Time-Periodic Weighted L^p Estimates

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Prelude

Remark (Fourier and Poincaré inequality)

If $u : \mathbb{R} \to \mathbb{R}$ is 2π-periodic and smooth, then

$$\widehat{\partial_t u}(k) = ik\widehat{u}, \quad \text{where } \widehat{u}(k) := \int_0^{2\pi} u(t)e^{-ikt}\,\frac{\mathrm{d}t}{2\pi}.$$

$\Rightarrow$ if $\widehat{u}(0) = \int_0^{2\pi} u(t)\,\frac{\mathrm{d}t}{2\pi} = 0$, Plancherel gives

$$\|u\|^2_{L_2(0,2\pi)} = \sum_{k\in\mathbb{Z}} |\widehat{u}(k)|^2 \leq \sum_{k\in\mathbb{Z}} k^2|\widehat{u}(k)|^2 = \|\partial_t u\|^2_{L_2(0,2\pi)},$$

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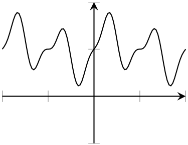

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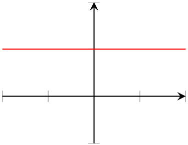

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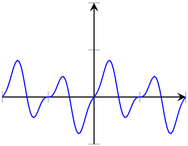

✗ no Poincaré ✗ no Poincaré ✓ Poincaré

Prelude II: Weighted, Anisotropic, Mixed-Norm Spaces

- Measure space $(S, \mathcal{A}, \mu)$. Weights $W(S)$: $w : S \to (0, \infty)$ measurable.

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$$\|f\|_{L_{\mathbf{p}}(S,\mathbf{w};E)} := \left(\int_{S_2} \left(\int_{S_1} \|f(x)\|_E^{p_1} w_1(x_1)\, \mathrm{d}\mu_1(x_1) \right)^{p_2/p_1} w_2(x_2)\, \mathrm{d}\mu_2(x_2) \right)^{1/p_2} .$$

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- anisotropic length on $\mathbb{R} \times \mathbb{R}^n$: For $\mathbf{a} \in (0, \infty)^2$, let $|\tau, \xi|_{\mathbf{a}} := (|\tau|^{\frac{2}{a_1}} + |\xi|^{\frac{2}{a_2}})^{1/2}$.

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Definition (Bessel Potential and Triebel-Lizorkin Spaces)

Let $\mathbf{w} \in W(\mathbb{R}) \times W(\mathbb{R}^n)$, $\mathbf{a} \in (0, \infty)^2$, $\mathbf{p} \in [1, \infty)^2$, $q \in [1, \infty)$, $s \in \mathbb{R}$.

$$H_{\mathbf{p}}^{s,\mathbf{a}}(\mathbb{R} \times \mathbb{R}^n, \mathbf{w}) := \{ f \in \mathscr{S}'(\mathbb{R} \times \mathbb{R}^n) \mid \mathscr{F}^{-1}\left[|\tau, \xi|_{\mathbf{a}}^s \widehat{f}\right] \in L_{\mathbf{p}}(\mathbb{R} \times \mathbb{R}^n, \mathbf{w}) \}$$
$$F_{\mathbf{p},q}^{s,\mathbf{a}}(\mathbb{R} \times \mathbb{R}^n, \mathbf{w}) := \{ f \in \mathscr{S}'(\mathbb{R} \times \mathbb{R}^n) \mid \mathscr{F}^{-1}\left[2^{js} \phi_j \widehat{f}\right] \in L_{\mathbf{p}}(\mathbb{R} \times \mathbb{R}^n, \mathbf{w}; \ell_q(\mathbb{N})) \}.$$

$(\phi_j)_{j \in \mathbb{N}_0}$ **a**-anisotropic Littlewood-Paley decomposition:

$$\operatorname{supp} \phi_0 \subset \{|\tau, \xi|_{\mathbf{a}} \le 2\}, \qquad \operatorname{supp} \phi_n \subset \{2^{n-1} \le |\tau, \xi|_{\mathbf{a}} \le 2^{n+1}\}.$$

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Example $\mathbf{w} = (1, 1)$, $\mathbf{a} = (2, 1)$, $\mathbf{p} = (p, p)$, $s = 2$:

$$H^{s,\mathbf{a}}_{p}(\mathbb{R} \times \mathbb{R}^n) := H^{s,\mathbf{a}}_{\mathbf{p}}(\mathbb{R} \times \mathbb{R}^n, \mathbf{w}) = W^1_p(\mathbb{R}; L_p(\mathbb{R}^n)) \cap L_p(\mathbb{R}; W^2_p(\mathbb{R}^n))$$

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Lemma (Extrapolation, García-Cuerva and Rubio de Francia '83, ...)

$\|Tf\|_{L_p(\mathbb{R}\times\mathbb{R}^n,w)} \leq c\|f\|_{L_p(\mathbb{R}\times\mathbb{R}^n,w)}$ *for one* $p \in (1,\infty)$, *all* $w \in A_p(\mathbb{R}\times\mathbb{R}^n)$

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Lemma (Anisotropic Hörmander-Mikhlin (Lorist '20))

$\mathrm{M}\in C^\infty(\mathbb{R}^{n+1}\setminus\{0\})$, $\mathbf{a}\in(0,\infty)^2$ *such that for all* $(\alpha_1,\alpha_2)\in\mathbb{N}_0\times\mathbb{N}_0^n$

$$|\tau,\xi|_{\mathbf{a}}^{a_1\alpha_1+a_2|\alpha_2|}\cdot\partial_\tau^{\alpha_1}\partial_\xi^{\alpha_2}\mathrm{M}(\tau,\xi)\leq C.$$

Then M *is* $L_{\mathbf{p}}(\mathbb{R}\times\mathbb{R}^n,\mathbf{w})$*-multiplier for all* $\mathbf{p}\in(1,\infty)^2$, $\mathbf{w}\in A_{p_1}(\mathbb{R})\times A_{p_2}(\mathbb{R}^n)$.

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Mapping $\mathrm{tr}: u\mapsto u_{|x_3=0}$ extends to bounded linear operator

$$\mathrm{tr}: W^2_p(\mathbb{R}^3)\to W_p^{2-\frac{1}{p}}(\mathbb{R}^2).$$

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Lemma (Trace Theorem (Hummel-Lindemulder '21))

- $\mathbf{p}\in(1,\infty)^2$, $\mathbf{a}\in(0,\infty)^2$, $\gamma\in(-1,p_2-1)$ *and* $s>\frac{a_2}{p_2}(1+\gamma)$.
- $\mathbf{w}\in A_{p_1}(\mathbb{R})\times A_{p_2}(\mathbb{R}^n)$ *such that* $w_2=(w_2',|x_n|^\gamma)$.

Mapping $\operatorname{tr}: u\mapsto u_{|x_n=0}$ *extends to bounded linear operator*

$$\operatorname{tr}: H^{s,\mathbf{a}}_{\mathbf{p}}(\mathbb{R}\times\mathbb{R}^n,\mathbf{w}) \to F^{s-\frac{a_2}{p_2}(1+\gamma),\mathbf{a}}_{\mathbf{p},p_2}(\mathbb{R}\times\mathbb{R}^{n-1},(w_1,w_2')).$$

Example $\mathbf{w}=(1,1)$, $\mathbf{a}=(2,1)$, $\gamma=0$, $\mathbf{p}=(p,p)$, $s=2$: Trace operator maps

$$W^1_p(\mathbb{R};L_p(\mathbb{R}^n))\cap L_p(\mathbb{R};W^2_p(\mathbb{R}^n)) \to W_p^{1-\frac{1}{2p}}(\mathbb{R};L_p(\mathbb{R}^{n-1}))\cap L_p(\mathbb{R};W_p^{2-\frac{1}{p}}(\mathbb{R}^{n-1}))$$

Preliminaries

- Elliptic/Time-Periodic boundary value problems:

$$\text{(E)} \begin{cases} \mathscr{A} u = f & \text{in } \mathbb{R}^n_+, \\ \partial_n^j u = g_j & \text{on } \partial\mathbb{R}^n_+, \end{cases} \qquad \text{(TP)} \begin{cases} Au := (\partial_t + \mathscr{A})u = f & \text{in } \mathbb{R} \times \mathbb{R}^n_+, \\ \partial_n^j u = g_j & \text{on } \mathbb{R} \times \partial\mathbb{R}^n_+, \\ u(t+2\pi, x) = u(t,x). \end{cases}$$

with $m \in \mathbb{N}$, $j \in \{0, \dots, m-1\}$ and

$$\mathscr{A} := \sum_{|\alpha| \le 2m} a_\alpha D^\alpha, \quad a_\alpha \in \mathbb{C}, D_l = (-i\partial_l).$$

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- Expect: $\mathscr{A}$ ✓ $\Rightarrow$ $i\mathscr{A}, -\mathscr{A}$ ✓ for (E); $\mathscr{A}$ ✓ $\Rightarrow$ $-\mathscr{A}$ ✓ for (TP).

Classical Approach to Time-Periodic Problems

$$\text{(IVP)}\begin{cases} Au := (\partial_t + \mathscr{A})u = f & \text{in } (0,\infty)\times\mathbb{R}^n_+, \\ \partial_n^j u = g_j & \text{on } (0,\infty)\times\partial\mathbb{R}^n_+, \\ u(0) = u_0. \end{cases}$$

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- Poincaré Map Approach:

$$\mathcal{M}(u_0) := u(2\pi)$$

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$$\text{(IVP)} \begin{cases} Au := (\partial_t + \mathscr{A})u = f & \text{in } (0,\infty) \times \mathbb{R}^n_+, \\ \partial_n^j u = g_j & \text{on } (0,\infty) \times \partial\mathbb{R}^n_+, \\ u(0) = u_0. \end{cases}$$

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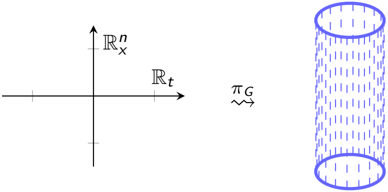
  Existence of the limit leads to a time-periodic solution.

Direct Approach to Time-Periodic Problems

- Introduce the locally compact abelian group

$$G := \mathbb{R}/2\pi\mathbb{Z} \times \mathbb{R}^n =: \mathbb{T} \times \mathbb{R}^n$$



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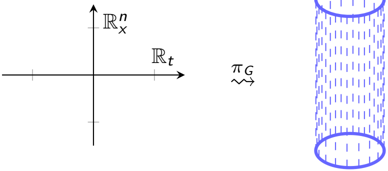
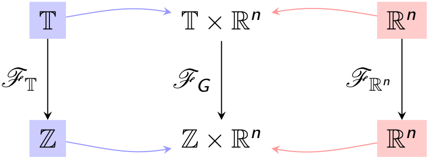


- Dual group $\widehat{G} = \mathbb{Z} \times \mathbb{R}^n$, compatible with Fourier transform $\mathscr{F}_G$:



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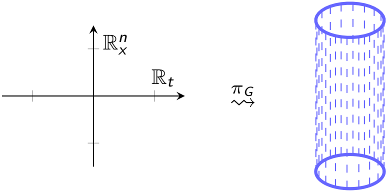

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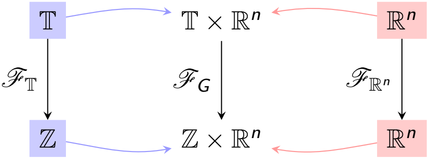

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$$u : G \to \mathbb{R}, \quad \mathcal{P}u(x) := \int_\mathbb{T} u(t,x)\,\mathrm{d}t$$

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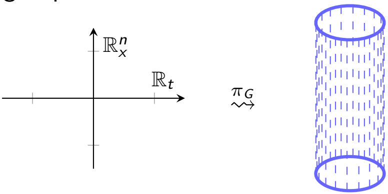

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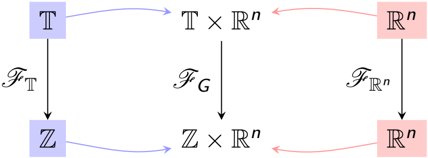

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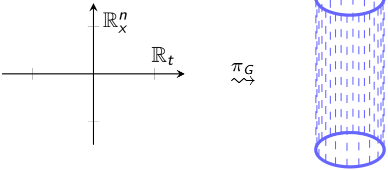

Definition (Time-periodic Bessel Potential Space on $G = \mathbb{T} \times \mathbb{R}^n$)

Let $\mathbf{w} \in W(\mathbb{T}) \times W(\mathbb{R}^n)$, $\mathbf{a} \in (0,\infty)^2$, $\mathbf{p} \in (1,\infty)^2$, $s \in \mathbb{R}$.

$$H^{s,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w}) := \{ f \in \mathcal{P}_\perp \mathscr{S}'(G) \mid \mathscr{F}_G^{-1}\big[|k,\xi|^s_{\mathbf{a}} \widehat{f}\,\big] \in L_{\mathbf{p}}(G,\mathbf{w}) \}$$

$$\|f\|_{H^{s,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w})} := \Big\| \mathscr{F}_G^{-1}\big[|k,\xi|^s_{\mathbf{a}} \widehat{f}\,\big] \Big\|_{L_{\mathbf{p}}(G,\mathbf{w})}$$

Time-Periodic Heat Equation

- Example ($n \geq 3$):

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$$\begin{cases} (\partial_t - \Delta)u = f & \text{in } \mathbb{R} \times \mathbb{R}^n, \\ u(t + 2\pi, x) = u(t, x). \end{cases}$$

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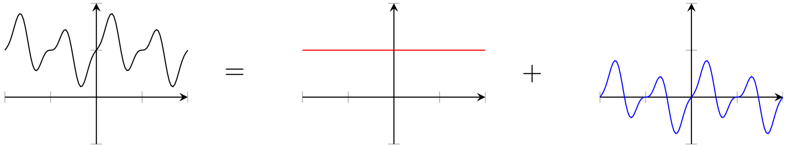

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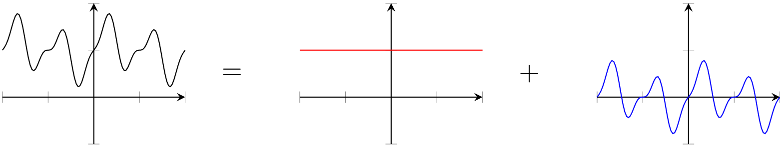

- Transference of multipliers:

Theorem (de Leeuw, Edwards&Gaudry)

If $\mathrm{m} = \mathrm{M}_{|\mathbb{Z}\times\mathbb{R}^n}$ for an $L_p(\mathbb{R}\times\mathbb{R}^n)$-multiplier M, then m is an $L_p(G)$-multiplier.

Time-Periodic Heat Equation

Theorem (Kyed-S. '17)

$p \in (1,\infty)$, $\mathbf{a} := (2,1)$. *For* $f \in L_{p,\perp}(G)$ *there is unique solution* $u \in H^{2,\mathbf{a}}_{p,\perp}(G)$ *to*

$$(\partial_t - \Delta)u = f \qquad \text{in } G.$$

It satisfies

$$\|u\|_{H^{2,\mathbf{a}}_p(G)} \sim \|u\|_p + \|\partial_t u\|_p + \sum_{i,j=1}^n \|\partial^2_{ij} u\|_p \lesssim \|f\|_p.$$

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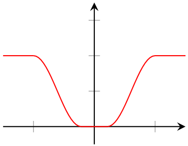
- $\mathrm{m} = \mathrm{M}_{|\mathbb{Z}\times\mathbb{R}^n}$ for $\mathrm{M}(\eta,\xi) := \dfrac{\varphi(\eta)}{i\eta + |\xi|^2}$ with $\varphi :=$ 

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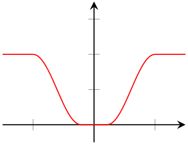
- $\mathrm{m} = \mathrm{M}_{|\mathbb{Z}\times\mathbb{R}^n}$ for $\mathrm{M}(\eta,\xi) := \dfrac{\varphi(\eta)}{i\eta + |\xi|^2}$ with $\varphi :=$ 
- Sanity check: M, $\eta\mathrm{M}$ and $\xi^\alpha\mathrm{M}$ bounded for $|\alpha| \le 2$.

Weighted Time-Periodic Heat Equation

Theorem (S. '23+)

Let $\mathbf{p} \in (1,\infty)^2$, $\mathbf{a} := (2,1)$ *and* $\mathbf{w} \in A_{p_1}(\mathbb{R}) \times A_{p_2}(\mathbb{R}^n)$ *with* w_1 2π*-periodic. For* $f \in L_{\mathbf{p},\perp}(G,\mathbf{w})$ *there is a unique solution* $u \in H^{2,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w})$ *to*

$$(\partial_t - \Delta)u = f \quad \text{in } G. \qquad \text{It satisfies } \|u\|_{H^{2,\mathbf{a}}_{\mathbf{p}}(G,\mathbf{w})} \lesssim \|f\|_{L_{\mathbf{p}}(G,\mathbf{w})}.$$

Rests on:

- Anis. Hörmander-Mikhlin cond. $\Rightarrow$ boundedness in $L_p(\mathbb{R}^n, w)$.
- Weighted Transf. Principle ($G := \mathbb{T} \times \mathbb{R}^n$, $H := \mathbb{R} \times \mathbb{R}^n$, $\Phi = \iota$, $\widehat{\Phi} = \pi_G$).
- Extrapolation.

Theorem (S. '23+)

For LCA groups G, H*, and continuous homomorphism* $\Phi : \widehat{G} \to \widehat{H}$*, define dual homomorphism* $\widehat{\Phi} : H \to G$ *via* $(\widehat{\Phi}(x), \chi) = (x, \Phi(\chi))$, $x \in H, \chi \in \widehat{G}$*. If* $\widehat{\Phi}$ *is surjective,* $p \in (1,\infty)$ *and* $w \in W(G)$*, then* $\mathrm{m} := \mathrm{M} \circ \Phi$ *is an* $L_p(G,w)$*-multiplier whenever* M *is continuous and an* $L_p(H, w \circ \widehat{\Phi})$*-multiplier.*

Theorem of Agmon, Douglis and Nirenberg

$\mathscr{P} := \sum_{|\alpha|=2m} a_\alpha D^\alpha.$

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$$z \mapsto \mathscr{P}(\xi', z)$$

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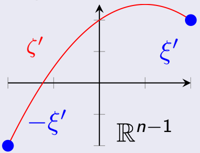
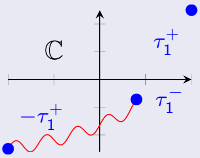
Remark ((i)⇒(ii) if $n > 2$)

Degree of $\mathscr{P}$ even: $\mathscr{P}(\xi', z) = \mathscr{P}(-\xi', -z)$.



roots of $\mathscr{P}(\zeta', z)$ can't cross real axis



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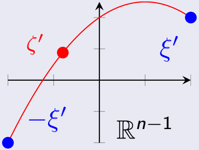
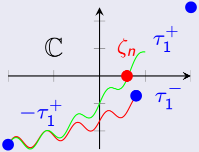


roots of $\mathscr{P}(\zeta', z)$ can't cross real axis
$(\zeta', \zeta_n) \in \mathbb{R}^n \setminus \{0\} \Rightarrow 0 = \mathscr{P}(\zeta', \zeta_n) \neq 0$



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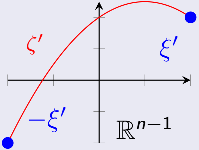

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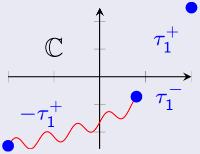


Theorem of Agmon, Douglis and Nirenberg

Theorem (Agmon, Douglis & Nirenberg, 1959)

If $\mathscr{A}$ is properly elliptic, then

$$\|u\|_{W_p^{2m}(\mathbb{R}^n_+)} \lesssim \|\mathscr{A}u\|_{L_p(\mathbb{R}^n_+)} + \sum_{j=0}^{m-1} \|\partial_n^j u\|_{W_p^{2m-j-\frac{1}{p}}(\partial\mathbb{R}^n_+)} + \|u\|_{L_p(\mathbb{R}^n_+)}.$$

Remark (Generalizations)

- Ω bounded and smooth
- Boundary operators $\mathscr{B}_j$ that "cover" $\mathscr{A}$ (roughly speaking: all linearly independent)
- x-dependency: $\mathscr{A}(x, D)$, $\mathscr{B}_j(x, D)$.

Existence and Uniqueness

Theorem (Kyed-S. '19, S. '23+)

Let $\mathscr{A}$ *be properly elliptic and* $\mathscr{P}(\xi) \notin i\mathbb{R}$ *for all* $\xi \in \mathbb{R}^n \setminus \{0\}$.
Let $\mathbf{p} \in (1,\infty)^2$, $\mathbf{a} := (2m, 1)$, $\gamma \in (-1, p_2 - 1)$ *and*
$\mathbf{w} := (w_1, w_2) \in A_{p_1}(\mathbb{R}) \times A_{p_2}(\mathbb{R}^n)$, *where* w_1 *is* 2π*-periodic and* $w_2 = (w_2', |x_n|^\gamma)$.
For $f \in L_{\mathbf{p},\perp}(\mathbb{T} \times \mathbb{R}^n_+, \mathbf{w})$, $g_j \in F^{2m-j-\frac{1+\gamma}{p_2},\mathbf{a}}_{\mathbf{p},p_2,\perp}(\mathbb{T} \times \partial\mathbb{R}^n_+, (w_1, w_2'))$ *there is a unique solution*

$$u \in H^{2m,\mathbf{a}}_{\mathbf{p},\perp}(\mathbb{T} \times \mathbb{R}^n_+, \mathbf{w})$$

to

$$\begin{cases} Pu := (\partial_t + \mathscr{P})u = f & \text{in } \mathbb{T} \times \mathbb{R}^n_+ \\ \partial_n^j u = g_j & \text{on } \mathbb{T} \times \partial\mathbb{R}^n_+. \end{cases}$$

It satisfies

$$\|u\|_{H^{2m,\mathbf{a}}_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})} \lesssim \|f\|_{L_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})} + \sum_{j=0}^{m-1} \|g_j\|_{F^{2m-j-\frac{1+\gamma}{p_2},\mathbf{a}}_{\mathbf{p},p_2}(\mathbb{T}\times\partial\mathbb{R}^n_+,(w_1,w_2'))}.$$

Time-Periodic ADN Theorem

Corollary

If $\mathscr{A}$ is properly elliptic and $\mathscr{P}(\xi) \notin i\mathbb{R}$ for all $\xi \in \mathbb{R}^n \setminus \{0\}$, then

$$\|u\|_{H^{2m,\mathbf{a}}_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})} \lesssim \|Au\|_{L_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})} + \sum_{j=0}^{m-1} \|\partial_n^j u\|_{F^{2m-j-\frac{1+\gamma}{p_2},\mathbf{a}}_{\mathbf{p},p_2}(\mathbb{T}\times\partial\mathbb{R}^n_+,(w_1,w_2'))} + \|u\|_{L_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})}.$$

Remark (Generalizations as in elliptic case)

- Ω bounded and smooth
- Boundary operators $\mathscr{B}_j$ that "cover" A (roughly speaking: all linearly independent)
- x-dependency: $A(x, D)$, $\mathscr{B}_j(x, D)$.

Proof of Main Theorem

Lemma

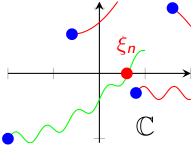
Let $P(\eta,\xi) := i\eta + \mathscr{P}(\xi)$. *For* $(\eta,\xi') \in \mathbb{R}\times\mathbb{R}^{n-1}\setminus\{(0,0)\}$,

$$z \mapsto P(\eta,\xi',z)$$

has exactly m *roots* $\rho_j^{\pm}(\eta,\xi') \in \mathbb{C}_{\pm}$ *in the upper/lower complex plane. Hence the polynomial* $P(\eta,\xi)$ *factors into* $P = \mathrm{M}_+\mathrm{M}_-$, *where*

$$\mathrm{M}_{\pm}(\eta,\xi) := \prod_{j=1}^{m}\left(\xi_n - \rho_j^{\pm}(\eta,\xi')\right)$$

are functions in (η,ξ) *(but not polynomials!).*



$$0 = P(\eta,\xi',\xi_n) = i\eta + \mathscr{P}(\xi',\xi_n) \Rightarrow \mathscr{P}(\xi',\xi_n) \in i\mathbb{R}$$

Proof of Main Theorem

Lemma

Polynomial $P(\eta,\xi)$ *factors into functions* $\mathsf{M}_+(\eta,\xi)$ *and* $\mathsf{M}_-(\eta,\xi)$.

Lemma

Put $\mathsf{m}_\pm := \mathsf{M}_{\pm}{}_{|\mathbb{Z}\times\mathbb{R}^n}$. *The operators*

$$P_\pm u := \mathscr{F}_G^{-1}\big[\mathsf{m}_\pm \widehat{u}\big], \quad P_\pm^{-1} u := \mathscr{F}_G^{-1}\big[\mathsf{m}_\pm^{-1} \widehat{u}\big]$$

extend uniquely to bounded and mutually inverse operators:

$$P_\pm : H^{s,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w}) \to H^{s-m,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w}), \quad P_\pm^{-1} : H^{s-m,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w}) \to H^{s,\mathbf{a}}_{\mathbf{p},\perp}(G,\mathbf{w}).$$

Proof of Main Theorem

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Proof.

$(\eta,\xi) \mapsto \frac{\mathsf{M}_\pm(\eta,\xi)}{|\eta,\xi|^m}$ and $(\eta,\xi) \mapsto \frac{|\eta,\xi|^m}{\mathsf{M}_\pm(\eta,\xi)}$ fulfill anisotropic Hörmander-Mikhlin condition. □

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- Observe that $Pu = P_-P_+u \rightsquigarrow$ in whole space $\boxed{u = P_+^{-1}P_-^{-1}f}$.

Proof of Main Theorem

Proof of Main Theorem, $g_j = 0$.

- Extend f by zero to $f \in L_{\mathbf{p},\perp}(G, \mathbf{w})$ and set

$$u := P_+^{-1} \mathbb{1}_{\mathbb{T}\times\mathbb{R}^n_+} P_-^{-1} f \qquad L_{\mathbf{p},\perp}(G, \mathbf{w}) \qquad H^{m,\mathbf{a}}_{\mathbf{p},\perp}(G, \mathbf{w})$$

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- Claim: u is the unique solution in $H^{m,\mathbf{a}}_{\mathbf{p},\perp}(G, \mathbf{w})$ to

$$\operatorname{supp} u \subset \mathbb{T} \times \overline{\mathbb{R}^n_+}, \qquad \text{and} \qquad \operatorname{supp}(f - Pu) \subset \mathbb{T} \times \overline{\mathbb{R}^n_-}.$$

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- Observe $u = \mathscr{F}_G^{-1}\mathrm{m}_+^{-1}\mathscr{F}_G \square$ with supp $\square \subset \mathbb{T}\times\overline{\mathbb{R}^n_+}$.
- Paley-Wiener: $\mathscr{F}_G^{-1}\mathrm{m}\,\mathscr{F}_G \square$ respects this support property, if $z \mapsto \mathrm{m}(k,\xi',z)$ admits bounded holomorphic extension to $\mathbb{C}_-$.
  Roots of $\mathrm{m}_+(k,\xi',z)$ are in $\mathbb{C}_+ \Rightarrow$ Paley-Wiener applies to $\mathrm{m} := \mathrm{m}_+^{-1}$.

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- Similar idea for $\operatorname{supp}(f - Pu)$.

Proof of Main Theorem

Proof of Main Theorem, $g_j = 0$.

- Extend f by zero to $f \in L_{\mathbf{p},\perp}(G, \mathbf{w})$ and set

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- Similar idea for $\operatorname{supp}(f - Pu)$.
- Use equation to improve to $\|u\|_{H^{2m,\mathbf{a}}_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})} \lesssim \|f\|_{L_{\mathbf{p}}(\mathbb{T}\times\mathbb{R}^n_+,\mathbf{w})}$. □

Summary

Boundary Value Problems

- Direct approach to time-periodic L^p-estimates of A-D-N type.
- No detour via initial value problem needed.
- Did not (have to) talk about $\mathcal{R}$-boundedness.

Same strategy works for

- Time-periodic Stokes Operator: Maekawa-S. '18.
- Maximal L^p regularity $\Leftrightarrow$ Time-periodic maximal L^p regularity.
- Link between A-D-N and Ladyzhenskaya-Ural'tseva-Solonnikov.

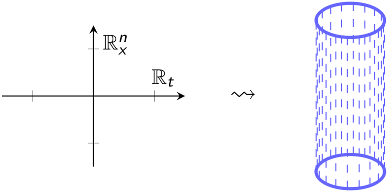
Outlook

- Fredholm properties for time-periodic bvp of A-D-N type.
- Systems of equations.
- Application to nonlinear problems.

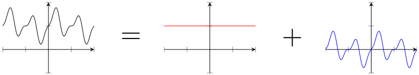
Thank you for your attention!

Time-periodic Agmon-Douglis-Nirenberg via

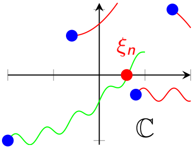
- Direct Fourier approach:



- Splitting off constants in time:



- Using distribution of roots to split $P = P_- P_+$:



- Representation formula: $u = \boxed{P_+^{-1} \boxed{\mathbb{1}_{\mathbb{T}\times\mathbb{R}^n_+} P_-^{-1} f}}$