitz vector field. The ODE

$$\begin{cases} \dot{\mathbf{x}}(t) = b(\mathbf{x}(t), t) & t > 0, \\ \mathbf{x}(0) = x \end{cases}$$

induces a flow, Φ^t, in $\mathbb{R}^d$ that maps the initial condition, $x \in \mathbb{R}^d$, at $t = 0$ to the solution at time $t > 0$.

Fix $m_0 \in \mathcal{P}(\mathbb{R}^d)$. For $0 \leq t \leq T$, let $m(\cdot, t) = \Phi^t \sharp m_0$ be the probability defined by

$$\int_{\mathbb{R}^d} \phi(x) m(x,t) dx = \int_{\mathbb{R}^d} \phi\left(\Phi^t(x)\right) m_0 dx.$$

Then

$$\begin{cases} m_t(x,t) + \operatorname{div}(b(x,t)m(x,t)) = 0, & (x,t) \in \mathbb{R}^d \times [0,T], \\ m(x,0) = m_0(x), & x \in \mathbb{R}^d, \end{cases}$$

Mean-field models I

- The mean-field game framework studies systems with an infinite number of competing rational agents.
- Each agent seeks to optimize an individual control problem that depends on statistical information about the whole population.
- The only information available to the agents is the probability distribution of the agents' states.

- For each time t, $m(x,t)$ is a probability density in $\mathbb{R}^d$ that gives the distribution of the agents
- We set

$$\tilde{L}(x,v,t) = L(x,v,m(\cdot,t)).$$

  and denote the Legendre transform of L by H.
- Each agent seeks to minimize a control problem whose value function solves

$$-u_t + H(x, D_x u, m) = 0.$$

According to the Verification Theorem, if u is a solution, $b = -D_p H(x, D_x u(x,t), m)$, determines the optimal strategy. Because all agents are rational, they use this strategy.

Hence, u and m are determined by

$$\begin{cases} -u_t + H(x, D_x u, m) = 0 \\ m_t - \operatorname{div}(D_p H m) = 0. \end{cases}$$

We supplement this system with terminal value function $u(x, T) = u_T$ and the initial distribution $m(x, 0) = m_0$.

Variations

- Second-order (noise)
- Stationary (with or without discount)
- Boundary conditions....

Monotone operators

Let W be a Hilbert space. A map $F : \mathcal{D}(F) \subset W \to W$ is monotone if

$$(F(x) - F(y), x - y)_W \geq 0$$

for all $x, y \in \mathcal{D}(F)$.

Stationary MFGs

If $H(x,p)$ is convex in p and g is increasing, the operator

$$F\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - H(x, Du) + g(m) \\ m - \operatorname{div}(D_pHm) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$ (here, we take **periodic** boundary conditions in space, for example).

Time-dependent MFGs

If $H(x,p)$ is convex in p and g is increasing, the operator

$$F\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} u_t - H(x, Du) + g(m) \\ m_t - \operatorname{div}(D_pHm) - 1 \end{bmatrix}$$

monotone in its domain $D \subset L^2 \times L^2$.

Applications

This point of view has yieded several results

- Uniqueness (Lasry and Lions)
- Numerical methods (AlMulla, Ferreira, and G.)
- Weak Solutions (Ferreira, G., and Tada)
- Uniqueness of weak solutions (Ferreira, G., and Voskanyan)

Uniqueness

Given two distinct solutions $F(m,u)=0$ and $F(\tilde{m},\tilde{u})=0$, we have

$$0 = \left(F \begin{bmatrix} m \\ u \end{bmatrix} - F \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix}, \begin{bmatrix} m \\ u \end{bmatrix} - \begin{bmatrix} \tilde{m} \\ \tilde{u} \end{bmatrix} \right) \geq 0.$$

With strong monotonicity (>0), we would get uniqueness. Often, it is possible to work around this difficulty.

Variational inequalities

Let $F : D(F) \subset W \to W$ is monotone. Then, for $w \in W$, $1 \implies 2 \implies 3$, where

1) $F(w) = 0$
2) w solves the variational inequality

$$(F(w), z - w) \geq 0, \qquad \forall z \in W.$$

3) w is a **weak solution** of the variational inequality; that is

$$(F(z), z - w) \geq 0$$

for all $z \in D(F)$.

Moreover, $3 \implies 2 \implies 1$ under continuity assumptions and $D(F)$ large enough.

Weak solutions

A weak solution of the MFG is a pair (m,u), $m \geq 0$, such that

$$\left\langle \begin{bmatrix} \eta \\ v \end{bmatrix} - \begin{bmatrix} m \\ u \end{bmatrix}, F \begin{bmatrix} \eta \\ v \end{bmatrix} \right\rangle_{\mathcal{D}'(\mathbb{T}^d)\times\mathcal{D}'(\mathbb{T}^d), C^\infty(\mathbb{T}^d)\times C^\infty(\mathbb{T}^d)} \geq 0$$

for all $(\eta, v) \in \mathcal{C}^\infty(\mathbb{T}^d;\mathbb{R}^+) \times \mathcal{C}^\infty(\mathbb{T}^d)$.

MFGs and variational inequalities

Consider the MFG corresponding to

$$F\begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} -u - H(x, Du) + g(m) \\ m - \operatorname{div}(D_pHm) - 1 \end{bmatrix}.$$

where H is periodic in x and convex in the second variable, and g is increasing.

Existence of weak solutions

Main Theorem (Ferreira, G.)

Under suitable but general Assumptions, there exists a weak solution, $(m, u) \in \mathcal{D}'(\mathbb{T}^d) \times \mathcal{D}'(\mathbb{T}^d)$, $m \geq 0$, to the MFG

$$F \begin{bmatrix} m \\ u \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Moreover, $(m, u) \in \mathcal{M}_{ac} \times W^{1,\gamma}$ for some $\gamma > 1$ and $\int_{\mathbb{T}^d} m \, dx = 1$.

Scope

First-order, second-order, degenerate elliptic, and congestion problems satisfying monotonicity conditions.

Example

Theorem (Ferreira, G.)

Let κ be a standard mollifier, $\alpha > 0$. Then, there exists a weak solution $u \in H^1$, $m \in L^{\alpha+1}$, $m \geq 0$ to

$$\begin{cases} u + \frac{|Du|^2}{2} + V(x) = m^\alpha + \kappa * m \\ m - \operatorname{div}(mDu) = 1. \end{cases}$$

That is, for all $(\eta, v) \in C^\infty$, $\eta > 0$, we have

$$\int \left(v + \frac{|Dv|^2}{2} + V(x) - \eta^\alpha - \kappa * m\right)(\eta - m) + \int (\eta - \operatorname{div}(\eta Dv) - 1)(v - u) \geq 0.$$

Example - further properties

Theorem (Ferreira, G.)

There exists a weak solution (u, m) such that

$$\begin{cases} -u - \dfrac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m \geq 0, & \text{in } \mathcal{D}' \\ m - \operatorname{div}(mDu) - 1 = 0, \text{a.e.}. \end{cases}$$

Moreover, if $\alpha > \max\left(\frac{d-4}{2}, 0\right)$

$$\Big(-u - \frac{|Du|^2}{2} + V(x) + m^\alpha + \kappa * m\Big)m = 0$$

almost everywhere.

Weak-strong uniqueness

Weak-strong uniqueness

Let (u, m) and $(\tilde{u}, \tilde{m})$ be, resp. a strong (smooth) and a weak solution ($u \in H^1$ and $m \in L^2$) of

$$\begin{cases} u + \frac{|Du|^2}{2} + V(x) = g(m) \\ m - \operatorname{div}(mDu) = 1, \end{cases}$$

with periodic boundary conditions. Suppose $m > 0$. Then $(u, m) = (\tilde{u}, \tilde{m})$.

Difficulties

But in some aspects this approach is not totally satisfactory.

- For the existence of weak solutions, we had prove the existence of weak solutions with more regularity so that traces can be defined.
- For numerical methods, boundary conditions and positivity constraints were handled in an ad hoc fashion.

Boundary conditions

For example, for

$$\begin{bmatrix} u_t - H(Du, x) + g(m) \\ m_t - \operatorname{div}(D_p H(Du, x) m) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

the standard initial-terminal conditions on $u(x, T)$ and $m(x, 0)$ are not well defined (in L^2) since there is no trace.

Goals

- Develop a functional framework for monotone operators in the context of MFGs
- Construct "gradient-flow" like PDEs for the approximation of solutions of MFGs

Derivatives and gradients

Given a Hilbert space W with inner product $(\cdot,\cdot)_W$ and a differentiable function $G : W \to \mathbb{R}$, we have two objects of interest:

$$DG : W \to W'$$

and

$$\nabla G : W \to W.$$

Riesz operator and gradient flows

The Riesz operator $R : W' \to W$ is the map that transforms DG into ∇G

$$\nabla G = RDG.$$

Example: $W = \mathbb{R}^d$, the Riesz operator is the transposition operator.

Thanks to this correspondence, we can define the gradient flow

$$\dot{\mathbf{x}} = -\nabla f(\mathbf{x}).$$

There is no such thing as a "derivative" flow!

Example

Let $f \in L^2(\mathbb{T}^d)$ and consider the functional in $H_0^1(\mathbb{T}^d)$

$$u \mapsto \int_{\mathbb{T}^d} fu.$$

Then, the Riesz operator is $Rf = (-\Delta)^{-1} f$ because

$$((-\Delta)^{-1} f, u)_{H_0^1} = \int_{\mathbb{T}^d} \nabla u \cdot \nabla (-\Delta)^{-1} f = \int_{\mathbb{T}^d} fu = (Rf, u)_{H_0^1}.$$

Consider the functional

$$u \mapsto \frac{1}{2}\int |\nabla u|^2.$$

The corresponding L^2 gradient flow is

$$u_t = \Delta u,$$

whereas the H_0^1 gradient flow is

$$u_t = -u.$$

While both equations are well posed, the existence of solutions for the heat equation requires a substantially complex analysis that the one for the last equation.

Let u^* solve $\Delta u^* = 0$. Then, for the L^2 gradient flow:

$$\frac{d}{dt}\frac{1}{2}\|u - u_*\|_{L^2}^2 = \int (u - u^*)\Delta u = -\int |\nabla u|^2$$

and for the H_0^1:

$$\frac{d}{dt}\frac{1}{2}\|u - u_*\|_{H_0^1}^2 = -\int \nabla(u - u^*) \cdot \nabla u = -\int |\nabla u|^2.$$

Monotone operator and contracting flows

Given a monotone operator the monotone flow

$$\dot{\mathbf{x}} = -F(\mathbf{x})$$

is contracting in W because for any two solutions $\mathbf{x}$ and $\mathbf{y}$

$$\frac{d}{dt}\frac{1}{2}\|\mathbf{x}-\mathbf{y}\|^2 = -(F(\mathbf{x})-F(\mathbf{y}),\mathbf{x}-\mathbf{y}) \leq 0.$$

Monotone flows for MFGs

For stationary MFGs, this suggests using the flow

$$\begin{bmatrix} m_s \\ u_s \end{bmatrix} = -\begin{bmatrix} -H(Du,x)+g(m) \\ -\operatorname{div}(D_pH(Du,x)m) \end{bmatrix}$$

(with H convex and g monotone increasing)

Monotone flows and positivity preservation

Mean-field games are only monotone operators if $m > 0$, so it is natural to ask whether monotone flows preserve positivity. This is unfortunately false...

Lack of positivity preservation

Set $H(x,p)=\frac{p^2}{2}+\sin(2\pi x)$. Then, the monotone flow is

$$\begin{bmatrix}\dot{\boldsymbol{m}}\\ \dot{\boldsymbol{u}}\end{bmatrix}=\begin{bmatrix}\frac{\boldsymbol{u}_x^2}{2}+\sin(2\pi x)\\ (\boldsymbol{m}\boldsymbol{u}_x)_x\end{bmatrix}.$$

Let $(m_0,0)$ to be the initial point and $\int_0^1 m_0 dx=1$. It is easy to check that $(\boldsymbol{m},\boldsymbol{u})=(m_0+\sin(2\pi x)\,t,0)$. However, $\boldsymbol{m}(t)$ becomes negative in some regions as $t\to+\infty$.

Boundary conditions in numerical methods

For time-dependent problems the flow for $u(x,t,s)$ and $m(x,t,s)$

$$\begin{bmatrix} m_s \\ u_s \end{bmatrix} = -\begin{bmatrix} u_t - H(Du,x) + g(m) \\ m_t - \operatorname{div}(D_pH(Du,x)m) \end{bmatrix}$$

does not preserve initial-terminal boundary conditions.

MFGs with common noise

For price models with common noise, we have to study

$$\begin{bmatrix} -du + H(x, \varpi + Du) = Z dW_t \\ m_t - \operatorname{div}(m D_p H(x, \varpi + Du)) \\ \int m D_p H(x, \varpi + Du) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -Q(t) \end{bmatrix}$$

with initial-terminal conditions on $u(x, T)$ and $m(x, 0)$.

Formal monotonicity

The map

$$\begin{bmatrix} m(x,t) \\ u(x,t) \\ \varpi(t) \end{bmatrix} \to \begin{bmatrix} du - H(x, \varpi + Du)dt - ZdW_t \\ m_t - \operatorname{div}(mD_pH(x, \varpi + Du)) \\ \int mD_pH(x, \varpi + Du) \end{bmatrix}$$

is formally monotone in $L^2(\mathbb{R} \times [0,T] \times \Omega) \times L^2(\mathbb{R} \times [0,T] \times \Omega) \times L^2([0,T] \times \Omega)$ but really no obvious interpretation of its range...

Monotone operators in dual spaces

A map $F : \mathcal{D}(F) \subset W \to W'$ is monotone if

$$\langle F(x) - F(y), x - y \rangle_{W' \times W} \geq 0$$

for all $x, y \in \mathcal{D}(F)$.
By the Riesz representation theorem, $RF : \mathcal{D}(F) \subset W \to W$ is a monotone operator because

$$(RF(x) - RF(y), x - y)_W = \langle F(x) - F(y), x - y \rangle_{W' \times W} \geq 0.$$

Monotone flows

Using the Riesz operator, we can define a monotone flow

$$\dot{\mathbf{x}} = -RF(\mathbf{x})$$

which is a contraction in W.

Example

If we regard the stationary MFG as an operator from $L^2 \times H_0^1 \to (L^2)' \times (H_0^1)'$, we obtain the flow

$$\begin{bmatrix} m_t \\ u_t \end{bmatrix} = - \begin{bmatrix} -H(Du,x) + g(m) \\ -(-\Delta)^{-1} \operatorname{div}(D_pH(Du,x)m) \end{bmatrix}$$

Examples

Consider the quartic mean-field game with congestion

$$\begin{cases} \frac{u_x^4}{2m^{1/2}} + \cos^2(2\pi x)u_x + \sin(2\pi x) - \ln m = \overline{H} \\ -(2u_x^3 m^{1/2})_x - (\cos^2(2\pi x)m)_x = 0. \end{cases}$$

and the associated monotone flow

$$\begin{cases} u_t = \frac{u_x^4}{2m^{1/2}} + \cos^2(2\pi x)u_x + \sin(2\pi x) - \ln m - \overline{H} \\ m_t = (2u_x^3 m^{1/2})_x - (\cos^2(2\pi x)m)_x = 0. \end{cases}$$

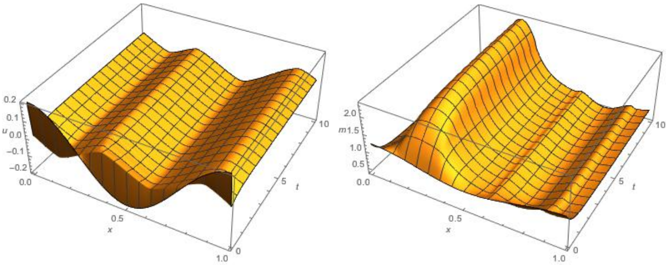


Fig.: Quartic congestion model: u and m

Hessian Riemannian flow

A modification that preserves positivity is

$$\begin{bmatrix} \dot{\boldsymbol{m}} \\ \dot{\boldsymbol{u}} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{m}\left(-H(x, P + D_x \boldsymbol{u}) + \overline{\boldsymbol{H}}(P) + g(\boldsymbol{m})\right) \\ -\operatorname{div}(D_p H(x, P + D_x \boldsymbol{u})\boldsymbol{m}) \end{bmatrix},$$

where

$$\overline{\boldsymbol{H}}(P) = \frac{\int_{\mathbb{T}^d} \left(\boldsymbol{m} H(x, P + D_x \boldsymbol{u}) - \boldsymbol{m} g(\boldsymbol{m})\right) dx}{\int_{\mathbb{T}^d} \boldsymbol{m}}.$$

Time dependent problems

Together with J. Saude, we developed a discretization for the initial-terminal value problem solving a discrete version of

$$\begin{bmatrix} m_s \\ u_s \end{bmatrix} = -\begin{bmatrix} (I-\partial_{tt})^{-1}\left(u_t - H(Du,x) + g(m)\right) \\ (I-\partial_{tt})^{-1}\left(m_t - \operatorname{div}(D_pH(Du,x)m)\right) \end{bmatrix},$$

the inverse of $I-\partial_{tt}$ in the first component is taken with zero initial conditions and in the second with zero terminal conditions.

This approach for time dependent problems takes care of the boundary conditions but not of the positivity of m, therefore we introduced the ad hoc modifications such as

$$\begin{bmatrix} m_s \\ u_s \end{bmatrix} = -\begin{bmatrix} m(I-\partial_{tt})^{-1}\left(u_t - H(Du,x) + g(m)\right) \\ (I-\partial_{tt})^{-1}\left(m_t - \operatorname{div}(D_pH(Du,x)m)\right) \end{bmatrix},$$

but then contraction becomes an issue....

Goals

- Develop a systematic way to construct monotone flows
- Establish the wellposednes of these problems
- Discretizations and numerical methods

Derivatives

Let $h: W \to \mathbb{R}$ be convex. The directional derivative of h is

$$dh(x;y) = \frac{d}{d\epsilon} h\left(x + \epsilon y\right)\bigg|_{\epsilon=0}, \quad x, y \in W.$$

If h is differentiable,

$$dh(x;y) = \langle dh(x), y\rangle, \quad y \in W,$$

where $dh(x)$ denotes the derivative of $h(x)$.

Hessians

The second variation is defined by

$$d^2h(x;y,z) = \frac{d}{d\epsilon}\left\langle dh(x+\epsilon y), z\right\rangle\Big|_{\epsilon=0}, \quad x,y,z \in W.$$

If dh is differentiable, we set

$$d^2h(x;y,z) = \left\langle \mathcal{H}(x)y, z\right\rangle, \quad y,z \in W,$$

where $\mathcal{H}(x) : W \to W'$ is linear and positive definite in the sense that

$$\langle \mathcal{H}(x)y, y\rangle_{W'\times W} > 0, \quad y \in W,\ y \neq 0.$$

Hessian-induced Riesz operator

- If $\mathcal{H}: W \to W'$ is invertible, the map $\mathcal{H}^{-1}: W' \to W$ is a generalization of the Riesz operator.
- In fact, if
$$h(x) = \frac{1}{2}\|x\|^2,$$
$\mathcal{H}^{-1} = R.$

Alvarez method for constrained optimization

- Consider
$$\min\{G(x) \mid x \in \overline{E},\ Ax = b\},$$
where $\overline{E}$ is the closure of an open, nonempty, convex set $E \subset \mathbb{R}^n$.
- Alvarez et al introduced a Riemannian metric derived from the Hessian matrix, $\nabla^2 h$, of a convex function h, on E.
- Then, they used the steepest descent flow to generate trajectories in the relative interior of the feasible set, $\mathcal{F} := E \cap \{x \mid Ax = b\}$.
- By choosing h conveniently, the steepest descent flow is well posed, never leaves the admissible set, and leads to a local minimum.

The authors sought a trajectory $\mathbf{x}(t)$ solving

$$\begin{cases} \dot{\mathbf{x}} + \nabla_{\mathcal{H}} G(\mathbf{x}) = 0, \\ \mathbf{x}(0) = x^0 \in \mathcal{F}, \end{cases}$$

where $\nabla_{\mathcal{H}} G(x)$ is the projection w.r.t. of the gradient of G into the admissible directions.

HRF

More concretely, the HRF is

$$\dot{\mathbf{x}} = -\mathcal{H}^{-1}(\mathbf{x})\left(I - A^{\times}\left(A\mathcal{H}^{-1}(\mathbf{x})A^{\times}\right)^{-1} A\mathcal{H}^{-1}(\mathbf{x})\right) DG(\mathbf{x}).$$

Bregman divergence

In particular,

$$\frac{d}{dt} A\dot{\mathbf{x}} = 0$$

and the Bregman divergence of h

$$d(x, x^*) = h(x^*) - h(x) - \langle dh(x), x^* - x \rangle$$

is a Liapunov function for any minimizer x^*; that is,

$$\frac{d}{dt}\left(h(x^*) - h(x) - (\nabla h(\mathbf{x}), x^* - \mathbf{x})\right) \leq 0.$$

Monotone HRF

Let F be a monotone operator. We define

$$F_{\mathcal{H}}(x) = \left(I - \mathcal{H}(x)^{-1} A^{\times} \left(A\mathcal{H}(x)^{-1} A^{\times} \right)^{-1} A \right) \mathcal{H}(x)^{-1} F(x),$$

Bregman dissipation

If x^* solve $F(x^*)=0$ then

$$\frac{d}{dt}\left(h(x^*)-h(\mathbf{x})-(\nabla h(\mathbf{x}),x^*-\mathbf{x})\right)\leq 0,$$

and

$$\frac{d}{dt}A\mathbf{x}=0.$$

Example - the convex function

Let $\theta : \mathrm{dom}(\theta) \subset [0, +\infty) \to \mathbb{R}$ and set

$$h(m, u) = \int_{\mathbb{T}^d} \theta(m(x))dx + \frac{1}{2}\|m\|^2_{W^{k,2}(\mathbb{T}^d)} + \frac{1}{2}\|u\|^2_{W_0^{k,2}(\mathbb{T}^d)}.$$

Example - Hessian

Then

$$\left\langle \mathcal{H}\begin{bmatrix}\eta\\ v\end{bmatrix}, \begin{bmatrix}\eta\\ v\end{bmatrix}\right\rangle = \int_{\mathbb{T}^d} \theta''(m(x))\eta^2 dx + \frac{1}{2}\|\eta\|^2_{H^k(\mathbb{T}^d)} + \frac{1}{2}\|v\|^2_{H^j_0(\mathbb{T}^d)}.$$

Example - stationary MFG

$$\begin{cases} H(Du(x)) + V(x) = g(m) + \overline{H}, \\ -\operatorname{div}\left(m(x)DH(Du(x))\right) = 0. \end{cases} \qquad x \in \mathbb{T}^d$$

with $m \geq 0$ and

$$\int_{\mathbb{T}^d} m = 1$$

Example - the regularized stationary MFG

For example $\theta(m) = m \ln m$, $(m,u) \in L^2 \times H^1_0$, we have the operator

$$\begin{bmatrix} \frac{m}{m+1}\left(-H(x,Du)+g(m)-\frac{\int_{\mathbb{T}^d}\frac{m}{m+1}(-H(x,Du)+g(m))dx}{\int_{\mathbb{T}^d}\frac{m}{m+1}dx}\right) \\ \Delta^{-1}\operatorname{div}(mD_pH(x,Du)) \end{bmatrix}.$$

Bregman dissipation

For any stationary solution (m^*, u^*), we have

$$\frac{d}{dt}\int_{\mathbb{T}^d}\left(m^*\log\left(\frac{m^*}{m}\right)-m^*+m\right)dx+\frac{1}{2}\|m-m^*\|^2_{L^2(\mathbb{T}^d)}+\frac{1}{2}\|u-u^*\|^2_{H^1_0(\mathbb{T}^d)}\leq 0.$$

In general,

$$\frac{d}{dt}\int_{\mathbb{T}^d}\theta(m^*)-\theta(m)-\theta'(m)(m^*-m)dx+\|m-m^*\|^2_{H^k(\mathbb{T}^d)}+\|u-u^*\|^2_{H^j_0(\mathbb{T}^d)}\leq 0.$$

So,

$$\int_{\mathbb{T}^d}\theta(m^*)-\theta(m)-\theta'(m)(m^*-m)$$

is bounded.

Positivity preservation

If $\theta(m)=\frac{1}{m^p}$ we have

$$\int_{\mathbb{T}^d}\frac{1}{(m^*)^p}-\frac{1}{m^p}+\frac{p}{m^{p+1}}(m^*-m)=\int_{\mathbb{T}^d}\frac{1}{(m^*)^p}+\frac{1-p}{m^p}+\frac{pm^*}{m^{p+1}}$$

is bounded. For suitable p, if m^* is strictly positive this gives pointwise bounds on m (for k large enough).

Summary

- We developed a general approach to construct regularized monotone flows for MFGs
- With suitable choices of spaces, these flows are well posed and preserve positivity of the density
- The effective numerical implementation of these methods is a topic of ongoing research.