Euler and Navier-Stokes equations: quasi-periodic solutions and inviscid limit

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The equation

Incompressible Navier-Stokes equations

$$\begin{cases} \partial_t U + U \cdot \nabla U - \nu \Delta U + \nabla P = \delta f(\omega t, x), \quad x \in \mathbb{T}^n \quad (n = 2, 3) \\ \mathrm{div}(U) = 0 \end{cases}$$

Quasi-periodic external force

$$f \in C^\infty(\mathbb{T}^d \times \mathbb{T}^n, \mathbb{R}^n)$$

$\omega = (\omega_1, \ldots, \omega_d) \in \mathbb{R}^d$ is the external frequency vector and $\delta \ll 1$ is a small parameter and $\nu > 0$ is the viscosity

Unknowns

$$U : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R}^n \quad \textbf{velocity field}$$

$$P : \mathbb{R} \times \mathbb{T}^n \to \mathbb{R} \quad \textbf{pressure}$$

Our Goal

- **Quasi-periodic solutions of the Euler equation**
- **Quasi-periodic solutions of the Navier-Stokes equation and study their behavior in the small viscosity regime $\nu \simeq 0$ (inviscid limit)**

Definition: quasi-periodic solution with d frequencies

$$\partial_t u = X(u)\,, \quad u \in \mathcal{H} \quad \text{(phase space)}.$$

A **quasi-periodic solution**, is a solution of the form $u(t) = U(\omega t)$ where $U : \mathbb{T}^d \to \mathcal{H}$, $\omega \in \mathbb{R}^d (= \textit{frequency vector})$ is irrational

$$\omega \cdot \ell \neq 0\,, \ \forall \ell \in \mathbb{Z}^d \setminus \{0\}$$

$\Longrightarrow$ the linear flow $\{\theta_0 + \omega t\}_{t \in \mathbb{R}}$ is DENSE on $\mathbb{T}^d$. The torus embedding $\mathbb{T}^d \to \mathcal{H}$, $\varphi \mapsto U(\varphi)$ satisfies

$$\omega \cdot \partial_\varphi U(\varphi) - X(U(\varphi)) = 0, \quad \omega \cdot \partial_\varphi := \sum_{i=1}^{d} \omega_i \partial_{\varphi_i}\,.$$

For $\delta = 0$, $\nu = 0$

Incompressible Euler equation wthout forcing term

$$\begin{cases} \partial_t U + U \cdot \nabla U + \nabla P = 0, & x \in \mathbb{T}^n \\ \mathrm{div}(U) = 0 \end{cases}$$

Families of particular equilibrium solutions

Given $P_0 \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$,

$$U(t,x) \equiv \zeta \quad \textbf{constant velocity field},$$
$$P(t,x) \equiv P_0 \quad \textbf{constant pressure}$$

Our results

- **Quasi-periodic solutions of Euler and Navier-Stokes equations** bifurcating from **constant velocity field** and **constant pressure**
- **Study the convergence of quasi-periodic solutions of Navier-Stokes to the ones of Euler in the inviscid limit** $\nu \to 0$

Our goal

Quasi-periodic solutions

We look for quasi-periodic solutions oscillating with frequency ω, $U(\omega t, x), P(\omega t, x)$, $U : \mathbb{T}^d \times \mathbb{T}^n \to \mathbb{R}^3$, $P : \mathbb{T}^d \times \mathbb{T}^n \to \mathbb{R}$ with

$$U(\omega t, x) = \zeta + \varepsilon u(\omega t, x) \quad \textbf{perturbations of constant vector fields},$$

$$P(\omega t, x) = P_0 + \varepsilon p(\omega t, x) \quad \textbf{perturbations of constant pressures},$$

$$\varepsilon = \sqrt{\delta}$$

Equation for u and p

$$\begin{cases} \omega \cdot \partial_\varphi u + \zeta \cdot \nabla u - \nu \Delta u + \varepsilon \Big(u \cdot \nabla u + \nabla p - f(\varphi, x) \Big) = 0 \\ \operatorname{div}(u) = 0 \end{cases}$$

Parameters

To impose non-resonance conditions, we use the parameters $(\omega, \zeta) \in \Omega$, $\Omega \subseteq \mathbb{R}^{d+n}$ is a bounded open set.

The small-divisors problem

The linearized operator at the origin $(\varepsilon, \nu = 0)$

$$L := \omega \cdot \partial_\varphi + \zeta \cdot \nabla = \mathrm{diag}_{\ell,j} \mathrm{i}(\omega \cdot \ell + \zeta \cdot j)$$

Small divisors

For a.e. (ω, ζ) the sequence $\{\omega \cdot \ell + \zeta \cdot j\}_{(\ell,j)}$ accumulates to 0! We can impose (for a set of large Lebesgue measure)

$$|\omega \cdot \ell + \zeta \cdot j| \geq \frac{\gamma}{\langle \ell, j \rangle^\tau}, \quad \forall (\ell, j) \neq (0,0)$$

$\gamma \ll 1$, $\tau \gg 1$. Then

$$L^{-1} = \mathrm{diag}_{\ell,j} \frac{1}{\mathrm{i}(\omega \cdot \ell + \zeta \cdot j)}$$

Loss of derivatives

$$L^{-1} : H^s \to H^{s-\tau}$$

Implicit function theorems and fixed point arguments fail.

The Euler equation: local and global solutions

- **Local existence of smooth solution.** Kato (1972), (1975) .
- **Global existence of smooth solutions in 2D.** Beale-Kato-Maida (1984). **Double exponential growth of Sobolev norms**

$$.\|u(t)\|_s \lesssim \exp(C\exp(Ct)), \quad C \gg 0\,.$$

Key fact in 2D: the vorticity $v := \nabla \times u$ satisfies $\|v(t)\|_{L^\infty_x} \lesssim \|v_0\|_{L^\infty_x}$

- **Global existence of weak solutions in 2D.** Yudovich (1963)
- **Global weak solutions in 3D.** Schnirelman (2000), De Lellis-Szekelyhidi (2009), . . .
- **Unbounded orbits in 2D on the disc.** Kinselev-Sverak (2014). Unbounded orbits $\|u(t)\|_s \simeq \exp(C\exp(Ct))$, $C \gg 0$
- **Ill-posedness.** For low regularity $H^{\frac{n}{2}+1}$, $n = 2, 3$, Bourgain-Li (2015)
- **Non-uniqueness results.** Buckmaster-Vicol (2018), Vishik (2018), Abritton-Brué-Colombo (2022)

Inviscid limit

Navier-Stokes and Euler equations

$$\begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = f \\ \operatorname{div}(u_\nu) = 0, \end{cases} \qquad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = f \\ \operatorname{div}(u) = 0, \end{cases}$$

General goal: Analyze the behaviour of fluids in the **vanishing viscosity regime** $\nu \to 0$.

- Approximate weakly viscous fluids with ideal fluids (important also for connections with turbulence)
- Study the convergence (in appropriate topologies)

$$(u_\nu, p_\nu) \to (u, p) \qquad \text{as} \qquad \nu \to 0$$

- Study the convergence (in appropriate topologies) of the corresponding **vorticities**

$$v_\nu = \nabla \times u_\nu, \quad v = \nabla \times u,$$

$$v_\nu \to v \qquad \text{as} \qquad \nu \to 0$$

Some results on the Inviscid limit

- **Smooth initial data.** Kato (1972, 1975) , Swann (1971), Constantin (1986), Masmoudi (2005). Convergence in H^s
- **Vortex patches.** Constantin-Wu (1995)-(1996) Convergence in L^2. See also Abidi-Danchin (2004), Masmoudi (2005)
- **Non smooth initial vorticity.** Constantin-Drivas-Elgindi (2022) $v_0 \in L^\infty(\mathbb{T}^2)$, Ciampa-Crippa-Spirito (2021) $v_0 \in L^p(\mathbb{T}^2)$. **Convergence in L^p**

Theorem. Kato, Swanna, Masmoudi . . .

Let $\varphi \in H^s$, $s > \frac{n}{2} + 1$.

$$\begin{cases} \partial_t u_\nu + u_\nu \cdot \nabla u_\nu - \nu \Delta u_\nu + \nabla p_\nu = f \\ \mathrm{div}(u_\nu) = 0 \\ u_\nu(0,x) = \varphi(x) \end{cases} \qquad \begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = f \\ \mathrm{div}(u) = 0 \\ u(0,x) = \varphi(x) \end{cases}$$

Then

$$\|u_\nu - u\|_{L^\infty([0,T],H^s)} \to 0 \quad \text{as} \quad \nu \to 0$$
$$\|u_\nu(t) - u(t)\|_{H^{s-2}} \lesssim \nu t, \quad \forall t \in [0, T],$$
$$\|u_\nu(t) - u(t)\|_{H^{s'}} \lesssim (\nu t)^{\frac{s-s'}{2}}, \quad \forall t \in [0, T], \quad s' < s$$

Construction of nonlinear waves in fluid mechanics

Water Waves: free boundary Euler equation in an ocean

- **Periodic Traveling waves (2D).** Stokes (1847), Nekrasov (1921), Levi-Civita (1925), Struik (1926)
- **Periodic standing waves (2D).** Iooss-Plotnikov-Toland (2005), Alazard-Baldi (2015)
- **Quasi-periodic standing waves (2D).** Berti-M. (2020), Baldi-Berti-Haus-M. (2020)
- **Quasi-periodic traveling waves (2D).** Berti-Franzoi-Maspero (2020)-(2021), Feola-Giuliani (2021)
- **Bi-periodic diamond waves (3D).** Craig-Nicholls (2000) (no small divisors) Iooss-Plotnikov (2009), (2011) (small divisors).

2D Euler and quasi-geostrophic equation

- **periodic vortex patches.** Burbea (1982), Hmidi-Mateu-Verdera (2013), Cordoba-Castro-Serrano, Hmidi-Mateu (2016)
- **quasi-periodic vortex patches:** Hassainia-Hmidi-Masmoudi, Berti-Hassainia-Masmoudi, Hassainia-Roulley, Hmidi-Roulley (2021)-(2022)

Vorticity formulation

Equation for the vorticity

$$v := \begin{cases} \nabla \times u = \partial_{x_2} u_1 - \partial_{x_1} u_2 & n = 2 \\ \nabla \times u = \begin{pmatrix} \partial_{x_2} u_3 - \partial_{x_3} u_2 \\ \partial_{x_3} u_1 - \partial_{x_1} u_3 \\ \partial_{x_1} u_2 - \partial_{x_2} u_1 \end{pmatrix} & n = 3 \end{cases}$$

$$\omega \cdot \partial_\varphi v + \zeta \cdot \nabla v - \nu \Delta v + \varepsilon \Big(u \cdot \nabla v - F(\varphi, x) \Big) = 0, \quad n = 2$$

$$\omega \cdot \partial_\varphi v + \zeta \cdot \nabla v - \nu \Delta v + \varepsilon \Big(u \cdot \nabla v - v \cdot \nabla u - F(\varphi, x) \Big) = 0, \quad n = 3,$$

$$F := \nabla \times f.$$

u expressed in terms of v (Biot-Savart law)

$$u := \mathcal{U}(v) = \begin{cases} (-\Delta)^{-1} \mathcal{J} \nabla v, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & n = 2 \\ (-\Delta)^{-1} (\nabla \times v), \quad n = 3 \end{cases}$$

Main results: KAM for the Euler equation (2D, 3D)

Reversibility assumption

We assume that the external force

$$f \in C^\infty(\mathbb{T}^d \times \mathbb{T}^n, \mathbb{R}^n), \quad f(\varphi, x) = -f(-\varphi, -x)\,. \tag{1}$$

Theorem. Baldi-M. Advances in Math. (2021)

There exist $s = s(d) \gg 0$ large enough, such that for every forcing term f satisfying (1) there exist $\varepsilon_0 = \varepsilon_0(f, d) \ll 1$, such that for every $\varepsilon \in (0, \varepsilon_0)$ the following holds. There exists a Borel set $\Omega_\varepsilon \subset \Omega$ of asymptotically full Lebesgue measure, i.e. $\lim_{\varepsilon \to 0} |\Omega \setminus \Omega_\varepsilon| = 0$, such that for every $(\omega, \zeta) \in \Omega_\epsilon$ there exists $v = v(\cdot\,; \omega, \zeta) \in H^s$ $v = \mathrm{odd}(\varphi, x)$, that solve the Euler equation

$$\begin{cases} \omega \cdot \partial_\varphi v + \zeta \cdot \nabla v + \varepsilon\Big(u \cdot \nabla v - F\Big) = 0 & n = 2 \\ \omega \cdot \partial_\varphi v + \zeta \cdot \nabla v + \varepsilon\Big(u \cdot \nabla v - v \cdot \nabla u - F\Big) = 0 & n = 3 \end{cases}$$

with $\|v\|_s \lesssim_s \varepsilon^b$ for some $b \in (0, 1)$.

Main results: KAM in the inviscid limit for 2D Navier-Stokes

Theorem. Franzoi-M. (2022)

Let $f \in \mathcal{C}^\infty(\mathbb{T}^d \times \mathbb{T}^2, \mathbb{R}^2)$ satisfying $f(\varphi, x) = -f(-\varphi, -x)$ and let $F := \nabla \times f$. There exist $s := s(d) \gg 0$, $\overline{\mu} := \overline{\mu}(d) \gg 0$ large enough, $\varepsilon_0 := \varepsilon_0(f, s, d) \ll 1$ small enough such that, for every $\varepsilon \in (0, \varepsilon_0)$ and for any value of the viscosity parameter $\nu > 0$, the following holds. Let $v_e(\cdot; \lambda) \in H_0^{s+\overline{\mu}}(\mathbb{T}^{d+2})$, $\lambda = (\omega, \zeta) \in \Omega_\varepsilon$ be the family of solutions of the Euler equation provided in the previous theorem. Then, there exists a Borel set $\mathcal{O}_\varepsilon \subseteq \Omega_\varepsilon$, satisfying $\lim_{\varepsilon \to 0} |\mathcal{O}_\varepsilon| = |\Omega|$ such that, for any $\lambda = (\omega, \zeta) \in \mathcal{O}_\varepsilon$, there exists a unique quasi-periodic solution $v_\nu(\cdot; \lambda) \in H_0^s(\mathbb{T}^{d+2})$, $\lambda \in \mathcal{O}_\varepsilon$, of the Navier Stokes equation

$$\omega \cdot \partial_\varphi v_\nu + \zeta \cdot \nabla v_\nu - \nu \Delta v_\nu + \varepsilon \Big(u_\nu \cdot \nabla v_\nu - F(\varphi, x) \Big) = 0$$

satisfying the estimate

$$\sup_{\lambda \in \mathcal{O}_\varepsilon} \| v_\nu(\cdot; \lambda) - v_e(\cdot; \lambda) \|_s \lesssim_s \nu \, .$$

As a consequence, for any value of the parameter $\lambda \in \mathcal{O}_\varepsilon$, the quasi-periodic solutions of the Navier Stokes equation v_ν converge to the ones of the Euler equation v_e in $H_0^s(\mathbb{T}^{d+2})$ in the limit $\nu \to 0$.

Functional equation

We look for zeroes of the nonlinear operator

The nonlinear map $\mathcal{F}_\nu$

$$\mathcal{F}_\nu(v,\varepsilon) := \omega \cdot \partial_\varphi v + \zeta \cdot \nabla v - \nu \Delta v + \varepsilon \Big(\mathcal{U}(v) \cdot \nabla v - F(\varphi, x) \Big),$$

$$\mathcal{U}(v) := (-\Delta)^{-1} \mathcal{J} \nabla v \quad F := \nabla \times f .$$

Space of zero average functions

we look for $v \in H^s_0$ where

$$H^s_0 := \Big\{ v \in H^s(\mathbb{T}^d \times \mathbb{T}^2) : \int_{\mathbb{T}^2} v(\varphi, x)\, dx = 0 \Big\} \quad \text{since} \quad \mathcal{F}_\nu : H^{s+2}_0 \to H^s_0$$

at $\nu = 0$, we have the Euler solution v_e, $\mathcal{F}_0(v_e, \varepsilon) = 0$.

Goal:

for any $\nu > 0$ and for $\varepsilon \ll 1$ small enough (uniform w.r.t. to the viscosity), we construct $u_\nu(\varphi, x)$ solution of the Navier-Stokes equation of the form

$$u_\nu(\varphi, x) = u_e(\varphi, x) + O(\nu) \quad \text{as} \quad \nu \to 0 .$$

Main difficulty

Singular limit problem

On has to control the effect of $-\nu\Delta$ **globally in time!**
small parameter $\nu \to 0$ vs. **highest order derivative** $-\Delta$.

Naive approaches fail

a fixed point argument would require $\varepsilon\nu^{-1} \ll 1$. **not ok to pass to the limit** $\nu \to 0$**!**. Indeed

$$\mathcal{F}_\nu(v, \varepsilon) = L_\nu v + \varepsilon \mathcal{N}(v) = 0, \quad L_\nu := \omega \cdot \partial_\varphi + \zeta \cdot \nabla - \nu\Delta \,.$$

$$L_\nu = \operatorname{diag}_{\substack{\ell \in \mathbb{Z}^d \\ j \in \mathbb{Z}^2 \setminus \{0\}}} \mathrm{i}(\omega \cdot \ell + \zeta \cdot j) + \nu |j|^2$$

one has $|\mathrm{i}(\omega \cdot \ell + \zeta \cdot j) + \nu|j|^2| \geq \nu|j|^2$, and hence L_ν is invertible (and gains two space derivatives) and has size $L_\nu^{-1} = O(\nu^{-1})$

$$\mathcal{F}_\nu(v, \varepsilon) = 0 \iff v = -\varepsilon L_\nu^{-1} \mathcal{N}(v) \,.$$

in order to implement a fixed point argumen one has to require $\varepsilon\nu^{-1} \ll 1$

Outline of the strategy

1. Inversion of the linearized Navier-Stokes operator at the Euler solution

- **Small divisors and Normal form reduction**
- **Uniform estimates w.r. to the viscosity ν.**
- **smallness condition on ε independent of the viscosity ν**

2. Construction of an approximate solution up to order $O(\nu^2)$.

The inverse of the linearized Navier-Stokes operator has size $O(\nu^{-1})$. We need an approximate solution up to order $O(\nu^2)$ of the form

$$v_{app} = v_e + \nu v_1 \, .$$

3. Fixed point argument

Implement a fixed point argument (in a neighborhood of v_{app}). Construct a solution

$$v \simeq v_{app} \quad \text{as} \quad \nu \to 0$$

The linearized Navier-Stokes operator

$$\mathcal{F}_\nu(v,\varepsilon) := \omega\cdot\partial_\varphi v + \zeta\cdot\nabla v - \nu\Delta v + \varepsilon\Big(\mathcal{U}(v)\cdot\nabla v - F(\varphi,x)\Big),$$

$$\mathcal{U}(v) := (-\Delta)^{-1}\mathcal{J}\nabla v \quad F := \nabla\times f.$$

We have $v_e(\varphi,x) = \mathrm{odd}(\varphi,x)$ is a **reversible solution** of the Euler equation

$$\mathcal{F}_0(v_e,\varepsilon) = \omega\cdot\partial_\varphi v_e + \zeta\cdot\nabla v_e + \varepsilon\Big(\mathcal{U}(v_e)\cdot\nabla v_e - F(\varphi,x)\Big) = 0$$

Linearize Navier-Stokes at u_e

$$\mathcal{L}_\nu = \mathcal{L}_e - \nu\Delta\,,$$
$$\mathcal{L}_e = \omega\cdot\partial_\varphi + (\zeta + \varepsilon u(\varphi,x))\cdot\nabla + \varepsilon\mathcal{R}\,,$$
$$u := \mathcal{U}(v_e), \quad \mathcal{R}[h] := \mathcal{U}[h]\cdot\nabla v_e$$

$\mathcal{L}_e$ is exactly the linearized Euler operator at u_e.

Reversible structure of the linearized Euler operator $\mathcal{L}_e$

$$\mathcal{L}_e : X\to Y, \quad \mathcal{L}_e : Y \to X\,,$$
$$X = \{v\in L^2 : v = \mathrm{even}(\varphi,x)\}, \quad Y := \{v\in L^2 : v = \mathrm{odd}(\varphi,x)\}\,.$$

Invertibility of $\mathcal{L}_\nu$: Neumann series approach fails!

$$\mathcal{L}_\nu = \mathcal{D}_\nu + \varepsilon \mathcal{P},$$

$$\mathcal{D}_\nu := \omega \cdot \partial_\varphi + \zeta \cdot \nabla - \nu \Delta = \mathrm{diag}_{\substack{\ell \in \mathbb{Z}^d \\ j \in \mathbb{Z}^2 \setminus \{0\}}} \mathrm{i}(\omega \cdot \ell + \zeta \cdot j) + \nu |j|^2$$

$$\mathcal{P} := u(\varphi, x) \cdot \nabla + \mathcal{R}$$

since

$$|\mathrm{i}(\omega \cdot \ell + \zeta \cdot j) + \nu |j|^2| \geq \nu |j|^2$$

then

$$\mathcal{D}_\nu^{-1} \sim \nu^{-1} (-\Delta)^{-1}$$

and

$$\mathcal{L}_\nu = \mathcal{D}_\nu \Big(\mathrm{Id} + \varepsilon \mathcal{D}_\nu^{-1} \mathcal{P} \Big) .$$

But

$$\varepsilon \mathcal{D}_\nu^{-1} \mathcal{P} = O(\varepsilon \nu^{-1}) .$$

In order to apply the Neumann series **one has to require** $\varepsilon \ll \nu$
this is not possible!

Key idea

Implement a normal form procedure that move the size of the perturbation from $O(\varepsilon)$ to $O(\varepsilon\nu)$

Normal form reduction of $\mathcal{L}_\nu$

Normal form Theorem

For any $\nu > 0$, for $\varepsilon \ll 1$ small enough (**independent of the viscosity ν**) there exists a a set $\mathcal{Q}_\varepsilon \subset \Omega$ (satisfying $|\mathcal{Q}_\varepsilon| \to |\Omega|$ as $\varepsilon \to 0$) such that for any value of the parameter $(\omega, \zeta) \in \mathcal{Q}_\varepsilon$ there exists a bounded, invertible and parity preserving map $\Phi \in H^s_0 \to H^s_0$ $(s \gg 0)$ such that the following holds.

$$\Phi^{-1}\mathcal{L}_e\Phi = \mathcal{D}_\infty\,,$$

$$\mathcal{L}_{\infty,\nu} := \Phi^{-1}\mathcal{L}_\nu\Phi = \Phi^{-1}(\mathcal{L}_e - \nu\Delta)\Phi = \mathcal{D}_\infty - \nu\Delta + \mathcal{R}_{\infty,\nu}$$

where the operator $\mathcal{D}_\infty$ is diagonal with purely imaginary eigenvalues and independent of ν

$$\mathcal{D}_\infty = \mathrm{diag}_{\ell,j}\mu_\infty(\ell,j)$$

with

$$\mu_\infty(\ell,j) = \mathrm{i}(\omega\cdot\ell + \zeta\cdot j + r_j^\infty), \quad r_j^\infty = O(\varepsilon|j|^{-1})\,.$$

The operator $\mathcal{R}_{\infty,\nu}$ is an **unbounded operator of order two and size $O(\varepsilon\nu)$** $\mathcal{R}_{\infty,\nu} \sim \varepsilon\nu\Delta$. More precisely

$$\|(-\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\|_{\mathcal{B}(H^s_0)} \lesssim_s \varepsilon\nu$$

Basic tool: pseudo-differential calculus

Classes of symbols and operators

Let $m \in \mathbb{R}$. We say that a smooth function

$$a : \mathbb{T}^d \times \mathbb{T}^2 \times \mathbb{R}^2 \to \mathbb{C}$$

is in the class $\mathcal{S}^m$ if for $s \gg 0$ (large enough) and $\alpha \in \mathbb{N}^2$,

$$\|\partial_\xi^\alpha a(\cdot,\cdot,\xi)\|_{H^s_{\varphi,x}} \lesssim_{s,\alpha} \langle \xi \rangle^{m-|\alpha|} . \tag{2}$$

An operator $\mathcal{A}$ is in the class $\mathcal{OPS}^m$ if there exists $a \in \mathcal{S}^m$ such that

$$\mathcal{A}u(x) = \mathrm{Op}(a)u(x) = \sum_{\xi \in \mathbb{Z}^2} a(\varphi, x, \xi)\widehat{u}(\xi)e^{\mathrm{i}x\cdot\xi} \quad \forall u \in C^\infty(\mathbb{T}^2)$$

Norms

If $\mathcal{A} \in \mathcal{OPS}^m$, we define for any $s \geq 0$, $\beta \in \mathbb{N}^2$, the norm

$$|\mathcal{A}|_{m,s,\beta} := \sup_{|\alpha| \leq \beta} \sup_{\xi \in \mathbb{R}^2} \|\partial_\xi^\alpha a(\cdot,\xi)\|_s \langle \xi \rangle^{-m+|\alpha|} < \infty$$

Normal form procedure

The linearized Navier-Stokes operator

$$\mathcal{L}_\nu := \mathcal{L}_e - \nu\Delta\,, \quad \mathcal{L}_e := \omega\cdot\partial_\varphi + \zeta\cdot\nabla + \varepsilon u(\varphi,x)\cdot\nabla + \varepsilon\mathcal{OPS}^{-1}$$

- **Pseudo-differential normal form.** Conjugate $\mathcal{L}_\nu$ to a diagonal normal form up to a **smoothing operator of size $O(\varepsilon)$** and up to an **unbounded term of size $O(\varepsilon\nu)$**.

$$\mathcal{P}_\nu = \omega\cdot\partial_\varphi + \mathcal{D} - \nu\Delta + \varepsilon\mathcal{OPS}^{-M} + \varepsilon\nu\mathcal{OPS}^{2}, \quad M \gg 0$$
$$\mathcal{D} := \mathrm{diag}_{j\neq 0}\mathrm{i}\mu_j, \quad \mu_j \in \mathbb{R}$$

- **Perturbative reducibility scheme (with weak non-resonance conditions).** Complete the diagonalization of the term $\varepsilon\mathcal{OPS}^{-M}$ by reducing quadratically its size and obtaining

$$\mathcal{L}_\nu^{(\infty)} = \mathcal{D}_\infty - \nu\Delta + \varepsilon\nu\mathcal{OPS}^{2}, \quad \mathcal{D}_\infty = \text{diagonal}$$

- **2nd Melnikov** $\quad |\omega\cdot\ell + \mu_j - \mu_{j'}| \geq \frac{\gamma}{\langle\ell\rangle^\tau |j|^\tau |j'|^\tau}, \quad (\ell,j,j') \neq (0,j,j)\,.$

Inversion of $\mathcal{L}_\nu$ (1)

Since $\mathcal{L}_\nu$ is conjugated to $\mathcal{L}_{\infty,\nu}$ it is enough to invert the latter one.

The diagonal operator $\mathcal{D}_\infty - \nu\Delta$

$$\mathcal{D}_\infty - \nu\Delta = \mathrm{diag}_{\ell,j}\mu_\infty(\ell,j) + \nu|j|^2\,.$$

Reversible structure $\Longrightarrow \mu_\infty(\ell,j)$ purely imaginary

Lower bounds for the eigenvalues

$$|\mu_\infty(\ell,j) + \nu|j|^2| \geq \nu|j|^2, \quad \forall(\ell,j) \in \mathbb{Z}^d \times (\mathbb{Z}^2 \setminus \{0\})$$

The lower bound $\nu|j|^2$ is a gain of two space derivatives

Estimate of $(\mathcal{D}_\infty - \nu\Delta)^{-1}$

One has that $(\mathcal{D}_\infty - \nu\Delta)^{-1} \sim \nu^{-1}(-\Delta)^{-1}$. More precisely

$$\|(\mathcal{D}_\infty - \nu\Delta)^{-1} \circ (-\Delta)\|_{\mathcal{B}(H^s_0)} \lesssim \nu^{-1}$$

Inversion of $\mathcal{L}_\nu$ (2)

Then we can invert **by Neumann series** and **uniformly w.r. to ν**, the operator

$$\mathcal{L}_{\infty,\nu} = \mathcal{D}_\infty - \nu\Delta + \mathcal{R}_{\infty,\nu}, \quad \|(-\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\|_{\mathcal{B}(H^s_0)} \lesssim_s \nu\varepsilon\,.$$

Indeed

$$\mathcal{L}_{\infty,\nu} = (\mathcal{D}_\infty - \nu\Delta) \circ \Big(\mathrm{Id} + (\mathcal{D}_\infty - \nu\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\Big)$$

One has

$$\begin{aligned}\|(\mathcal{D}_\infty - \nu\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\|_{\mathcal{B}(H^s_0)} &= \|(\mathcal{D}_\infty - \nu\Delta)^{-1} \circ (-\Delta) \circ (-\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\|_{\mathcal{B}(H^s_0)} \\ &\lesssim \|(\mathcal{D}_\infty - \nu\Delta)^{-1} \circ (-\Delta)\|_{\mathcal{B}(H^s_0)} \|(-\Delta)^{-1} \circ \mathcal{R}_{\infty,\nu}\|_{\mathcal{B}(H^s_0)} \\ &\lesssim \nu^{-1}\nu\varepsilon \lesssim \varepsilon\end{aligned}$$

As a consequence

Inversion of $\mathcal{L}_{\infty,\nu}, \mathcal{L}_\nu$.

For $\varepsilon \ll 1$ small enough, uniformly w.r. to $\nu > 0$, $\mathcal{L}_{\infty,\nu}, \mathcal{L}_\nu$ are invertible and $\mathcal{L}^{-1}_{\infty,\nu}\,,\ \mathcal{L}^{-1}_\nu \sim \nu^{-1}(-\Delta)^{-1}$. More precisely

$$\|\mathcal{L}^{-1}_{\infty,\nu} \circ (-\Delta)\|_{\mathcal{B}(H^s_0)}\,,\ \|\mathcal{L}^{-1}_\nu \circ (-\Delta)\|_{\mathcal{B}(H^s_0)} \lesssim \nu^{-1}\,.$$

Approximate solution (1)

$$\mathcal{F}_\nu(v) := (\omega \cdot \partial_\varphi + \zeta \cdot \nabla) v - \nu \Delta v + \varepsilon \Big(\mathcal{U}(v) \cdot \nabla v - F \Big),$$
$$\mathcal{U}(v) := (-\Delta)^{-1} \mathcal{J} \nabla v \,.$$

Goal

construct an approximate solution of the form $v_{app} = v_e + \nu v_1$ of $\mathcal{F}_\nu(v) = 0$ **up to order** $O(\nu^2)$. This is necessary to close a fixed point argument, since the inverse of the linearized Navier-Stokes operator at the Euler solution has size $\mathcal{L}_\nu^{-1} = O(\nu^{-1})$. Hence we determine $v_1(\varphi, x)$ in such a way that

$$\mathcal{F}_\nu(v_e + \nu v_1) = O(\nu^2) \,.$$

Invertibility of the linearized Euler operator $\mathcal{L}_e$.

the linearized Euler operator $\mathcal{L}_e$ is conjugated to the diagonal operator $\mathcal{D}_\infty = \mathrm{diag}_{\ell, j} \mu_\infty(\ell, j)$, $\mu_\infty(\ell, j) = \mathrm{i}(\omega \cdot \ell + \zeta \cdot \nabla) + O(\varepsilon |j|^{-1})$. By imposing

$$|\mu_\infty(\ell, j)| \geq \frac{\gamma}{\langle \ell, j \rangle^\tau}, \quad \gamma \ll 0, \quad \tau \gg 0$$

one gets that $\mathcal{L}_e$ is invertible (with loss of derivatives) and $\mathcal{L}_e^{-1} \in \mathcal{B}(H_0^{s+\tau}, H_0^s)$

Approximate solution (2)

Using that v_e is a solution of the forced Euler equation

$$\begin{aligned}
\mathcal{F}_\nu(v_e + \nu v_1) &= \mathcal{F}_\nu(v_e) + \nu D\mathcal{F}_\nu(v_e)[v_1] + O(\nu^2) \\
&= \omega \cdot \partial_\varphi v_e + \zeta \cdot \nabla v_e + \varepsilon\Big(\mathcal{U}(v_e)\cdot\nabla v_e - F\Big) \\
&+ \nu\Big(-\Delta v_e + \mathcal{L}_e[v_1] - \nu\Delta v_1\Big) + O(\nu^2) \\
&= \nu\Big(-\Delta v_e + \mathcal{L}_e[v_1]\Big) + O(\nu^2)
\end{aligned}$$

$O(\nu)$: Equation for v_1

$$\mathcal{L}_e[v_1] = \Delta v_e \implies v_1 := \mathcal{L}_e^{-1}\Delta v_e$$

Lemma

$s, \mu \gg 0$ and $v_e \in H_0^{s+\mu}$ be a solution of the forced Euler equation satisfying $\|v_e\|_{s+\mu} \lesssim \varepsilon^{\mathtt{a}}$, $\mathtt{a} \in (0,1)$. Then $v_{app} = v_e + \nu v_1 \in H_0^s$ satisfies

$$\|v_1\|_s \lesssim \varepsilon^{\mathtt{a}}\gamma^{-1}, \quad \|\mathcal{F}_\nu(v_{app})\|_s \lesssim \varepsilon^{\mathtt{a}}\gamma^{-1}\nu^2$$

Fixed point argument (1)

We look for solutions $v = v_{app} + \psi = v_e + \nu v_1 + \psi$ of

$$\mathcal{F}_\nu(v) = (\omega \cdot \partial_\varphi + \zeta \cdot \nabla)v - \nu\Delta v + \varepsilon\Big(\mathcal{N}(v) - F\Big) = 0$$

$$\mathcal{N}(v) := \mathcal{U}(v) \cdot \nabla v, \quad \mathcal{U}(v) := (-\Delta)^{-1}\mathcal{J}\nabla v$$

where

$$\psi \in \mathcal{B}_s(\nu) := \Big\{g \in H^s_0 : \|g\|_s \leq \nu\Big\}$$

Fixed point equation: $\psi = \mathcal{S}_\nu(\psi)$

$$\mathcal{S}_\nu(\psi) := -\mathcal{L}_\nu^{-1}\Big(\mathcal{F}_\nu(v_{app}) + \varepsilon\,\nu\, d\mathcal{N}(v_1)[\psi] + \varepsilon\mathcal{N}(\psi)\Big)$$

Lemma

For any $\nu > 0$, for $\varepsilon \ll 1$ small enough, the map $\mathcal{S}_\nu : \mathcal{B}_s(\nu) \to \mathcal{B}_s(\nu)$ is a contraction.

Fixed point argument (2)

$$\mathcal{S}_\nu(\psi) := -\mathcal{L}_\nu^{-1}\Big(\mathcal{F}_\nu(v_{app}) + \varepsilon\,\nu\, d\mathcal{N}(v_1)[\psi] + \varepsilon\mathcal{N}(\psi)\Big)\,.$$

If

$$\|\psi\|_s \leq \nu$$

Then

$$\|\mathcal{L}_\nu^{-1}\mathcal{F}_\nu(v_{app})\|_s \lesssim \nu^{-1}\varepsilon^{a}\gamma^{-1}\nu^2 \lesssim \varepsilon^{a}\gamma^{-1}\nu$$

$$\Big\|\mathcal{L}_\nu^{-1}\Big(\varepsilon\,\nu\, d\mathcal{N}(v_1)[\psi]\Big)\Big\|_s \lesssim \nu^{-1}\varepsilon\nu\|\psi\|_s \lesssim \varepsilon\nu$$

$$\Big\|\mathcal{L}_\nu^{-1}\Big(\varepsilon\,\mathcal{N}(\psi)\Big)\Big\|_s \lesssim \nu^{-1}\varepsilon\|\psi\|_s^2 \lesssim \varepsilon\nu$$

Then for $\varepsilon \ll 1$, one has $\|\mathcal{S}_\nu(\psi)\|_s \leq \nu$. **The contraction argument works uniformly w.r. to the viscosity**.

Thanks for the attention!

The Normal form transformation (1)

The normal form transformation Φ which diagonalizes $\mathcal{L}_e$ has the form

$$\Phi = \mathcal{A} \circ \mathcal{B}$$

where

- **Change of variable induced by a diffeomorphism of $\mathbb{T}^2$**

$$\mathcal{A} : h(\varphi, x) \mapsto h(\varphi, x + \alpha(\varphi, x)) \quad \text{with} \quad \|\alpha\|_s \lesssim \varepsilon$$

- **Close to the identity transformation**

$$\|\Psi - \mathrm{Id}\|_{\mathcal{B}(H^s_0)} \lesssim \varepsilon$$

Compute $\Psi^{-1}\nu\Delta\Psi$

One has indeed

$$\Psi^{-1}(-\nu\Delta)\Psi = -\nu\Delta + \varepsilon\nu\mathcal{OPS}^2$$

The Normal form transformation (2)

Conjugation of the viscous term by $\mathcal{A}$

One computes

$$\mathcal{A}^{-1}(-\nu\Delta)\mathcal{A} = -\nu\Delta + \sum_{i,j} b_{ij}(\varphi, x)\partial_{x_i x_j} + \sum_k c_k(\varphi, x)\partial_{x_k}$$

with $\quad \|b_{ij}\|_s, \|c_k\|_s \lesssim \varepsilon\nu\,.$

Conjugation of the viscous term by Ψ

Since

$$\Psi^{\pm 1} - \mathrm{Id} = O(\varepsilon)$$

one has

$$\Psi^{-1}(\nu\Delta)\Psi - \nu\Delta \sim \varepsilon\nu\Delta$$

or more precisely

$$\left\|(-\Delta)^{-1} \circ \left(\Psi^{-1}(\nu\Delta)\Psi - \nu\Delta\right)\right\|_{\mathcal{B}(H^s_0)} \lesssim \varepsilon\nu$$

3D Navier-Stokes and Euler

$$\mathcal{F}_\nu : H^{s+2}(\mathbb{T}^d \times \mathbb{T}^3, \mathbb{R}^3) \to H^s(\mathbb{T}^d \times \mathbb{T}^3, \mathbb{R}^3)$$

$$\mathcal{F}_\nu(v) := \omega \cdot \partial_\varphi v + \zeta \cdot \nabla v - \nu \Delta v + \varepsilon \Big(\mathcal{U}(v) \cdot \nabla v - v \cdot \nabla \mathcal{U}(v) - F(\varphi, x) \Big),$$

$$\mathcal{U}(v) := (-\Delta)^{-1}(\nabla \times v), \quad F := \nabla \times f.$$

The linearized operator

$$\mathcal{L}_\nu := \mathcal{L}_e - \nu \Delta, \quad \mathcal{L}_e = \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \varepsilon a(\varphi, x) \cdot \nabla + \varepsilon \mathcal{R},$$

$$a := \mathcal{U}(v_e) = (-\Delta)^{-1}(\nabla \times v), \quad \mathcal{R}[h] := \mathcal{U}(h) \cdot \nabla v_e - h \cdot \nabla \mathcal{U}(v_e) - v_e \cdot \nabla \mathcal{U}(h).$$

$\mathcal{L}_e$ is reversible like in $2D$

Remark: 3×3 blocks

Unperturbed normal form operator

$$\zeta \cdot \nabla : H_0^1(\mathbb{T}^3, \mathbb{R}^3) \to L_0^2(\mathbb{T}^3, \mathbb{R}^3),$$

$$\mathrm{i}\zeta \cdot j, \quad j \in \mathbb{Z}^3 \setminus \{0\} \quad \textbf{eigenvalues}$$

For any $j \in \mathbb{Z}^3 \setminus \{0\}$, we have **multiplicity three**, indeed the eigenspace E_j of the eigenvalue $(\mathrm{i}\zeta \cdot j)$ is given by

$$E_j := \mathrm{span}\Big\{(e^{\mathrm{i}j\cdot x}, 0, 0), (0, e^{\mathrm{i}j\cdot x}, 0), (0, 0, e^{\mathrm{i}j\cdot x})\Big\}$$

multiplicity 3 $\Longrightarrow$ 3×3 blocks

3×3 block-diagonal operators

$\mathcal{Z} : L^2(\mathbb{T}^3, \mathbb{R}^3) \to L^2(\mathbb{T}^3, \mathbb{R}^3)$ is 3×3 block diagonal if

$$\mathcal{Z}u(x) = \sum_{\xi \in \mathbb{Z}^3} Z(\xi)\widehat{u}(\xi)e^{\mathrm{i}x\cdot\xi}, \quad Z(\xi) \in \mathrm{Mat}_{3\times3}(\mathbb{C}) .$$

Normal form Theorem

Theorem

Let $\nu > 0$. For any $s \gg 0$ large enough and $\varepsilon \ll 1$, small enough, there exists a Borel set $\mathcal{O}_\varepsilon \subseteq \Omega$ such that for any $(\omega, \zeta) \in \mathcal{O}_\varepsilon$, there exists a bounded, invertible, parity preserving map $\Phi : H_0^s \to H_0^s$ such that

$$\Phi^{-1}\mathcal{L}_e\Phi = \mathcal{D}_\infty$$

$$\Phi^{-1}\mathcal{L}_\nu\Phi = \mathcal{D}_\infty - \nu\Delta + \varepsilon\nu\mathcal{OPS}^2$$

where $\mathcal{D}_\infty$ is a time independent, 3×3 **block-diagonal operator** with blocks $D_\infty(\ell, j)$. Every block $D_\infty(\ell, j)$ has eigenvalues $\mu_1(\ell, j), \mu_2(\ell, j), \mu_3(\ell, j)$

$$\mu_i(\ell, j) = \mathrm{i}(\omega \cdot \ell + \zeta \cdot j) + O(\varepsilon|j|^{-1}), \quad i = 1, 2, 3$$

Problem: The reversibility structure does not imply that $\mu_i(j)$ are purely imaginary. One has that

$$Re(\mu_i(\ell, j)) = O(\varepsilon|j|^{-1}).$$

Eigenvalues of $\mathcal{D}_\infty - \nu\Delta$

$$\mu_i(\ell, j) + \nu|j|^2, \quad (\ell, j) \in \mathbb{Z}^d \times (\mathbb{Z}^3 \setminus \{0\})$$

Lower bound

$$|\mu_i(\ell, j) + \nu|j|^2| \geq \left| Re\Big(\mu_i(\ell, j) + \nu|j|^2\Big) \right| \geq \nu|j|^2 + O(\varepsilon|j|^{-1})$$

Problem

$$|\mu_i(\ell, j) + \nu|j|^2| \geq C\nu|j|^2$$

if $\varepsilon \ll \nu$

Reduction to constant coefficients of the highest order

Feola-Giuliani-M.-Procesi (2019),Baldi-M. (2021)

Transport operator

$$\mathcal{T} := \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \varepsilon a(\varphi, x) \cdot \nabla \,,$$

$$a = \mathrm{even}(\varphi, x), \quad \int_{\mathbb{T}^2} a(\varphi, x)\, dx = 0, \quad \mathrm{div}(a) = 0$$

Theorem

Let $\gamma \in (0,1)$, $\tau \gg 0$

$$(\omega, \zeta) \in DC(\gamma, \tau) := \left\{ (\omega, \zeta) \in \Omega : |\omega \cdot \ell + \zeta \cdot j| \geq \frac{\gamma}{\langle \ell, j \rangle^\tau}, \quad \forall (\ell, j) \in \mathbb{Z}^{\nu+3} \setminus \{0\} \right\}.$$

For $\varepsilon \gamma^{-1} \ll 1$ and for any $(\omega, \zeta) \in DC(\gamma, \tau)$, there exists an invertible diffeomorphism of the torus $x \mapsto x + \alpha(\varphi, x)$ such that the operator

$$\mathcal{A} h(\varphi, x) := h(\varphi, x + \alpha(\varphi, x))$$

satisfies

$$\mathcal{A}^{-1} \mathcal{T} \mathcal{A} = \omega \cdot \partial_\varphi + \zeta \cdot \nabla.$$

The zero-th order term

The full conjugation $\mathcal{A}^{-1}\mathcal{L}_\nu\mathcal{A}$

For any $\nu > 0$, for $\varepsilon\gamma^{-1} \ll 1$ small enough, for any $(\omega, \zeta) \in DC(\gamma, \tau)$, one has that

$$\mathcal{L}_e^{(1)} := \mathcal{A}^{-1}\mathcal{L}_e\mathcal{A} = \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \varepsilon\mathcal{R}^{(1)}$$

where the reversible operators $\mathcal{R}^{(1)} \in \mathcal{OPS}^{-1}$. Moreover, the linearized Navier-Stokes operator $\mathcal{L}_\nu$ satisfies

$$\begin{aligned}\mathcal{L}_\nu^{(1)} &= \mathcal{A}^{-1}\mathcal{L}_\nu\mathcal{A} = \mathcal{A}^{-1}\Big(\mathcal{L}_e - \nu\Delta\Big)\mathcal{A} \\ &= \mathcal{L}_e^{(1)} - \nu\Delta + \mathcal{R}_\nu^{(1)}\end{aligned}$$

where $\mathcal{R}_\nu^{(1)} \sim \nu\varepsilon(-\Delta)$, more precisely

$$\|(-\Delta)^{-1} \circ \mathcal{R}_\nu^{(1)}\|_{\mathcal{B}(H_0^s)} \lesssim \varepsilon\nu$$

Since $\mathcal{A} : h(\varphi, x) \mapsto h(\varphi, x + \alpha(\varphi, x))$ and $\alpha = O(\varepsilon)$, one can compute

$$\mathcal{A}^{-1}(-\Delta)\mathcal{A} = -\Delta + \sum_{i,j} b_{ij}(\varphi, x)\partial_{x_i x_j} + \sum_k c_k(\varphi, x)\partial_{x_k}$$

with $\quad b_{ij}, c_k = O(\varepsilon)\,.$

The lower order terms

$$\mathcal{L}_e^{(1)} = \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \mathcal{R}^{(1)}, \quad \mathcal{R}^{(1)} = \mathrm{Op}(R^{(1)}) \in \mathcal{OPS}^{-1}$$

Normalize the order -1. We consider $\Phi = \mathrm{Id} + \varepsilon \mathcal{S}$, $\mathcal{S} = \mathrm{Op}(S) \in \mathcal{OPS}^{-1}$. Then

$$\Phi^{-1}\mathcal{L}_e^{(1)}\Phi = \omega \cdot \partial_\varphi + \zeta \cdot \nabla + \varepsilon \mathrm{Op}\Big((\omega \cdot \partial_\varphi + \zeta \cdot \nabla)S + R_2\Big) + \varepsilon \mathcal{OPS}^{-2}$$

We have that $\mathcal{R}_2\mathcal{S} \in \mathcal{OPS}^{-2}$ and we solve for $(\omega, \zeta) \in DC(\gamma, \tau)$

Homological equation

$$(\omega \cdot \partial_\varphi + \zeta \cdot \nabla)S + R^{(1)} = \langle R^{(1)} \rangle_{\varphi,x}$$

$$S(\varphi, x, \xi) = -\sum_{(\ell,j)\neq(0,0)} \frac{\widehat{R}^{(1)}(\ell, j, \xi)}{\mathrm{i}(\omega \cdot \ell + \zeta \cdot j)} e^{\mathrm{i}(\ell\cdot\varphi + j\cdot x)}$$

Conjugation of $\mathcal{L}_\nu^{(1)}$

Since $\Phi^{\pm 1} - \mathrm{Id} = O(\varepsilon)$, one has

$$\Phi^{-1}\Big(-\nu\Delta + \mathcal{R}_\nu^{(1)}\Big)\Phi = -\nu\Delta + O(\varepsilon\nu\Delta)$$

KAM reducibility

Starting point for the reducibility

$$\mathcal{P} := \omega \cdot \partial_\varphi + \mathcal{N} + \varepsilon \mathcal{R}\,, \quad \mathcal{P}_\nu := \mathcal{P} - \nu\Delta + \mathcal{Q}_\nu$$

$$\mathcal{N} := \zeta \cdot \nabla + \varepsilon \mathcal{Z} = \mathrm{diag}_{j\neq 0}\mathrm{i}\Big(\zeta \cdot j + O(\varepsilon |j|^{-1})\Big)\,,$$

$$\mathcal{R} \in \mathcal{OPS}^{-M} \quad \text{for} \quad M \gg 0, \quad \|(-\Delta)^{-1} \circ \mathcal{Q}_\nu\|_{\mathcal{B}(H^s_0)} \lesssim \varepsilon\nu\,.$$

Theorem: reducibility of $\mathcal{P}$ (neglect the viscous part)

For $\varepsilon \ll 1$ small enough and for a large set of parameters (ω, ζ), there exists an invertible, parity preserving map $\Phi_\infty \in \mathcal{B}(H^s_0)$ such that $\|\Phi^{\pm 1} - \mathrm{Id}\|_{\mathcal{B}(H^s_0)} \lesssim \varepsilon$ and

$$\Phi_\infty^{-1}\mathcal{P}\Phi_\infty = \mathcal{D}_\infty, \quad \mathcal{D}_\infty = \mathrm{diag}_{\ell,j}\mu_\infty(\ell, j)$$

$$\mu_\infty(\ell, j) = \mathrm{i}\Big(\omega \cdot \ell + \zeta \cdot j + O(\varepsilon |j|^{-1})\Big)$$

Conjugation of $\mathcal{P}_\nu$

$$\Phi_\infty^{-1}\mathcal{P}_\nu\Phi_\infty = \Phi_\infty^{-1}\mathcal{P}\Phi_\infty - \Phi_\infty^{-1}\Big(-\nu\Delta + \mathcal{Q}_\nu\Big)\Phi_\infty$$

$$= \mathcal{D}_\infty - \nu\Delta + O(\varepsilon\nu\Delta)$$

The reducibility step

We look for a transformation close to the identity $\Phi = \mathrm{Id} + \varepsilon \Psi$ in order to move the perturbation at the order $O(\varepsilon^2)$.

$$\mathcal{P}_+ = \Phi^{-1} \mathcal{P} \Phi = \omega \cdot \partial_\varphi + \mathcal{N} + \varepsilon(\omega \cdot \partial_\varphi \Psi + [\mathcal{N}, \Psi] + \mathcal{R}) + O(\varepsilon^2)$$

- **Homological equation** $\omega \cdot \partial_\varphi \Psi + [\mathcal{N}, \Psi] + \mathcal{R} =$ diagonal operator
- **Matrix representation** $\mathbb{T}^d \to \mathcal{B}\Big(L^2(\mathbb{T}^2)\Big), \quad \varphi \mapsto \Psi(\varphi), \, \varphi \mapsto \mathcal{R}(\varphi)$

$$\Psi \equiv (\widehat{\Psi}(\ell)_j^{j'})_{\ell \in \mathbb{Z}^d, j, j' \in \mathbb{Z}^2}, \quad \mathcal{R} \equiv (\widehat{\mathcal{R}}(\ell)_j^{j'})_{\ell \in \mathbb{Z}^d, j, j' \in \mathbb{Z}^2}$$

- **Solution** $\widehat{\Psi}(\ell)_j^{j'} = -\dfrac{\widehat{\mathcal{R}}(\ell)_j^{j'}}{\mathrm{i}(\omega \cdot \ell + \mu_j - \mu_{j'})}, \quad (\ell, j, j') \neq (0, j, j)$
- **Second Melnikov conditions**

$$|\omega \cdot \ell + \mu_j - \mu_{j'}| \geq \frac{\gamma}{\langle \ell \rangle^\tau |j|^\tau |j'|^\tau}, \quad \forall (\ell, j, j') \neq (0, j, j)$$