Some existence results for mean field games

David Ambrose

April 3, 2023

Drexel

Introduction

- I will begin by discussing the vortex sheet and the result of Duchon and Robert.
- This is an existence result in the Wiener algebra.
- In addition to problems which are elliptic in space-time such as these, the method also applies to parabolic problems.
- We will show how the method applies to the mean field games PDE system.
- We will then look at a mean field games system arising from a specific application.

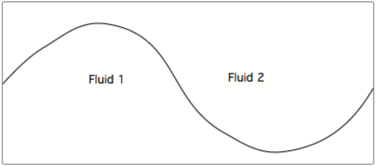
References

Some of my works on nonseparable mean field games:

- **D.M. Ambrose. Strong solutions for time-dependent mean field games with non-separable Hamiltonians. J. Math. Pures Appl., 113:141-154, 2018.**
- **D.M. Ambrose. Existence theory for a time-dependent mean field games model of household wealth. Appl. Math. Optim., 83:2051-2081, 2021.**
- D.M. Ambrose. Existence theory for non-separable mean field games in Sobolev spaces. Indiana U. Math. J., 71:611-647, 2022.
- D.M. Ambrose and A.R. Meszaros. Well-posedness of mean field games master equations involving non-separable local Hamiltonians. Trans. Amer. Math. Soc., 376:2481-2523, 2023.

The Vortex Sheet Problem

- We consider two infinitely deep, horizontally periodic fluids, separated by a sharp interface.



- The fluid velocities are given by the irrotational, incompressible Euler equations in each fluid region:

$$\mathbf{v}_{i,t} + \mathbf{v}_i \cdot \nabla \mathbf{v}_i = -\nabla p_i,$$

$$\mathrm{div}(\mathbf{v}_i) = 0,$$

$$\mathbf{v}_i = \nabla \phi_i.$$

- The fluids have densities ρ_1 and ρ_2. If $\rho_2 = 0$, then this is the water wave case.

The Vortex Sheet IVP Is Ill-Posed

- If $\rho_1\rho_2 > 0$, then the initial value problem is ill-posed.
- There have been several proofs of this over the years: Caflisch & Orellana 1989, Lebeau 2002, Lebeau & Kamotski 2005, Wu 2006. Ill-posedness is also implied by, but not explicitly discussed in, the work of Duchon & Robert 1988.
- The ill-posedness can be seen from linear theory: consider $y(\alpha, t) = \epsilon\eta(\alpha, t)$. Then, the linearized equation of motion is

$$\eta_{tt} = -\eta_{\alpha\alpha} + \tau H(\eta_{\alpha\alpha\alpha}),$$

  where H is the Hilbert transform and τ is the (non-negative) coefficient of surface tension.
- If $\tau = 0$, then the problem is elliptic in space-time, and has an ill-posed initial value problem.
- This ill-posedness is really the same thing as the Kelvin-Helmholtz instability (the problem is so unstable as to be ill-posed).

The Duchon-Robert Formulation

- In just about all studies of the vortex sheet, the irrotationality assumption is used to reduce the dimension by one; that is, only quantities on the interface need to be considered.
- Consider the interface to be a graph, $(x, y(x))$. Let $\Omega(x) = 1 + \omega(x)$ be the vortex sheet strength.
- Denote $v = y_x$. They write the evolution equations as

  $$v_t - \Lambda\omega = F(v, \omega)_x, \qquad \omega_t - \Lambda v = G(v, \omega)_x,$$

  where F and G are nonlinear terms stemming from the Biot-Savart integral.
- Here, $\Lambda = \sqrt{-\partial_{xx}} = H\partial_x$, and thus $\hat{\Lambda}(\xi) = |\xi|$.
- Again, the linearization is elliptic in space-time, and we see the ill-posedness at the linear level.

The Duchon-Robert Result

- Specify half the data: $v(x,0) = v_0(x)$. If v_0 is sufficiently small in a certain function space (the Wiener Algebra), then there exists a solution (v, ω) to the initial value problem for all time. This solution is analytic at all positive times.
- Method of proof: write a Duhamel formula which integrates forward in time from $t = 0$ and backwards in time from $t = \infty$:

$$v = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G),$$

$$-\omega = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) + \frac{1}{2}I^-(F+G).$$

- Consider function spaces on space-time domain $\mathbb{R} \times [0, \infty)$; in such spaces, prove Lipschitz bounds on F and G, and prove that I^+, I^- are bounded linear operators.
- Put this all together using the contraction mapping theorem, to get existence of solutions in these function spaces.

The Wiener algebra

- The Wiener algebra is the space of functions with Fourier transform/series in L^1 or ℓ^1.
- This has a simple algebra property:

$$\|fg\| = \int_{\mathbb{R}} |\mathcal{F}(fg)(\xi)|\ d\xi \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |\hat{f}(\xi-\eta)\hat{g}(\eta)|\ d\eta d\xi \leq \|f\|\|g\|.$$

- We need a space-time version of this. Let $\alpha > 0$ and let the space $\mathcal{B}_\alpha$ be the set of functions for which

$$\|f\|_\alpha = \int_{\mathbb{R}} \sup_{t\in[0,\infty)} e^{\alpha t|\xi|} \left|\hat{f}(t,\xi)\right|\ d\xi.$$

- The algebra property is inherited by $\mathcal{B}_\alpha$.
- If $f \in \mathcal{B}_\alpha$, then for all $t \in (0,\infty)$, f is analytic with radius of analyticity at least αt.

The operators and their boundedness

- The operator S was just the semigroup for the linear part, $S(t) = e^{-t\Lambda}$.
- The operators I^+ and I^- are defined by

$$(I^+h)(t,\cdot) = \int_0^t e^{(s-t)\Lambda} h_x(s,\cdot)\,ds,$$

$$(I^-h)(t,\cdot) = \int_t^\infty e^{(t-s)\Lambda} h_x(s,\cdot)\,ds.$$

- We can calculate the $\mathcal{B}_\alpha$ norm of $I^\pm h$, and we find that it is bounded by a constant times $\|h\|_{\mathcal{B}_\alpha}$.
- The operator I_0 is just $I^-(0)$, so it is an integral over the whole time interval $[0,\infty)$.

The contraction mapping

- Our fixed-point formulation again is

$$v = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) - \frac{1}{2}I^-(F+G),$$

$$-\omega = Sv_0 + \frac{1}{2}SI_0(F+G) + \frac{1}{2}I^+(F-G) + \frac{1}{2}I^-(F+G).$$

- I^+ and I^- are bounded linear operators on $\mathcal{B}_\alpha$ for $\alpha \in (0,1)$.
- The nonlinearities F and G are series expansions of the kernel in the Birkhoff-Rott integral, and all terms are quadratic or higher.
- If we just had $F(v) = v^2$, we would get $\|I^+(F(v_1) - F(v_2))\|_\alpha \leq C\|F(v_1) - F(v_2)\|_\alpha \leq C\|v_1 - v_2\|_\alpha \|v_1 + v_2\|_\alpha$. For small v_1, v_2, this is contracting.
- The real F, G are like this but just more complicated.

Adaptation to parabolic problems

- Let A be a Fourier multiplier operator with negative symbol. Let N be some nonlinearity. Then $u_t = Au + N(u)$, is a nonlinear forward parabolic equation.
- We can write the Duhamel formula,

$$u(t,\cdot) = e^{At}u_0 + \int_0^t e^{(t-s)A}N(u(s,\cdot))\, ds.$$

- If N is quadratic, say, and involves at most as many derivatives as A does, then you get the same properties as in elliptic problems.
- That is, using space-time versions of the Wiener algebra, we can show that the Duhamel integral is bounded on these spaces, and you can get existence via a contraction for small data.
- Applications: 2D Kuramoto-Sivashinksy (with Anna Mazzucato), epitaxial growth, dissipative Constantin-Lax-Majda (with Pavel Lushnikov, Michael Siegel, and Denis Silantyev), mean field games.

The Mean Field Games System

- The following is the mean field games system of coupled PDEs:

$$u_t + \Delta u + \mathcal{H}(t, x, Du, m) = 0,$$

$$m_t - \Delta m + \operatorname{div}(m\mathcal{H}_p(t, x, Du, m)) = 0.$$

- We can take $x \in \mathbb{T}^n$ and $t \in [0, T]$, for some T.
- This is supplemented with boundary conditions. We specify $m(0, x) = m_0(x)$, and $u(T, x) = G(x, m(T, x))$ for a given payoff function. Special case: $u(T, x) = u_T(x)$.
- In most existence theory in the literature, $\mathcal{H}(t, x, p, m) = H(t, x, p) + F(t, x, m)$, and H is taken to be convex and F is taken to be monotone in m. These assumptions may not hold for systems arising in practice, and are unnecessary for our approach.

Reformulating for the Duchon-Robert Method

- Project away the means: let $w = \mathbb{P}u$ and let $\mu = m - \bar{m}$.
- Let $\mathbb{P}\mathcal{H}(t, x, Dw, \mu) = \Xi(t, x, Dw, \mu) = \mathbb{P}(b(t, x)\mu + \Upsilon(t, x, Dw, \mu))$.
- Let $\Theta(t, x, Dw, \mu) = \mathcal{H}_p(t, x, Du, m)$.
- We get the following system:

$$w_t + \Delta w + \mathbb{P}(b\mu) + \mathbb{P}(\Upsilon(\cdot, \cdot, Dw, \mu)) = 0,$$

$$\mu_t - \Delta\mu + \operatorname{div}(\mu\Theta(\cdot, \cdot, Dw, \mu)) + \bar{m}\operatorname{div}(\Theta(\cdot, \cdot, Dw, \mu)) = 0.$$

- The reason for separating out a linear term in the w equation: say, for example, that $\mathcal{H} = m|Du|^4 + m^3$. Then, we have $m^3 = (\mu + \bar{m})^3$, which has a term linear in μ. Otherwise, Υ will be assumed to satisfy a nonlinear estimate.

The Duhamel Formulation

- Write down our integral operators

$$I^+(f)(t) = \int_0^t e^{\Delta(t-s)} f(s,\cdot)\, ds$$

  and

$$I^-(f)(t) = \int_t^T e^{\Delta(s-t)} f(s,\cdot)\, ds.$$

  Also, let $I_T f = I^+(f)(T)$.
- We get the following Duhamel formula for the forward equation for μ :

$$\mu(t,\cdot) = e^{\Delta t}\mu_0 + I^+(\mathrm{div}(\mu\Theta(\cdot,\cdot,Dw,\mu)))(t,\cdot) \\ + \bar{m}(I^+(\mathrm{div}(\Theta(\cdot,\cdot,Dw,\mu))))(t,\cdot),$$

- Define $A(\mu, w) = \mu(T,\cdot)$; this involves I_T.

Continuing the Duhamel Formulation

- Then, using the payoff boundary condition, we integrate backward in time from time T, finding the following Duhamel formula for the backward equation for w :

$$\begin{aligned} w(t,\cdot) = e^{\Delta(T-t)}\tilde{G}(\cdot, A(\mu,w)) - I^{-}(\mathbb{P}\Upsilon(\cdot,\cdot,Dw,\mu))(t) \\ - I^{-}(\mathbb{P}(be^{\Delta\cdot}\mu_0))(t) - I^{-}(\mathbb{P}(bI^{+}\mathrm{div}(\mu\Theta(\cdot,\cdot,Dw,\mu))(\cdot)))(t) \\ - \bar{m}I^{-}(\mathbb{P}(bI^{+}\mathrm{div}(\Theta(\cdot,\cdot,Dw,\mu))(\cdot)))(t). \end{aligned}$$

- These equations for μ and w give a fixed point problem. We seek fixed points in some function space.

Function Spaces: The Wiener algebra

- We saw the Wiener algebra before, and a space-time version adapted to the time interval $[0, \infty)$.
- We need a new space-time version for a finite time interval, $[0, T]$:

$$\|f\|_{\mathcal{B}_{\alpha,j}} = \sum_{k\in\mathbb{Z}^n} \sup_{t\in[0,T]} \left| |k|^j e^{\beta(t)|k|} \hat{f}(k) \right| .$$

- Here, $j \in \mathbb{N}$, and $\beta(s) = \begin{cases} 2\alpha s/T, & s \in [0, T/2], \\ 2\alpha - 2\alpha s/T, & T \in [T/2, T]. \end{cases}$
- These spaces are related to the dissertation work of my student, Timur Milgrom. The spaces are still Banach algebras.

Bounds for the Linear Operators

- Our operators I^+ and I^- are bounded linear operators between $\mathcal{B}_{\alpha,j}$ and $\mathcal{B}_{\alpha,j+2}$, for any j and for any $\alpha \in [0, T/2)$:

$$\|I^{\pm}\|_{\mathcal{B}_{\alpha,j} \to \mathcal{B}_{\alpha,j+2}} \leq \frac{2T}{T - 2\alpha} + 2.$$

- The gain of two derivatives here comes from the presence of the Laplacian, and is a version of parabolic smoothing.

The Contraction Mapping Argument

- We find a fixed point of a mapping associated to the Duhamel formulation.
- This requires showing that the associated mapping is a local contraction.
- We make Lipschitz assumptions on G and on $\mathcal{H}$ via Υ and Θ, on our Wiener-type spaces.
- So, for example, we might assume

$$\|\Upsilon(\cdot,\cdot,a,b)-\Upsilon(\cdot,\cdot,y,z)\|_{\mathcal{B}_{\alpha,j}} \leq M(a,b,y,z)\left[\|(a,b)-(y,z)\|_{(\mathcal{B}_{\alpha,j})^{n+1}}\right],$$

  where M is a function which is continuous and which satisfies $M(0,0,0,0)=0$.
- Together with estimates for I^{+} and I^{-}, we get a local contraction (local about the origin).

A mean field games existence theorem

- Let $T > 0$ be given. Let $\alpha \in (0, T/2)$ be given. Let Υ, G, and Θ satisfy the appropriate Lipschitz properties.
- There exists $\delta > 0$ such that if $\|\mu_0\| < \delta$, then there exist w and μ in $\mathcal{B}_{\alpha,2}$ so that u and m solve the mean field games system.
- For all $t \in (0, T)$, the functions u and m are analytic.
- So, we get existence of smooth solutions, as long as the initial m is a small perturbation of the uniform distribution.

About applied problems

- Problems arising from applications are frequently non-separable, but may not fit exactly into the framework we have just described.
- Two key examples are the model for natural resource extraction of Chan and Sircar, and the model for household savings and wealth of Achdou, Buera, Lasry, Lions, and Moll.
- We will now describe the household wealth problem and describe existence theory (actually, a nonexistence result) for it.

The household wealth problem

- Households seek to maximize utility from consumption, v. The distribution of households is g. Each household decides an allocation of their income on savings vs. consumption, and these decisions influence others through determination of the interest rate.
- The independent variables are $a \in [a_{min}, \infty)$, wealth/debt and $z \in [z_{min}, z_{max}]$, income.
- We have the mean field games system

$$\partial_t v + \frac{\sigma^2(z)}{2}\partial_{zz} v + \mu(z)\partial_z v + (z + r(t)a)\partial_a v + H(\partial_a v) - \rho v = 0,$$

$$\partial_t g - \frac{1}{2}\partial_{zz}(\sigma^2(z)g) + \partial_z(\mu(z)g) + \partial_a((z + r(t)a)g) + \partial_a(H_p(\partial_a v)g) = 0.$$

- The system comes with data, and with moment constraints:

$$g(0, \cdot) = g_0, \qquad v(T, \cdot) = v_T.$$

$$\int\int g\, dadz = 1, \qquad \int\int ag\, dadz = 0.$$

About the interest rate, $r(t)$

- The interest rate, $r(t)$, is not a given function. It is to be determined from v and g.
- It is the only source of coupling in the problem. If r were given, then the v equation would have no dependence on g in the v equation.
- That $r = r[v, g]$ means that this is a non-separable MFG system.
- Achdou et al. state that r is to be determined from the moment constraint $\int\int ag = 0$. We will spend significant attention on how this does or does not work.

Discussion of the problem

- Of course, the constraint $\int\int g = 1$ is part of the specification that g is a probability distribution. We should also have g non-negative; if g is initially non-negative, the equation for g will preserve this.
- The constraint $\int\int ag = 0$ is an equilibrium condition. It states that the positive and negative wealth in the system are in balance, so that all money borrowed by households under consideration is borrowed from other households in the system.
- The diffusion and drift coefficient functions σ and μ are given functions of z.
- The form of the system is not the same as those for which existence theory for mean field games systems has previously been developed.
- We will follow an argument similar to A '22, but rather than a coupled system of nonlinear heat equations, this has more the flavor of a coupled system of transport equations.

The constraint $\int ag = 0$ and the interest rate, r

- The only coupling between the equations is through the interest rate, $r(t)$.
- How is $r(t)$ to be determined, then? Through the constraint $\int\int ag(t,a,z)\ dadz = 0$.
- Say that this constraint is satisfied by the data $g(0,\cdot)$. Then we take the time derivative of the constraint, requiring $\int\int ag_t = 0$:
- Call $\mathcal{C} = \int\int ag\ dadz$.

$$\mathcal{C}_t = \int\int \frac{a}{2}\partial_{zz}(\sigma^2 g)\ dadz - \int\int a\partial_z(\mu g)\ dadz \\ - \int\int a\partial_a((z+ra)g)\ dadz - \int\int a\partial_a(H_p g)\ dadz.$$

- Assumptions on the diffusion and drift coefficients σ and μ make the first two terms vanish. The third and fourth terms can be integrated by parts.

First calculation of the interest rate, r

- Continuing our calculation, we have

$$\mathcal{C}_t = -\int a(z+ra)g\Big|_{a_{min}}^{a=\infty} dz + \int\int (z+ra)g \; dadz$$
$$- \int aH_p g\Big|_{a_{min}}^{a=\infty} dz + \int\int H_p g \; dadz.$$

- If g were supported at a_{min}, you could solve for r setting $\mathcal{C}_t = \mathcal{C} = 0$:

$$r = -\frac{\int\int (z+ra+H_p)g \; dadz + a_{min}\int (z+H_p)g\Big|_{a_{min}} dz}{a_{min}^2 \int g(a_{\min}, z) \; dz}.$$

When g is supported away from a_{min}

- We had the formula

$$\mathcal{C}_t = -\int a(z+ra)g\Big|_{a_{min}}^{a=\infty} dz + \int\int (z+ra)g\; dadz$$
$$- \int aH_p g\Big|_{a_{min}}^{a=\infty} dz + \int\int H_p g\; dadz.$$

- If we assume g is compactly supported on (a_{min}, ∞), then the first and third terms vanish.
- We introduce $\mathcal{Q} = \int\int (z+H_p)g\; dadz$. Then we have found

$$\mathcal{C}_t - r\mathcal{C} = \mathcal{Q}.$$

A relaxation

- Under our support assumption, $\mathcal{C} = 0$ implies $\mathcal{Q} = 0$, and thus $\mathcal{Q}_t = 0$.
- We will use $\mathcal{Q}_t = 0$ to determine the interest rate.
- We need to be careful, though: while $\mathcal{C} = 0$ implies $\mathcal{Q}_t = 0$, the reverse implication does not hold.
- If instead $\mathcal{Q} = \mathcal{Q}_0 \neq 0$, and $\mathcal{C}$ is initially zero, then $\mathcal{C}_t$ is initially nonzero and it is not true that $\mathcal{C} = 0$ for all t.

The interest rate r with support away from a_{min}

- We want to choose the interest rate r to enforce $\mathcal{Q}_t = 0$.
- We take the time derivative:

$$\mathcal{Q}_t = P - r \int\int (\partial_a v) g H_{pp}\, dadz,$$

  where P is a collection of terms which do not explicitly involve r.
- If $\int\int (\partial_a v) g H_{pp}\, dadz \neq 0$, then we can solve for r.
- This quantity is always nonzero, since we take $\partial_a v > 0$ since H_{pp} has a sign (inherited from u), and since g is a probability measure.
- If we let $K = -\int\int (\partial_a v) g H_{pp}\, dadz$, although we know K cannot equal zero, we still need to be careful about not letting it get arbitrarily small.
- We can express our interest rate as

$$r(t) = \rho + \frac{\int\int \left[\mu g - \frac{1}{2} H_{pp} \partial_z \left((\partial_z \partial_a v)\, \sigma^2 g \right) \right]\, dadz}{K}.$$

Existence and nonexistence: strategy

- We prove existence and uniqueness of some solutions for the relaxed problem by an energy method argument (not the Duchon-Robert method).
- As we have said, any solution of the original problem ($\mathcal{C} = 0$) is also a solution of our problem ($\mathcal{Q}_t = 0$).
- Since solutions of the relaxed problem exist and are unique, if there is a solution of the original problem with given data, then it must be the solution we find of the relaxed problem.
- We can verify in some cases that the solution of the relaxed problem has $\mathcal{C}_t \neq 0$ and thus does not satisfy $\mathcal{C} = 0$; therefore in these cases there is no solution of the original problem.

Conclusions

- The Duchon-Robert method allows one to prove existence of solutions for equations which are elliptic in space-time or which are parabolic.
- The parabolic case includes such things as coupled forward-backward parabolic systems.
- A contraction mapping is found on functions on space-time, so the entire temporal domain is treated at once.
- With the function spaces having exponential weights in the Wiener algebra, this implies analyticity of solutions at all times in the interior of the time interval, starting from non-analytic data.
- Although we did not discuss it, we also can prove existence theorems for mean field games by energy methods with Sobolev data.
- Problems arising from specific applications do not necessarily fit into the general framework but can be studied by similar means, such as for the household wealth problem.

Thanks for your attention.