# Vortex lines, anomalous dissipation, and intermittency

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SITE Talk series,
February 22

# Abstract



## Abstract

We develop a theory of circulation statistics in strong turbulence ($\nu \to 0$ in the NS equation), treated as a degenerate fixed point of a Hopf equation. We use spherical Clebsch variables to parametrize vorticity in the stationary singular (weak) Euler flow, characterized by two winding numbers. The singular vortex line is regularized by matching Burgers vortex. We compute anomalous dissipation, helicity, and Hamiltonian for our singular flow in the limit of strong turbulence. The Hamiltonian shows explicit dependence on the logarithm of the Reynolds number, leading to expansion in inverse powers of this logarithm. This leads to certain multi-fractal properties of our solution.

# Introduction



Richard Feynman wrote half a century ago in his famous "Lectures in Physics"

*"there is a physical problem that is common to many fields, that is very old, and that has not been solved. It is not the problem of finding new fundamental particles, but something left over from a long time ago—over a hundred years. Nobody in physics has really been able to analyze it mathematically satisfactorily in spite of its importance to the sister sciences. It is the analysis of circulating or turbulent fluids."*

# Introduction



The turbulence problem looks deceptively simple: find the limit of the solution of the Navier-Stokes equations when viscosity goes to zero at fixed energy flow into the system.

**Navier-Stokes:** $$\partial_t \vec{v} = -(\vec{v} \cdot \vec{\nabla})\vec{v} - \vec{\nabla} p + \nu \vec{\nabla}^2 \vec{v}; \quad (1)$$

**incompressibility:** $$\vec{\nabla} \cdot \vec{v} = 0; \quad (2)$$

In this limit, the Navier-Stokes equation tends to the Euler equation everywhere except some singular regions: vortex sheets and vortex lines, where large Laplacian could compensate the factor of $\nu$.

# Energy balance



The energy balance on a stationary solution of Navier-Stokes equations can be written as follows

**energy balance:** $$0 = \partial_t \int_V d^3 r \frac{1}{2} v^2 = -\mathcal{E}_d + \mathcal{E}_p; \quad (3)$$

**dissipation:** $$\mathcal{E}_d = \nu \int_V d^3 r \vec{\omega}^2; \quad (4)$$

**energy pumping:** $$\mathcal{E}_p = \int_{\partial V} J_n dS; \quad (5)$$

**energy current:** $$\vec{J} = -\vec{v}\left(p + \frac{1}{2}\vec{v}^2\right) + \nu \vec{\omega} \times \vec{v}; \quad (6)$$

**vorticity:** $$\vec{\omega} = \vec{\nabla} \times \vec{v}; \quad (7)$$

These singular regions with infinite $\vec{\omega}$ may lead to anomalous dissipation: a finite limit of the integral in $\mathcal{E}_d$ at $\nu \to 0$.

# Anomalous dissipation



This anomalous dissipation was present in analytic solutions of the Navier-Stokes pioneered by Burgers [1], [2] and generalized by Townsend[3].

**sheet:**
$$\vec{\omega} = 2\vec{n} \times \Delta\vec{v} \frac{\exp\left(-\frac{z^2}{2w^2}\right)}{w\sqrt{2\pi}}; \quad (8)$$

**width:**
$$w = \sqrt{\nu/(S_{xx} + S_{yy})}; \quad (9)$$

**tube:**
$$\vec{\omega} = \vec{n} \frac{\Gamma \exp\left(-\frac{x^2+y^2}{2w^2}\right)}{2\pi w}; \quad (10)$$

**width:**
$$w = \sqrt{\nu/S_{zz}}; \quad (11)$$

**direction:**
$$\vec{n} = (0, 0, 1); \quad (12)$$

where $\Delta\vec{v}$ is velocity gap, $\Gamma$ is velocity circulation, $S_{ij}$ is a background strain. The surface is approximated by an $xy$ plane, and the loop by a straight line in $z$ direction.

# Anomalous dissipation



The anomalous dissipation in singular regions was observed in DNS [4].

Three problems remained.

1. Find a matching Euler flow with a singular vortex line.
2. Compute anomalous contributions to Hamiltonian, Dissipation and other observables.
3. Understand the nature of spontaneous stochastization and find an analog of the Gibbs distribution in turbulence.

Here is where the microscopic turbulence theory got stuck in the middle of the previous century; most of the subsequent progress was phenomenological.

# Kelvinon

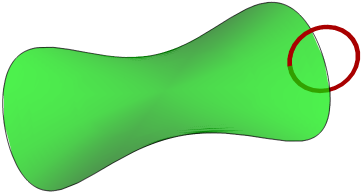

Figure: Kelvinon cycles. The $\beta$ cycle (red) around the $\alpha$ cycle (black) of the vortex sheet (green).

# Kelvinon



This singular flow is parametrized by a three-component unit vector $\vec{s} : \vec{s}^2 = 1$, or a point on a sphere $S_2$, and an arbitrary factor $Z$, which is a global constant in Euler flow.

$$\text{scale factor:} \quad \vec{\omega} = Z\tilde{\omega};\ \vec{v} = Z\tilde{v}; \tag{13}$$

$$\text{omega form:} \quad \tilde{\omega} = e_{abc} s_a \vec{\nabla} s_b \times \vec{\nabla} s_c; \tag{14}$$

$$\text{velocity:} \quad \tilde{v} = \vec{\nabla}\Phi - \vec{\nabla} \times \vec{\Psi} \tag{15}$$

$$\text{harmonic potential:} \quad \vec{\nabla}^2 \Phi = 0; \tag{16}$$

$$\text{Neumann at } S\text{:} \quad \partial_n \Phi(\vec{r}) = \left(\vec{\nabla} \times \vec{\Psi}(\vec{r})\right)_n; \tag{17}$$

$$\text{Biot-Savart:} \quad \vec{\Psi}(\vec{r}) = \int d^3 r' \frac{\tilde{\omega}(\vec{r}')}{4\pi|\vec{r} - \vec{r}'|}; \tag{18}$$

The flow belongs to a certain topological class with infinite vorticity at a closed line $C$ and a velocity gap at a surface $S$ bounded by this loop $C$.

# Mapping physical space onto target space

The stationary Euler equation for the Clebsch field is passive advection:

$$\vec{v} \cdot \vec{\nabla} s_i = 0; \tag{19}$$

The field $\vec{s}(\vec{r})$ maps our physical space $R_3$ except a discontinuity surface $S_C$ onto the two-dimensional target space $S_2$.
The canonical variables $\phi_1, \phi_2$ are two coordinates on a sphere related to Euler angles.

$$\phi_1 = Z s_3; \tag{20}$$

$$\phi_2 = \arg\left(s_1 + \imath\, s_2\right); \tag{21}$$

$$\vec{\omega} = \vec{\nabla}\phi_2 \times \vec{\nabla}\phi_1; \tag{22}$$

$$\vec{v} = -\phi_2 \vec{\nabla}\phi_1 + \vec{\nabla}\phi_3; \tag{23}$$

$$\vec{\nabla}^2 \phi_3 = \vec{\nabla} \cdot \left(\phi_2 \vec{\nabla}\phi_1\right); \tag{24}$$

The circulation around the loop $C$ equals the area inside an oriented curve $\gamma : \vec{s}(C)$ on a sphere.

# Mapping physical space onto target space



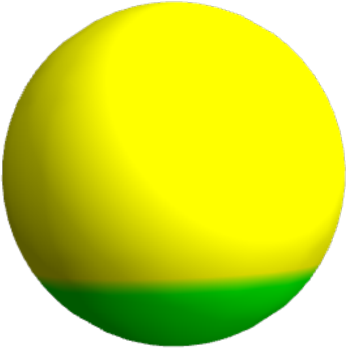

Figure: Surface $S_C$ mapped on a green region on $S_2$ where $s_3 < \cos\lambda$

# Singularities



The singularities at the surface and the loop arise because of the phase gap of the complex field

$$\psi = s_1 + \imath\, s_2. \tag{25}$$

When the point $\vec{r}$ encircles the loop at any point ($\beta$ cycle), this field acquires the phase $2\pi l$, and when it goes along $C$ ($\alpha$ cycle), it acquires the phase $2\pi m$ with some integer $l, m$.
This phase gap corresponds to the vector $\vec{s}(\vec{r})$ going $l$ or $m$ times around the horizontal circle on a sphere $S_2$.
Velocity diverges at the loop to provide finite circulation $\Gamma_\beta$, matching a Burgers vortex

$$v_\beta \to \frac{\Gamma_\beta}{2\pi\rho}; \tag{26}$$

in cylindrical coordinates $\rho, \beta, z$ with $z$ axis directed along the loop tangent vector at this point.

# Singularities



A tangent discontinuity of velocity at the surface $S_C \setminus C$ comes from the delta function in vorticity. The boundary values are

$$\vec{\omega} \to 2\pi l \delta(z) \vec{\sigma} \times \vec{\nabla}\phi_1; \tag{27}$$

$$\vec{v}^+ - \vec{v}^- = 2\pi l \vec{\nabla}\phi_1; \tag{28}$$

$$\phi_1^+ = \phi_1^-; \tag{29}$$

$$\phi_2^+ = \phi_2^- - 2\pi l; \tag{30}$$

$$\omega_n^+ = \omega_n^- = -\vec{\sigma} \cdot \vec{\nabla}\phi_2^\pm \times \vec{\nabla}\phi_1^\pm \tag{31}$$

Here $\vec{\sigma}$ is the local normal vector to the surface, and $z$ is the normal coordinate at this point.

# Helicity

The local limit of the Burgers vortex is the delta function in the plane normal to the local direction of the loop.

The global formula generalizing this delta function is

$$\vec{\omega}_C(\vec{r}) = \oint_C d\vec{r}_1 \Gamma_\beta \delta^3(\vec{r} - \vec{r}_1) \tag{32}$$

In our case, **the circulation $\Gamma_\beta$ is constant** (does not vary at the loop).

This is a consequence of the Euler equation (19) for the Clebsch field. The Helicity integral for this singular loop

$$\mathcal{H}_C = \int d^3 r \vec{\omega}_C \cdot \vec{v} = \oint_C d\vec{r}_1 \cdot \vec{v} \Gamma_\beta = \Gamma_\alpha \Gamma_\beta \tag{33}$$

# Helicity

Depending on the signs of winding numbers $m, l$, this oriented region would be either the upper $\Omega_+$ or the lower $\Omega_-$ part of the sphere $S_2$, separated by the horizontal circle $\theta = \lambda$.

Note that both sides of the surface are mapped on the same region ($\Omega_+$ or $\Omega_-$), with precisely the same area, providing the same circulation. The choice between $\Omega_\pm$ depends upon the signs of $m, l$.

There are four distinct possibilities

$$\sigma_m \equiv \operatorname{sign} m = \pm 1; \tag{34}$$

$$\sigma_l \equiv \operatorname{sign} l = \pm 1; \tag{35}$$

# Helicity

The circulation at the $\alpha, \beta$ cycles are proportional to these winding numbers.

$$\Gamma_\alpha = \oint_\alpha v_\alpha dr_\alpha = \int_{S_C^+} d\vec{\sigma} \cdot \vec{\omega}^+ = \int_{\Omega_+} d\phi_1 \wedge d\phi_2 = 2\pi m Z(1 + \sigma_m \cos\lambda) \tag{36}$$

$$\Gamma_\beta = \oint_\beta v_\alpha dr_\alpha = \int_{\Omega_-} d\phi_1 \wedge d\phi_2 = 2\pi l Z(1 + \sigma_l \cos\lambda); \tag{37}$$

We took advantage of the constant boundary value

$$s_3(C) = \cos\lambda. \tag{38}$$

Note that changing the sign of each number changes the value of the circulation and its sign.

# Helicity

There is also a contribution to helicity from the vortex surface.

The global generalization of the Burgers-Townsend formula for the singular vorticity inside the vortex sheet reads

$$\vec{\omega}_S(\vec{r}) = \int_{\vec{r}_1 \in S} d\vec{\sigma}(\vec{r}_1) \times \Delta \vec{v}(\vec{r}_1) \delta^3(\vec{r} - \vec{r}_1); \quad (39)$$

Substituting this into the helicity integral and using (28), we get

$$\mathcal{H}_S = \pi l Z^2 \int_{\vec{r}_1 \in S} d\vec{\sigma}(\vec{r}_1) \times \vec{\nabla} s_3(\vec{r}_1) \cdot \left( \tilde{v}(\vec{r}_1^{\,+}) + \tilde{v}(\vec{r}_1^{\,-}) \right); \quad (40)$$

# Energy balance fixes normalization constant Z



The free parameter $Z$ is a constant describing our **solution in the vicinity of our Kelvinon**. Other vortex structures are assumed to be far away, outside this volume.

The energy balance $\partial_t H = 0$, valid for any stationary solution of the Navier-Stokes equation, **relates this parameter to the energy flow $\mathcal{E}_p$** through the infinite boundary $\partial V$ of the volume $V$ occupied by vorticity of this solution.

**balance:** $$-Z^3\Sigma + \mathcal{E}_p = 0; \tag{41}$$

**anomaly:** $$\Sigma = \frac{\mu}{2}\oint_C |d\vec{r}|\tilde{S}_{tt}; \tag{42}$$

**parameter:** $$\mu = \pi l^2 (1 + \sigma_l \cos\lambda)^2 \tag{43}$$

# Energy balance fixes normalization constant Z

We introduce conventional energy flux per unit volume

$$\epsilon = \frac{\mathcal{E}}{V(C)} \tag{44}$$

where $V(C) \sim |C|^3$ is the volume occupied by our Kelvinon, assuming vorticity decays at the scale of distances $r \sim |C|$.

This assumption is valid at the scales $|C|$ below the intrinsic scale $w$ of the Kelvinon. In this limit, only one scale $|C|$ is left, which justifies the K41 dimensional analysis.

Below, we see that even in this region (lower part of the inertial range), logarithmic terms are modifying the K41 scaling laws.

# Energy balance fixes normalization constant Z



This leads us to the anomalous Hamiltonian:

$$Z = \left(\frac{\epsilon V(C)}{\Sigma}\right)^{\frac{1}{3}}; \tag{45}$$

$$H_R = Z^2 \left(\int_{|r-C|>R} \frac{\tilde{v}^2}{2} + 4\mu|C| \log R\right) + 2Z^2\mu \oint_C |d\vec{r}| \left(\gamma + \log \frac{Z\tilde{S}_{tt}}{4\nu}\right); \tag{46}$$

The spurious parameter $R$, which formally enters this Hamiltonian, drops from it in the limit $R \to 0$. The singular terms in the integral over external region $|\vec{r} - \vec{C}| > R$ exactly cancel the logarithmic term $\log R$, leaving a finite remainder.

# Asymptotic freedom

Let us find the asymptotic solution to the minimization problem at large Reynolds number $\frac{Z}{\nu}$.

We are going to bypass the hard problem of finding the spatial shape of the Clebsch field from the passive convection (19) with nonlinear velocity

$$\frac{\delta \tilde{H}_{Euler}}{\delta \phi} = \tilde{v} \cdot \vec{\nabla} s_3 = 0; \tag{47}$$

$$\frac{\delta \tilde{H}_{Euler}}{\delta s_3} = -\tilde{v} \cdot \vec{\nabla} \phi = 0; \tag{48}$$

which holds in the region $|\vec{r} - C| \geq R$ where Euler equation applies.

That includes the matching region $|\vec{r} - C| = R$ where the Burgers solution also holds. Here

$$\tilde{H}_{Euler} = \int_{|\vec{r}-C|>R} \frac{\tilde{v}^2}{2} \tag{49}$$

# Asymptotic freedom

The solution to this problem is needed for the subleading terms in the asymptotic expansion in inverse powers of the logarithm of Reynolds number. The relevant variable in the leading order of this expansion is the constant boundary value $s_3(C) = \cos\lambda$, which has to be determined by the minimization of the Hamiltonian.

Variation of the Euler part $\tilde{H}_{Euler}$ yields zero by (48)

$$\frac{\partial \tilde{H}_{Euler}}{\partial s_3(C)} = \oint_C |d\vec{r}| \frac{\delta \tilde{H}_{Euler}}{\delta s_3(\vec{r})} = 0 \tag{50}$$

We are left with derivatives of $Z$ and derivatives of the anomalous term

$$\frac{\partial H_R}{\partial s_3(C)} \to 2H_R \frac{\partial \log Z}{\partial s_3(C)} + \\ 4Z^2 l^2 \sigma_l (1 + \sigma_l \cos\lambda)|C| \log \frac{Z}{\nu}; \tag{51}$$

# Asymptotic freedom

Using

$$\frac{\partial \log Z}{\partial s_3(C)} = \frac{-2\sigma_l}{3(1+\sigma_l \cos\lambda)}; \tag{52}$$

$$H_R \to Z^2 \left( \tilde{H}_{Euler} + 2\mu |C| \log \frac{Z}{\nu} \right) \tag{53}$$

and dropping $O(1)$ terms added to $\log \frac{Z}{\nu}$ we obtain relation between $\mu$ and $Z$

$$\mu = \frac{h}{\log \frac{Z}{\nu}} \tag{54}$$

$$h = \frac{\tilde{H}_{Euler}}{|C|}. \tag{55}$$

# Asymptotic freedom



By dimensional counting $\tilde{H}_{Euler} \propto |C|$, so that $h = \mathit{const}$.

Combining this relation with the equation for $Z$, we find the implicit equation

$$|C| = \mathit{const}\; Z^{\frac{3}{4}} \left(\log Z\right)^{\frac{1}{4}}; \tag{56}$$

This corresponds to so-called beta function for running coupling constant $g = \frac{\mu}{h} = \frac{1}{\log \frac{Z}{\nu}}$

$$\frac{\partial g}{\partial \log |C|} = -\frac{4g^2}{3+g} \tag{57}$$

Our running coupling constant decreases inversely proportional to the logarithm of the Reynolds number.

# References



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