A VARIATIONAL APPROACH FOR FIRST ORDER KINETIC MFG (WITH LOCAL COUPLINGS)

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BACKGROUND AND MOTIVATION

- The theory of MFG was initiated roughly 15 years ago by
  - → LASRY – LIONS
  - → HUANG – MALHAMÉ – CAINES
- Aim : characterize limits of NASH equilibria of N-player (stochastic) differential games when $N \to +\infty$.

- First models : agents control their velocities

Simplest models : state space $\mathbb{R}^d$ (or $\mathbb{T}^d$)

- $T>0$ given time horizon
- $L : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ Lagrangian
- $g : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ final cost
- A typical agent predicts the evolution of the density of agents : $[0,T] \ni t \mapsto m_t \in \mathcal{P}(\mathbb{R}^d)$ and solves :

$$u(t;x_0) := \inf \left\{ \int_t^T L(x_s; \alpha_s; m_s)\,ds + g(x_T; m_T) \right\}$$

$$\text{s.t.} \begin{cases} \dot{x}_s = \alpha_s & s \in (t,T) \\ x_t = x_0 \end{cases} \qquad H(x,\cdot,m) := L(x,\cdot,m)^*$$

- Optimal control in feedback form: $\alpha_s := -D_pH(x_s; Du(t;x_s); m_s)$
- True evolution of m : $\partial_t m - \mathrm{div}(D_pH(\cdot, Du; m)m) = 0.$

MFG MODELS ON ACCELERATION CONTROL

- In the MFG community these were studied relatively recently
- Outside of the context of games, such models were intensively studied (flocking, swarming, etc.)
- Celebrated Cucker–Smale model to describe behavior of flocks (2007)

THE FRAMEWORK

- State and velocity space $\mathbb{M} \times \mathbb{R}^d$, $\mathbb{M} \in \{\mathbb{T}^d, \mathbb{R}^d\}$
- Time horizon : $T > 0$.

A typical agent predicts the evolution $[0,T] \ni t \mapsto m_t \in \mathcal{P}(\mathbb{M} \times \mathbb{R}^d)$ of the density of the agents and solves

$$u(t, x_0, v_0) := \inf \left\{ \int_t^T \underbrace{L}_{\text{Lagrangian}}(x_s, v_s, \alpha_s; m_s)\, ds + \underbrace{g}_{\text{final cost}}(x_T, v_T; m_T) \right\}$$

$$\text{s.t.} \quad \begin{cases} \dot{x}_s = v_s \\ \dot{v}_s = \alpha_s \\ x_t = x_0 \\ v_t = v_0 \end{cases}$$

- If all agents behave rationally, the Nash equilibria can be characterized by the solutions of the MFG System :

(MFG)

$$\begin{cases} -\partial_t u(t,x,v) - v\cdot D_x u(t,x,v) + H(x,v,D_v u(t,x,v),m_t) = 0 \\ \partial_t m(t,x,v) + v\cdot D_x m(t,x,v) - \mathrm{div}_v\big(m_t D_p H(x,v,D_v u,m_t)\big) = 0 \\ \qquad\qquad \text{in } (0,T)\times M\times\mathbb{R}^d\,; \\ m(0,\cdot,\cdot) = m_0\,;\quad u(T,\cdot,\cdot) = g(\cdot,\cdot,m_T) \\ \qquad\qquad \text{in } M\times\mathbb{R}^d \end{cases}$$

given initial agent distribution

$$H(x,v,\cdot,m) := L(x,v,\cdot,m)^*$$

LITERATURE REVIEW

- Nourian – Caines – Malhamé ; 2011
- Cannarsa – Mendico ; 2020
- Achdou – Mannucci – Marchi – Thou ; 2020 ; 2021
- Bardi – Cardaliaguet ; 2021
- Cardaliaguet – Mendico ; 2021
- Mendico ; 2021

All these consider Hamiltonians that depend nonlocally on m ; essentially quadratic in p.

- Mimikos-Stamatopoulos ; 2021 : 2^{nd} order model ; couplings depend locally on m.

1^{st} ORDER KINETIC MFG VIA VATIATIONAL TECHNIQUES

Standing assumptions

- Suppose : $\underset{\mathbb{M}\ \ \mathbb{R}^d\ \ \mathbb{R}^d\ \ \mathbb{R}}{H(x, v, p, m)} = H(x, v, p) - f(x, v, m)$

- H is cont; diff & convex in p; behaves as $|p|^r$, $r > 1$.

- $\mathcal{F}(x, v, m) := \int_0^m f(x, v, m')\, dm'$

- $\mathcal{G}(x, v, m) := \int_0^m g(x, v, m')\, dm'$

- $\mathcal{F}, \mathcal{G}$ cont; strictly convex & diff. in m

- $\mathcal{F} \sim m^q$; $q > 1$; $\mathcal{G} \sim m^s$; $1 < s \leq q$.

- $\mathcal{F}(x, v, m) = \mathcal{G}(x, v, m) = +\infty$ if $m < 0$.

- $m_0 \in C_b(\mathbb{M} \times \mathbb{R}^d) \cap \mathcal{P}(\mathbb{M} \times \mathbb{R}^d)$.

THE MINIMIZATION PROBLEMS

(A) minimize

$$\mathcal{A}(u) := \int_0^T \int_{M\times\mathbb{R}^d} \mathcal{F}^*(x, v, -\partial_t u - v\cdot D_x u + H(x, v, D_v u))\,dx\,dv\,dt$$

$$- \int_{M\times\mathbb{R}^d} u(0, x, v)\, m_0(x, v)\,dx\,dv + \int_{M\times\mathbb{R}^d} \mathcal{G}^*(x, v, u(T, x, v))\,dx\,dv$$

over $E_0 := \{ u \in C^1_b \;:\; |v|\,|D_x u| \in L^\infty \}$

(B) minimize

$$\mathcal{B}(m, w) := \int_0^T \int_{M\times\mathbb{R}^d} \mathcal{F}(x, v, m)\,dx\,dv\,dt + \int_0^T \int_{M\times\mathbb{R}^d} L\left(x, v, -\frac{w}{m}\right) dx\,dv\,dt$$

$$+ \int_{M\times\mathbb{R}^d} \mathcal{G}(x, v, m_T(x, v))\,dx\,dv$$

$$\text{s.t.} \begin{cases} \partial_t m + v\cdot D_x m + \mathrm{div}_v(w) = 0 & \text{in } \mathcal{D}' \\ m|_{t=0} = m_0 \end{cases}$$

LITERATURE ON VARIATIONAL MFG

- Benamou – Brenier ; 2000.
- Cardaliaguet ; 2015.
- Cardaliaguet – Graber ; 2015
- Cardaliaguet – Graber – Porretta – Tonon ; '15

Planning problems

- Graber – M. – Silva – Tonon , 2019
- Orrieri – Porretta – Savaré ; 2019

(other models of similar flavor :
Cardaliaguet – Carlier – Nazaret ; 2013
Dolbeault – Nazaret – Savaré ; 2009 ; etc)

GENERAL STRATEGY FOR EXISTENCE

① Establish duality between (A) & (B)

② By FENCHEL–ROCKAFELLAR, duality implies existence in (B).

③ Suitably relax (A).

*④ Show existence in (A) by the direct method of Calc. Var.

Consequence : • notion of weak solution

• uniqueness of m immediate

• uniqueness of u on $\mathrm{spt}(m)$

OVERCOMING THE CHALLENGES IN ④

- Most of the previous results on variational MFG consider $\mathbb{T}^d$ as a state space; i.e. compactness
- Also, typical assumption $m_0 > c_0 > 0$ in $\mathbb{T}^d$.
- In our case, since $v \in \mathbb{R}^d$, this is not feasible. Also, $m_0 \notin C_c(M \times \mathbb{R}^d)$.
- Need to overcome even these first technical challenges.

RELAXATION OF (A)

$$\tilde{\mathcal{A}}(u,\beta,\beta_T) := \int_0^T \int_{M\times\mathbb{R}^d} \mathcal{F}^*(x,v,\beta)\,dx\,dv\,dt - \int_{M\times\mathbb{R}^d} m_0 u_0\,dx\,dv + \int_{M\times\mathbb{R}^d} \mathcal{G}^*(x,v,\beta_T)\,dx\,dv$$

s.t.

$$\begin{cases} -\partial_t u - v\cdot D_x u + H(x,v,D_v u) \le \beta \\ u_T \le \beta_T \end{cases}$$

in sense of $\mathcal{D}'$; i.e. via testing with C^1_c functions

- Need a weak notion of trace in time for u (via a priori estimates ; $t \mapsto \int_{M\times\mathbb{R}^d} \phi(t,x,v)\,u(t,x,v)\,dx\,dv$ is locally BV ; $\forall\, \phi \in C^1_c$)

- In spirit, similar notions of traces were defined in [Cardaliaguet – Graber – Porretta – Tonon] and [Orrieri – Porretta – Savaré], but the lack of compactness and the presence of kinetic drift posed some difficulties.
- Moreover, need to give sense to $\langle m_0, u_0 \rangle$ for $m_0 \in C_b$ (not C_c!).

A PRIORI ESTIMATES FOR $(u_n, \beta_n; \beta_{T,n})_n$

- Immediate : $(\beta_n)_+^n$ bdd in $L^{q'}$
  $(\beta_{T,n})_+^n$ bdd in $L^{s'}$ (from the energy)
- Then $(u_n)_+^n$ bdd in L^1_{loc}
- Since we get boundedness of
  $$-\int_{M\times\mathbb{R}^d} u_{0,n}\, m_0 \, dx\, dv \, ,$$
  the strategy in previous references was to multiply the (HJB) eq. by m_0, then use $m_0 \geq c_0 > 0$, and get bounds on $(u_{0,n})_n$. In our case this is not feasible!

THE REACHABLE SET $\mathcal{U}_{m_0}$

- The main idea: one needs to know information on u, essentially only on sets that can be reached from $\mathrm{spt}(m_0)$ via the underlying control system.

$\mathcal{U}_{m_0} := \{$ all points from $[0,T]\times M\times\mathbb{R}^d$, that can be reached by

$$\begin{cases}\dot{x} = v \\ \dot{v} = a\end{cases} \quad \begin{pmatrix} x(0) \\ v(0)\end{pmatrix} \in \mathrm{spt}(m_0);\ a\in C([0,T];\mathbb{R}^d) \Big\}$$

Kalman rank $\Rightarrow$ $\mathcal{U}_{m_0} = \{0\}\times\mathrm{spt}(m_0) \cup (0,T]\times M\times\mathbb{R}^d$.

THE REMAINING A PRIORI ESTIMATES

- Instead of m_0, test the (HJB) inequality with $\phi \in C^1_c(\mathcal{U}_{m_0})$; $\phi \geq 0$; $0 \leq \phi_0 \leq m_0$.

- $(D_r u_n)_n$ bdd in $L^r_{loc}(\mathcal{U}_{m_0})$
- $(u_n)_n$ bdd in $L^1_{loc}(\mathcal{U}_{m_0})$
- $(\beta_n)^n_-$ bdd in $L^1_{loc}(\mathcal{U}_{m_0})$
- $(\beta_{T,n})^n_-$ bdd in $L^1_{loc}(M \times \mathbb{R}^d)$.

Proof of duality between $\tilde{\mathcal{A}}$ and $\mathcal{B}$ is very technical: several truncations and convolutions.

LAST PIECE FOR STRONG PRECOMPACTNESS

- Notice that so far, we didn't have any "strong control" (such as derivative) on $(u_n)_n$ in the x-direction.
- This is needed to hope for strong precompactness of $(u_n)_n$.
- To overcome this, use AVERAGING lemmas from kinetic theory.

IDEA BEHIND THE AVERAGING LEMMAS

$$-\partial_t u_n - v\cdot D_x u_n = \underbrace{\beta_n - H(x,v,D_v u_n)}_{\text{bdd in } L^1_{loc}(\mathcal{U}_{\infty})}$$

The averages $\rho_\phi[u](t,x) := \int_{\mathbb{R}^d} u(t,x,v)\phi(v)\,dv$
$\phi \in C_c^\infty(\mathbb{R}^d)$

enjoy additional frac. Sobolev reg
and/or strong L^p-compactness.

- Need equi-integrability of u in v.
- get $(\rho_\phi[u_n])_n$ is rel. strongly compact in L^1_{loc}.

$\Rightarrow$ $(u_n)_n$ str. precomp. in L^1_{loc}.
[cf. GOLSE–SAINT-RAYMOND, 2004; HAN-KWAN, 2010]

THE NOTION OF SOLUTION

<u>THM</u> [M.-Griffin-Pickering, '22] MFG has a weak solution.

- The continuity equation is satisfied in $\mathcal{D}'$.
- $u \in L^1_{loc}(\mathcal{U}_{m_0})$; $D_v u \in L^r_{loc}(\mathcal{U}_{m_0})$; $m|D_v u|^r \in L^1$
- $(u_0)_+ \in (L^\infty + L^{q'})$; $(u_0)_-$ is a locally finite Radon measure supported in $\{m_0 > 0\}$
- $\int_{M\times\mathbb{R}^d} m_0 u_0 (dx\,dv)$ is finite
- (HJB) ineq. in $\mathcal{D}'$
- Energy equality: $\int_{M\times\mathbb{R}^d} m_0 u_0(dx\,dv) - \int_{M\times\mathbb{R}^d} g(m_T)\, m_T\, dx\,dv$

$$= \int_0^T\!\!\int_{M\times\mathbb{R}^d} f(x,v,m)\, m\, dx\,dv\,dt$$

$$+ \int_0^T\!\!\int_{M\times\mathbb{R}^d} \left[D_p H(x,v,D_v u)\cdot D_v u - H(x,v,D_v u)\right] m\, dx\,dv\,dt$$

ADDIONAL SOBOLEV ESTIMATES

- Based on the ideas developped in [GRABER – M., 2018] and [GRABER – M. – SILVA – TONON, 2019] under additional strong monotonicity cond. on $f(x,v,\cdot)$ and $g(x,v,\cdot)$ and strong conv. on H in p, we can obtain diff. quotient estimates for $D_{x,v} m$ & $D_{x,v} D_v u$ in L^2_{loc}
- Initial estimates using $(tD_x + D_v)$.

Idea behind the Sobolev estimates:

- test the optimality of (u, m) in the variational problems by their translates in (t, x, v)
- derive differential quotient estimates from the energy
- This idea goes back to [Brenier, 1999] and [Ambrosio–Figalli, 2008] who studied the regularity of the pressure in weak solutions to incompressible Euler eqs.

How does this formally work?

Suppose we are in a simple setting (without acceleration control); i.e.

$$\mathcal{A}(u,\beta,\beta_T) := \int_0^T\int_{\mathbb{R}^d} \mathcal{F}^*(\beta_t(x))\,dx\,dt + \int_{\mathbb{R}^d} \mathcal{G}^*(\beta_T(x))\,dx - \int_{\mathbb{R}^d} u(0,x)\,m_0(x)\,dx.$$

$$\text{s.t.} \begin{cases} -\partial_t u + H(Du) \leq \beta \\ u(T,\cdot) \leq \beta_T \end{cases}$$

$$\mathcal{B}(m,w) := \int_0^T\int_{\mathbb{R}^d} L\left(-\frac{w_t(x)}{m_t(x)}\right) + \mathcal{F}(m_t(x))\,dx\,dt + \int_{\mathbb{R}^d} \mathcal{G}(m_T(x))\,dx$$

$$\text{s.t.} \begin{cases} \partial_t m + \mathrm{div}(w) = 0 \\ m|_{t=0} = m_0 \end{cases}$$

- all data are taken to be x-independent.
- let $\delta \in \mathbb{R}^d$, with $|\delta|$ small and consider

$$m^\delta := m(t, x+\delta) \quad ; \quad w^\delta := w(t; x+\delta).$$

- (m^δ, w^δ) is a competitor for $\mathcal{B}$

(let us forget for a moment that $m^\delta|_{t=0} \neq m_0$)

- if (m,w) is an optimizer for $\mathcal{B}$

$$\Longrightarrow \quad \mathcal{B}(m,w) \leq \mathcal{B}(m^\delta, w^\delta)$$

But $\mathcal{B}(m^\delta, w^\delta) = \mathcal{B}(m,w) + \frac{d}{d\delta}\Big|_{\delta=0} \mathcal{B}(m^\delta, w^\delta) \cdot \delta + O(|\delta|^2)$

$$\leq \mathcal{B}(m,w) + C|\delta|^2.$$

- Let (u, β, β_T) be optimal for $\mathcal{A}$, then

$$\mathcal{B}(m,w) + \mathcal{A}(u, \beta, \beta_T) = 0.$$

For any competitor $(\hat{m}, \hat{w})$ and $(\hat{u}, \hat{\beta}, \hat{\beta}_T)$ we have

$$(*)\quad \mathcal{A}(\hat{u}, \hat{\beta}, \hat{\beta}_T) + \mathcal{B}(\hat{m}, \hat{w}) = \int_0^T \int_{\mathbb{R}^d} \left[\mathcal{F}(\hat{m}) + \mathcal{F}^*(\hat{\beta}) - \hat{\beta}\hat{m} \right] dx\,dt$$
$$+ \int_{\mathbb{R}^d} \left[\mathcal{G}(\hat{m}_T) + \mathcal{G}^*(\hat{\beta}_T) - \hat{\beta}_T \hat{m}_T \right] dx$$
$$+ \int_0^T \int_{\mathbb{R}^d} \hat{m} \left[H(D\hat{u}) + H^*\left(-\frac{\hat{w}}{\hat{m}}\right) + D\hat{u} \cdot \frac{\hat{w}}{\hat{m}} \right] dx\,dt$$

Suppose strong quantified convexity for $\mathcal{F}$, H & $\mathcal{G}$

Young $\Rightarrow$ $\mathcal{F}(a) + \mathcal{F}^*(a^*) - a a^* \geq 0$,

but suppose $\exists$ $\mathcal{F}_1, \mathcal{F}_2$ functions and $c_0 > 0$

s.t. $\mathcal{F}(a) + \mathcal{F}^*(a^*) - a a^* \geq c_0 \left| \mathcal{F}_1(a) - \mathcal{F}_2(a^*) \right|^2$

<u>Ex</u> : $\mathcal{F}(a) = \frac{1}{q} a^q \Rightarrow \mathcal{F}_1(a) = a^{\frac{q}{2}}$; $\mathcal{F}_2(a^*) = (a^*)^{\frac{q'}{2}}$

Therefore

$$(*) \quad \mathcal{A}(\hat{u},\hat{\beta},\hat{\beta}_T)+\mathcal{B}(\hat{m},\hat{w}) \geq \int_0^T\int_{\mathbb{R}^d} c_0 \,|\mathcal{F}_1(\hat{m})-\mathcal{F}_2(\hat{\beta})|^2 dx\,dt$$

In particular, if (u,β,β_T) & (m,w) are minimizers $\Longrightarrow$ $\mathcal{F}_1(m)=\mathcal{F}_2(\beta)$ a.e.

- Use this in $\mathcal{B}(m^\delta,w^\delta) \leq \mathcal{B}(m,w)+C|\delta|^2$

add $\mathcal{A}(u,\beta,\beta_T)$ to both sides:

$$\Longrightarrow \mathcal{A}(u,\beta,\beta_T)+\mathcal{B}(m^\delta,w^\delta) \leq C|\delta|^2$$

By $(*)$ $$c_0\int_0^T\int_{\mathbb{R}^d} |\mathcal{F}_1(m^\delta)-\underset{\underset{\mathcal{F}_1(m)}{\shortparallel}}{\mathcal{F}_2(\beta)}|^2 dx\,dt \leq C|\delta|^2$$

$$\Longrightarrow c_0\int_0^T\int_{\mathbb{R}^d} |\mathcal{F}_1(m^\delta)-\mathcal{F}_1(m)|^2 dx\,dt \leq C|\delta|^2$$

$\Longrightarrow \mathcal{F}(m) \in H^1.$

Similar estimate can be obtained for Du and m_T.

- If x-dependence is also present for the data, need additional tricks.
- In the kinetic models, one cannot simply do "standard" perturbations in the form $x+\delta$, $v+\delta$. Instead one needs to "naturally" follow the control system
- Need special care for the perturbations in time and going to initial & final times.

Theorem [Griffin-Pickering – M. , 2022]

Under suitable regularity and strong quantified convexity assumptions on the data, one has the additional estimates

$$\| m^{\frac{q}{2}-1}(tD_x + D_v) m \|_{L^2_{loc}([0,T]\times M \times \mathbb{R}^d)} \leq C$$

and

$$\| m^{\frac{1}{2}}(tD_x + D_v) D_v u \|_{L^2_{loc}([0,T]\times M \times \mathbb{R}^d)} \leq C$$

and

$$\| m_T^{\frac{s}{2}-1} (TD_x + D_v) m_T \|_{L^2} \leq C.$$

THANK YOU FOR YOUR ATTENTION !