On rotational eddy viscosity models

Luigi C. Berselli

Università degli Studi di Pisa

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Joint project with
R. Lewandowski (Rennes) A. Kaltenbach & M. Růžička (Freiburg)

Qualitative/Quantitative results

In the study of the NSE equations

$$\partial_t \mathrm{v} + \nabla \cdot (\mathrm{v} \otimes \mathrm{v}) - \nu \Delta \mathrm{v} + \nabla p = \mathrm{f} \qquad \mathrm{div}\, \mathrm{v} = 0,$$

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Invariance of scaling of the Navier Stokes equations is studied $x' = Lx$ and $\mathsf{v}' = U\mathsf{v}$, obtaining the non-dimensional form

$$\partial_t \mathsf{v} + \nabla \cdot (\mathsf{v} \otimes \mathsf{v}) - \frac{1}{Re} \Delta \mathsf{v} + \nabla p = \mathsf{f}, \qquad Re = UL\,\nu^{-1}$$

and a criterion about the stability in terms of Re.

Qualitative/Quantitative results

1941 Kolmogorov's theory predicts that simulating turbulent flows by using the Navier-Stokes Equations

$$\begin{aligned} \partial_t \mathrm{v} + \nabla \cdot (\mathrm{v} \otimes \mathrm{v}) - \nu \Delta \mathrm{v} + \nabla p = \mathrm{f}, \\ \mathrm{div}\, \mathrm{v} = 0, \\ u(0,x) = u_0(x), \end{aligned}$$

requires $\mathcal{N} = O(Re^{9/4})$ degrees of freedom, where

$$Re = UL\,\nu^{-1}$$

denotes the Reynolds number and U and L are a typical velocity and length, respectively.

Qualitative/Quantitative results

This number $\mathcal{N}$ is too large, in comparison with computational capabilities (especially available memory) of actual computers, to perform a Direct Numerical Simulation (DNS).

> *"It must be admitted that the problems are too vast to be solved by a direct computational attack." (J. von Neumann, 1949)*

Indeed, for realistic flows (geophysical flows) $Re = O(10^8)$, yielding $\mathcal{N}$ of order 10^{18}....but also smaller values of Re are challenging.

Introduction to averaging

It is clear that we cannot resolve all (significant) scales of the flow, but we can try to decompose the velocity as

$$\mathrm{v} = \overline{\mathrm{v}} + \mathrm{v}'$$

where $\overline{\mathrm{v}}$ is the mean velocity (to be specified) and v' are the turbulent fluctuations (baptized in this way by Lord Kelvin).

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There are some fundamental issues:

- Is this physically sound?
- How to mathematically define averages.
- Write algebraic, partial, integro, pseudo differential (whatsoever) equations for the averages.

The basic (mathematical) goal

The basic goal is to have a system of PDE's involving only the averaged field, with good mathematical properties.

This his approach for describing and simulating the large scales, find its "modern" origin with J. Smagorinsky, J.-L. Lions, and O.A. Ladyžhenskaya $\sim$ 1960, with rather different motivations.

The analysis of these models was strictly linked with the theory of monotone operators of which the p-Laplacian is the fundamental model $\Delta_p u := \mathrm{div}\,(|\nabla u|^{p-2}\nabla u)$.

Smagorinsky model

The Smagorinsky model $\sim$ 1960 have been used earlier by von Neumann and Richtmyer $\sim$ 1950.

$$\begin{aligned} \partial_t \mathsf{v} - C_S \ell^2 \mathrm{div}\left(|\nabla \mathsf{v}| \nabla \mathsf{v}\right) + \nabla \cdot (\mathsf{v} \otimes \mathsf{v}) - \nu \Delta \mathsf{v} + \nabla p = \mathsf{f}, \\ \mathrm{div}\, \mathsf{v} = 0, \\ \mathsf{v}(0, x) = \mathsf{v}_0(x), \end{aligned}$$

where ℓ has the dimension of a length (smallest resolved scale)

For this model extensive testing of the numerical properties has been performed.

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A limitation (others to be discussed later on) is that it suggests that involves a single length scale, the Prandtl one.

A few remarks

The operator used by Smagorinsky (J.L. Lions for analysis) is not rotational invariant. Better expression would be

$$A = -C_S \ell^2 \mathrm{div}\left(|Dv|Dv\right) \qquad \text{with } Dv := \frac{\nabla v + \nabla v^T}{2}.$$

Theory pretty similar (Korn inequality), but fine properties of regularity for operator with ∇ or D are still being studied (B., Růžička, Mazya, Chianchi, Diening, Breit 2020-2021-2022....)
The existence theory uses the theory of monotone operators

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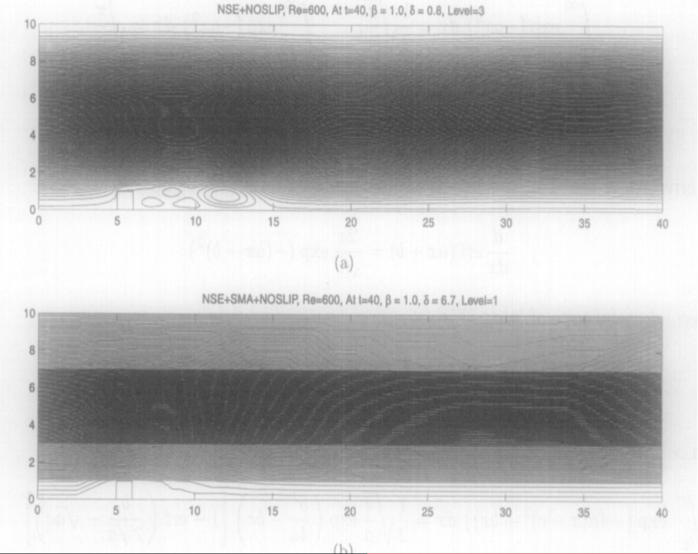
to pass to the limit in the Galerkin approximations. The generalization

$$A_p = -\mathrm{div}\left(|Dv|^{p-2}Dv\right)$$

was studied starting from Ladyžhenskaya for $p > 2$ as a possible approximation of NSE by more dissipative models

Prandtl and solid walls

Known numerical issues



Prandtl and solid walls

The Smagorinsky model fits with Dirichlet conditions and it is based on the assumption of a *constant* relevant length scale (too optimistic, special features appear near to the boundary layer).

It has been adapted (van Driest damping for channel flow)

$$-\nabla \cdot (\kappa^2 z^2 |D\mathrm{v}| D\mathrm{v}) \qquad \text{in } \{z > 0\} \subset \mathbb{R}^3$$

by imposing that the factor ℓ is a multiple of the distance from the flat boundary: $\ell = \ell(z) = \kappa z$, where $\kappa \in [0.35, 0.42]$ is the Von Kármán constant

Rotational models

The Boussinesq assumption suggests that –in average–fluctuations dissipate energy: additional turbulent viscosity $\nu_T \geq 0$, which is proportional to

- the mixing length
- the kinetic energy of fluctuations

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$$\nu_T^{rot} \sim \ell^2(x)|\omega(x)|,$$

where ℓ is a multiple of the distance from the boundary and $\omega = \mathrm{curl}\, \mathrm{v} = \nabla \times \mathrm{v}$.

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Rotational equation: $\mathrm{curl}\,(\nu_T^{rot}\omega)$

Rotational Smagorinsky (Baldwin & Lomax)

Hence arriving to the model

$$\begin{aligned} \partial_t \mathrm{v} - \nu_0 \,\mathrm{div}\, D\mathrm{v} + (\nabla \mathrm{v})\mathrm{v} + C_{BL} \mathrm{curl}\left(d^2|\omega|\omega\right) + \nabla q = f & \quad \text{in } \Omega, \\ \mathrm{div}\, \mathrm{v} = 0 & \quad \text{in } \Omega, \\ \mathrm{v} = 0 & \quad \text{on } \partial\Omega, \end{aligned} \tag{1}$$

where

$$d = d(x) = \mathrm{dist}(x, \partial\Omega).$$

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Since $\nu_0 << C_{BL}$ the role of the classical linear dissipative term should be disregarded: results valid in the limit is vanishing viscosities. (The presence of $\nu_0 > 0$ would make the analysis much simpler).

Modeling: alternate scalings

If one thinks of a flow as composed of eddies of different sizes

- in a region of large eddies the velocity and its curl changes are both $\mathcal{O}(1)$ of the typical distance;
- in a region of smaller eddies the velocity changes over a distance of $\mathcal{O}$(eddy length scale), so the local deformation is $\mathcal{O}$(1/eddy length scale)

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The BL model introduces a turbulent viscosity $\nu_T = (Cd)^2|\omega|$, where d is the (local) smallest resolved scale

$$\nu_T = \begin{cases} \mathcal{O}(d^2) & \text{in regions where } |\omega| = \mathcal{O}(1) \\ \mathcal{O}(d) & \text{in the smallest resolved scale where } |\omega| = \mathcal{O}(d^{-1}). \end{cases}$$

Modeling: alternate scalings

By extrapolation motivated by experiments the following alternate scaling has been also proposed

$$\nu_T = (C\delta)^{\alpha = p-1} |Dv|^{p-2} \qquad 1 < p < \infty.$$

It satisfies

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This is the only way to identify (by using only a typical length and the vorticity) a quantity with the dimensions of a viscosity.

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The dimensions of this quantity are $\nu_T \sim L^{\theta+\alpha} T^{2-\theta-p}$, hence to be dimensional consistent one has to solve the following system

$$\begin{cases} \theta + \alpha = 2 \\ 2 - \theta - p = -1 \end{cases} \Longrightarrow \qquad \theta = 3 - p \quad \text{and} \quad \alpha = p - 1.$$

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It turns out again that the "correct" exponent is $\alpha = p - 1$ and

$$S(v_*, d(x), \omega) \sim v_*^{3-p} d(x)^{p-1} |\omega|^{p-2} \omega.$$

The Rotational Smagorinsky: a PARABOLIC (in certain spaces) problem DEGENERATE at the boundary

The model we are considering is then the following

$$
\begin{aligned}
\partial_t \mathrm{v} + \omega \times \mathrm{v} + \operatorname{curl}\left(C_\alpha \ell^{\alpha} |\omega| \omega\right) + \nabla q = f \qquad & \text{in } (0,T)\times\Omega,\\
\omega = \operatorname{curl} \mathrm{v} \qquad & \text{in } (0,T)\times\Omega,\\
\operatorname{div} \mathrm{v} = 0 \qquad & \text{in } (0,T)\times\Omega,\\
\mathrm{v} = 0 \qquad & \text{on } (0,T)\times\partial\Omega,\\
\mathrm{v}(0) = \overline{\mathrm{v}_0} \qquad & \text{in } \Omega,
\end{aligned}
\tag{2}
$$

- Ω is a smooth bounded domain in $\mathbb{R}^3$
- ℓ is the Prandtl mixing length,
- $0 < \alpha < 2 = 3 - 1$ is a given exponent,
- $C_\alpha > 0$ is a calibration constant,
- v is the mean velocity,
- ω is the mean vorticity,
- $\overline{q}$ the Bernoulli pressure + potentials

The Rotational Smagorinsky: main result

Theorem (Existence in weighted spaces, weight=distance from boundary)

Let $\ell(x) = d(x, \partial\Omega)$ *and* $\alpha \in [0, 2)$, $0 < T < \infty$, $\mathrm{v}_0 \in L^2_\sigma(\Omega)$, *and* $\mathrm{f} \in L^{3/2}(0, T; (W^{1,3}_0(\Omega, d^\alpha)^*)$. *Then,* $\exists$ *a weak solution to* (2) *s.t.*

$$\mathrm{v} \in C([0, T]; L^2_\sigma(\Omega)) \cap L^3(0, T; W^{1,3}_{0,\sigma}(\Omega, d^\alpha)),$$

and for all $t \in [0, T]$

$$\frac{1}{2}\|\mathrm{v}(t)\|^2 + \int_0^t \int_\Omega C_\alpha d^\alpha\, |\omega|^3\, dx\, ds = \frac{1}{2}\|\mathrm{v}_0\|^2 + \int_0^t \langle f, \mathrm{v}\rangle_{W^{1,3}_0(\Omega, d^\alpha)} ds.$$

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The limitation $\alpha < 2$ seems to be intrinsic to the problem due to the fact that d^α is not anymore a Muckenhoupt weight for $\alpha \geq 2$

Classical abstract tools

We recall that for a Stationary problem

$$Au = f \qquad A : V \to V' \text{ (separable reflexive Banach)}$$

existence follows from

- A bounded: $\|Au\|_{V'}$ bounded if $\|u\|_V$ bounded.
- A hemi-continuous: $\lambda \mapsto < A(u + \lambda v), w >$ continuous
- A monotone: $< Au - Av, u - v > \geq 0$
- A coercive: $\frac{<Av,v>}{\|v\|} \to \infty$ if $\|v\| \to \infty$

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Often this framework is not enough and more general properties are needed.

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A is pseudo-monotone if $\forall\,(x_n)_{n\in\mathbb{N}} \subseteq X$ from

$$x_n \overset{n\to\infty}{\rightharpoonup} x \text{ in } X,$$

$$\limsup_{n\to\infty} \langle Ax_n, x_n - x\rangle_X \leq 0,$$

it follows that $\langle Ax, x - y\rangle_X \leq \liminf_{n\to\infty} \langle Ax_n, x_n - y\rangle_X$ for all $y \in X$.

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Bounded+coercive+pseudo-monotone $\Rightarrow$ existence

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This derives from some general properties: Let $A, B : V \to V^*$

- If A is monotone and hemicontinuous, then A is pseudo-monotone.
- If A is strongly continuous, then A is pseudo-monotone.
- If A and B are pseudo-monotone, then $A + B$ is pseudo-monotone.

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Strategy used by J.L. Lions for the p-Navier-Stokes equations: the treatment of "lower order terms" as a compact operator needs usually additional information on the time derivative.
The incorporation of the time derivative into the function space is however critical w.r.t the coercivity.

The evolution problem

For $I := (0, T)$, $T \in (0, \infty)$, and $p \in (1, \infty)$, we set

$$\mathcal{X} := L^p(I, V) \qquad \text{and} \qquad \mathcal{Y} := L^\infty(I, H).$$

$$\mathcal{W} := W^{1,p,p'}(I, V, V^*) := \left\{ \boldsymbol{u} \in \mathcal{X} \mid \exists \, \tfrac{d\boldsymbol{u}}{dt} \in L^{p'}(I, V^*) \right\},$$

the **Bochner–Sobolev space** w.r.t. the evolution triple (V, H, id).

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For $u_0 \in H$ and $\boldsymbol{f} \in L^{p'}(I, V^*)$, a solution $\boldsymbol{u} \in \boldsymbol{\mathcal{W}}$ of the evolution equation $\frac{d\boldsymbol{u}}{dt} + \boldsymbol{\mathcal{A}}\boldsymbol{u} = \boldsymbol{f}$ is

$$\int_I \left\langle \frac{d\boldsymbol{u}}{dt}(t) + \boldsymbol{\mathcal{A}}(\boldsymbol{u}(t)), \boldsymbol{v}(t) \right\rangle_V = \int_I \langle \boldsymbol{f}(t), \boldsymbol{v}(t)\rangle_V \, dt \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{X}},$$
$$\boldsymbol{u}_c(0) = u_0 \quad \text{in } H.$$

Tools for existence

Extension to time-dependent problems

Definition (Bochner pseudo-monotonicity)

$\boldsymbol{\mathcal{A}}:\boldsymbol{\mathcal{X}}\cap\boldsymbol{\mathcal{Y}}\to\boldsymbol{\mathcal{X}}^*$ is **Bochner pseudo-monotone** if for a sequence $(\boldsymbol{u}_n)_{n\in\mathbb{N}}\subseteq\boldsymbol{\mathcal{X}}\cap\boldsymbol{\mathcal{Y}}$ from

$$\begin{aligned} \boldsymbol{u}_n &\overset{n\to\infty}{\rightharpoonup} \boldsymbol{u} && \text{in } \boldsymbol{\mathcal{X}}, \\ \boldsymbol{u}_n &\overset{*}{\rightharpoonup} \boldsymbol{u} && \text{in } \boldsymbol{\mathcal{Y}} \quad (n\to\infty), \\ \boldsymbol{u}_n(t) &\overset{n\to\infty}{\rightharpoonup} \boldsymbol{u}(t) && \text{in } H \quad \text{for a.e. } t\in I, \end{aligned}$$

and

$$\limsup_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n, \boldsymbol{u}_n-\boldsymbol{u}\rangle_{\boldsymbol{\mathcal{X}}} \leq 0,$$

it follows that
$\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}, \boldsymbol{u}-\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{X}}} \leq \liminf_{n\to\infty} \langle \boldsymbol{\mathcal{A}}\boldsymbol{u}_n, \boldsymbol{u}_n-\boldsymbol{v}\rangle_{\boldsymbol{\mathcal{X}}}$ for every $\boldsymbol{v}\in\boldsymbol{\mathcal{X}}$.

Tools for existence

Definition (Bochner coercivity)

An operator $\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{X}}^*$ is called:

(i) **Bochner coercive with respect to $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}^*$ and** $u_0 \in H$ if there is a constant $M := M(\boldsymbol{f}, u_0, \boldsymbol{\mathcal{A}}) > 0$ such that for every $\boldsymbol{u} \in \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}}$ from

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_H^2 + \langle \boldsymbol{\mathcal{A}}\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{u}\chi_{[0,t]}\rangle_{\boldsymbol{\mathcal{X}}} \leq \frac{1}{2}\|u_0\|_H^2 \qquad \text{for a.e. } t \in I,$$

it follows that $\|\boldsymbol{u}\|_{\boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}}} = \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{X}}} + \|\boldsymbol{u}\|_{\boldsymbol{\mathcal{Y}}} \leq M$.

(ii) **Bochner coercive** if it is Bochner coercive with respect to $\boldsymbol{f}$ and u_0, for every $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}^*$ and $u_0 \in H$.

Tools for existence

Definition (Bochner coercivity)

An operator $\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{X}}^*$ is called:

(i) **Bochner coercive with respect to $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}^*$ and** $u_0 \in H$ if there is a constant $M := M(\boldsymbol{f}, u_0, \boldsymbol{\mathcal{A}}) > 0$ such that for every $\boldsymbol{u} \in \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}}$ from

$$\frac{1}{2}\|\boldsymbol{u}(t)\|_H^2 + \langle \boldsymbol{\mathcal{A}}\boldsymbol{u} - \boldsymbol{f}, \boldsymbol{u}\chi_{[0,t]}\rangle_{\boldsymbol{\mathcal{X}}} \leq \frac{1}{2}\|u_0\|_H^2 \qquad \text{for a.e. } t \in I,$$

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(ii) **Bochner coercive** if it is Bochner coercive with respect to $\boldsymbol{f}$ and u_0, for every $\boldsymbol{f} \in \boldsymbol{\mathcal{X}}^*$ and $u_0 \in H$.

Theorem (Abstract existence)

if $\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{X}}^*$ is bounded, Bochner pseudo-monotone, and Bochner coercive, then the corresponding evolution problem $\frac{d\boldsymbol{u}}{dt} + \boldsymbol{\mathcal{A}}\boldsymbol{u} = \boldsymbol{f}$ is solvable for any initial datum $u_0 \in H$.

Existence theorem: practical case

Proposition

Let $A : V \to V^*$ *and* $\exists\, p \in (1,\infty)$ *and* $\exists\, c_0, c_1 > 0$ *such that:*

(C.1) $\|Av\|_{V^*} \leq c_0 \|v\|_V^{p-1} \qquad \forall\, v \in V.$

(C.2) $A : V \to V^*$ *is pseudo-monotone.*

(C.3) $\langle Av, v\rangle_V \geq c_1 \|v\|_V^p \qquad \forall\, v \in V.$

Then, the induced operator $\boldsymbol{\mathcal{A}} : \boldsymbol{\mathcal{X}} \cap \boldsymbol{\mathcal{Y}} \to \boldsymbol{\mathcal{X}}^*$

$$\langle \boldsymbol{\mathcal{A}}\boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{\mathcal{X}}} := \int_I \langle A(\mathrm{u}(t)), \mathrm{v}(t)\rangle_V \, dt,$$

is well-defined, bounded, Bochner pseudo-monotone, Bochner coercive.

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is well-defined, bounded, Bochner pseudo-monotone, Bochner coercive.

It is enough to check properties for each fixed $t \in I$!

Functional setting: Weighted spaces

$$L^p(\Omega, \varrho) := \left\{ f : \ \Omega \to \mathbb{R}^n \text{ measurable } : \int_\Omega |f(x)|^p \, \varrho(x) \, \mathrm{d}x < \infty \right\}.$$

note that $L^p(\Omega, d^\alpha) \subset L^1(\Omega)$ if

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$$W^{k,p}(\Omega, \varrho) := \{ f : \ \Omega \to \mathbb{R}^n : \ D^\alpha f \in L^p(\Omega, \varrho) \text{ for all } \alpha \text{ s.t. } |\alpha| \leq k \},$$

$\left(\int_\Omega d^\alpha |\nabla \mathrm{f}|^p \, \mathrm{d}x \right)^{\frac{1}{p}}$ is an equivalent norm in $W_0^{1,p}(\Omega, d^\alpha)$, provided that

$$0 \leq \alpha < p - 1.$$

Definition

We say that a weight $\varrho \in L^1_{\text{loc}}(\mathbb{R}^3)$ belongs to the Muckenhoupt class A_p, for $1 < p < \infty$, if there exists C such that

$$\sup_{Q \subset \mathbb{R}^n} \left(\fint_Q \varrho(x)\,\mathrm{d}x \right) \left(\fint_Q \varrho(x)^{1/(1-p)}\,\mathrm{d}x \right)^{p-1} \leq C,$$

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The function $\varrho(x) = \big(d(x)\big)^{\alpha}$ is a Muckenhoupt weight of class A_p if and only if $-1 < \alpha < p - 1$.

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Lemma

For $-1 < \alpha < p - 1$ there exists a constant $C = C(\Omega, \alpha, p)$ such that

$$\int_\Omega d^\alpha |\nabla v|^p \,\mathrm{d}x \leq C \int_\Omega d^\alpha |\mathrm{curl}\, v|^p \,\mathrm{d}x \qquad \forall\, v \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha). \tag{3}$$

Hardy-Sobolev-CKN inequalities

We will extensively use: $p \in [1, n)$, $\alpha \neq p - 1$ and $q \in [p, \frac{np}{n-p}]$

$$\left(\int_\Omega d^{\frac{q}{p}(n-p+\alpha)-n}|f|^q \, dx\right)^{\frac{1}{q}} \leq c \left(\int_\Omega d^\alpha |\nabla f|^p \, dx\right)^{\frac{1}{p}}, \qquad \text{(HS)}$$

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Lemma

$(V,H,id) := \left(W^{1,3}_{0,\sigma}(\Omega, d^\alpha), L^2_\sigma(\Omega), id\right)$ *is an evolution triple:*

$$\alpha \in [0,2): \left(\int_\Omega |u|^2\,dx\right)^{1/2} \leq C_{\alpha,\Omega}\left(\int_\Omega d^\alpha|\operatorname{curl} u|^3\,dx\right)^{1/3}.$$

Functional setting

$$V := W^{1,3}_{0,\sigma}(\Omega, d^\alpha) \qquad \|v\|_V := \left(\int_\Omega d^\alpha |\mathrm{curl}\, v|^3\, dx\right)^{1/3}$$

$$H := L^2_\sigma(\Omega) \qquad \|v\|_H := \left(\int_\Omega |v|^2\, dx\right)^{1/2}$$

$$\mathcal{X} := L^3(I, V), \qquad \mathcal{Y} := L^\infty(I, H)$$

$$\mathcal{W} := \left\{\mathrm{u} \in L^3(I, V) | \exists \frac{d\mathrm{u}}{dt} \in L^{3/2}(I, V^*)\right\},$$

and define the operator $A := S + B : V \to V^*$

$$\langle Sv, w\rangle_V := \int_\Omega d^\alpha |\mathrm{curl}\, v| \mathrm{curl}\, v \cdot \mathrm{curl}\, w\, dx,$$

$$\langle Bv, w\rangle_V := \int_\Omega (\omega \times v) \cdot w\, dx \qquad \omega = \mathrm{curl}\, v.$$

Verification of the hypotheses

Lemma (Monotonicity of A)

For smooth enough vector field ω_i and for $\alpha \in \mathbb{R}^+$ it holds that

$$\int_\Omega (d^\alpha |\omega_1|^{p-2}\omega_1 - d^\alpha |\omega_2|^{p-2}\omega_2) \cdot (\omega_1 - \omega_2)\, \mathrm{d}x \geq 0,$$

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Lemma (Boundedness of B)

For all $\alpha \in [0,2)$ the operator $B : V \to V^*$ is bounded. It satisfies $\langle Bu, v\rangle_V \leq c\|u\|_V^2\|v\|_V$ and $\langle Bu, u\rangle_V = 0$, for all $u, v \in V$.

Lemma (Compactness of B)

For all $\alpha \in [0,2)$ the operator $B : V \to V^*$ is compact.

Sketch of the proof

multiplying and dividing a.e. $x \in \Omega$ by the positive function $d^{\alpha/3}$

$$\left| \int_\Omega (\mathrm{curl}\, v \times u) \cdot w \right| \leq \int_\Omega d^{-\alpha/6}|u|\, d^{\alpha/3}|\mathrm{curl}\, v|\, d^{-\alpha/6}|w|$$

$$\leq \left(\int_\Omega d^{-\alpha/2}|u|^3 \right)^{1/3} \left(\int_\Omega d^{\alpha}|\mathrm{curl}\, v|^3 \right)^{1/3} \left(\int_\Omega d^{-\alpha/2}|w|^3 \right)^{1/3}.$$

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From (HS), for $p \in [1,3)$, $\alpha \in [0,2)$, $u \in W^{1,p}_{0,\sigma}(\Omega, d^\alpha)$

$$\left(\int_\Omega d^{-\frac{\alpha}{2}}|u|^{\frac{p(6-\alpha)}{2(3-p+\alpha)}}\,dx\right)^{\frac{1}{q}} \leq c\left(\int_\Omega d^{\alpha}|\nabla u|^p\,dx\right)^{\frac{1}{p}},$$

with

$$q := \frac{p(6-\alpha)}{2(3-p+\alpha)} < p^*.$$

$\forall\, \alpha \in [0,2)\ \exists\, p \in (1+\alpha, 3)$ such that $3 < q < p^*$. Next, Ω bounded implies $V \hookrightarrow L^3(\Omega, d^{-\frac{\alpha}{2}})$.

Extension to other values of $p > 3$

The results remains valid (minor changes) for all $p > 3$.

Theorem

Let $p > 3$, $\alpha \in [0, p-1)$, $0 < T < \infty$, $\overline{\mathrm{v}_0} \in L^2_\sigma(\Omega)$, *and* $\mathrm{f} \in L^{p'}(0, T; (W^{1,p}_0(\Omega, d^\alpha)^*)$*. Then,* $\exists$ *a weak solution to*

$$\partial_t v + \omega \times v + \mathrm{curl}\left(d^\alpha |\omega|^{p-2}\omega\right) + \nabla \overline{q} = \mathrm{f} \qquad + \textit{BC and IC}$$

such that

$$\overline{\mathrm{v}} \in C([0, T]; L^2_\sigma(\Omega)) \cap L^p(0, T; W^{1,p}_{0,\sigma}(\Omega, d^\alpha))$$

$$\frac{1}{2}\|v(t)\|^2 + \int_0^t \int_\Omega C_\alpha d^\alpha |\omega|^p = \frac{1}{2}\|\overline{\mathrm{v}_0}\|^2 + \int_0^t \langle f, v\rangle_{W^{1,p}_0(\Omega, d^\alpha)}.$$

Extension to other values of $p < 3$?

The argument used to estimate B for $p = 3$ fails

$$\left|\int_\Omega (\mathrm{curl}\, v \times u) \cdot w\right|$$
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estimate the 1^{st}-3^{rd} with (HS) for $q = 2p' < p^* \leftrightarrow p > \frac{9}{5}$.

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Sharpness of Hardy–Sobolev $\rightarrow$ existence of weak sol. (if possible) needs different methods/formulations of the problem (very weak sol?)

Frame invariance and vortex identification

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The mean velocity $\overline{u}$, being an average of u transforms in the same way as u, while

$$\tau = -\overline{(u' \otimes u')}$$

should have the same invariance of u' which is not necessarily that of full-frame invariance (invariance with respect to $x^* = Q(t)x + b(t)$), but only extended Galilean invariance.

Restrictions on τ

- Reynolds stress models expression invariant for arbitrary translation accelerations and should only be affected by rotations of the reference frame through the intrinsic mean vorticity $\overline{W} = \overline{\omega} + 2\Omega$ (angular velocity)
- All frame-dependent effects must vanish in the limit of 2D turbulence
- Reynolds stress models must be consistent with the Taylor Proudman theorem (statistically steady turbulence in a rapidly rotating frame should be 2D)

Failure of the classical EV models

- Smagorinsky: $\nu_T = l^2|D\overline{u}|$, it is frame-indifferent for all mean flows. However, since it is frame-indifferent in 3D as well as in 2D, incapable of describing the effects of rotation in retarding the energy transfer process

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  In the limit of 2D there is a violation of material frame-indifference (Coriolis force can be included in the pressure in 2D). Next, the BL model predicts that there is an increase in turbulent dissipation corresponding to an increase in the rotation rate of the framing which violates constraint (violating of Taylor-Proudman theorem)

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Nevertheless these two models have appealing analytical properties fitting with the mathematical theory of monotone operators.

Frame invariance vs vortex identification

The heuristic behind problem of designing a LES model in the family of EV can be summarized as:

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> *A major drawback of the eddy viscosity subgrid-scale stress models is their inability to represent correctly, with a single universal constant, different turbulent fields in rotating or sheared flows, near solid walls, or in transitional regimes.*

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> *. . . the extra stress-tensor should be active only in the regions of "high vortex activity". . . .*

Other models or criteria

A first 2D-criterion Okubo-Weiss (1970-1990) about $Tr(\nabla u : \nabla u^T) < 0$ (elliptical point)
Then 3D Q-criterion Hunt, Wray, and Moin 1988 order to adaptively detect regions of intense vorticity as those such that

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Also this model suffers from growth when rotation increases & it is based on a non-invariant quantity (not much better changing to $\frac{1}{2}(|\overline{W}| - |D\overline{u}|)$)

Vortex identification

So two problems are mixing, one is the *vortex identification* and the other is the identification/modeling of an *eddy viscosity*.

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1. Both laminar and turbulent boundary layer flows possesses vorticity. (Blausius) Vorticity of laminar boundary layer flow is concentrated near the wall surface and should be viewed as irrotational since the streamlines and path-lines are all parallel and straight.
2. Vorticity doesn't directly represent rotation even though rigid body rotation must possess vorticity. Therefore, vorticity could be small while rotation is strong and vorticity could be large while rotation is weak or none

Vortex identification

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Moreover different observer in non-inertial reference frames could determine large/small vorticity in a non-consistent way.

If rotation is an indication of unstable regions, the LES models should be based on Lagrangian invariance .

Vortex identification

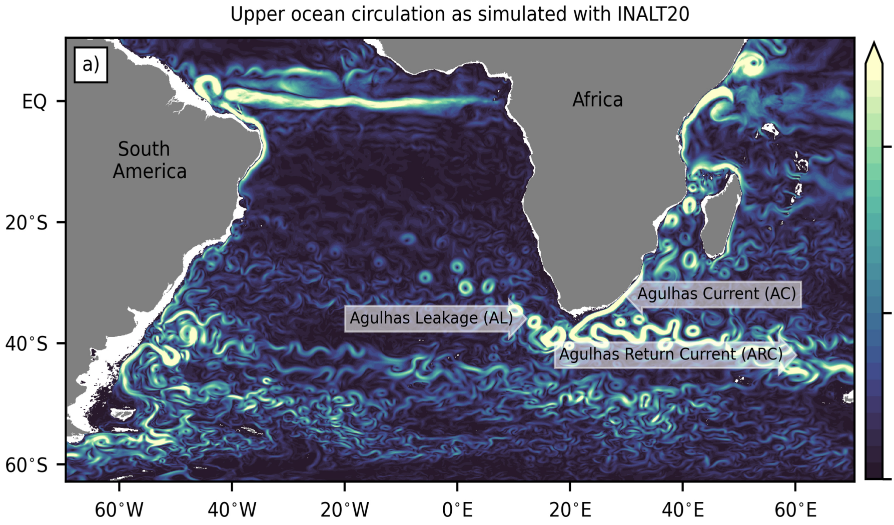
Here, we illustrate the detection of rotationally coherent eddy boundaries in velocity data derived from satellite-observed sea-surface heights η under the geostrophic approximation Haller et al. 2016 (ϕ, θ) longitude-latitude coordinate system.

$$\dot{\phi} = -\frac{g}{R^2 f(\theta)\cos(\theta)}\partial_\theta \eta$$

$$\dot{\theta} = \frac{g}{R^2 f(\theta)\cos(\theta)}\partial_\phi \eta$$

where g is the constant of gravity, R is the mean radius of the Earth and $f(\theta) = 2\Omega\sin(\theta)$ is the Coriolis parameter, with Ω denoting the Earth's mean angular velocity. The publicly available (Archiving, Validation and Interpretation of Satellite Oceanographic data) AVISO data are used.

Vortex identification



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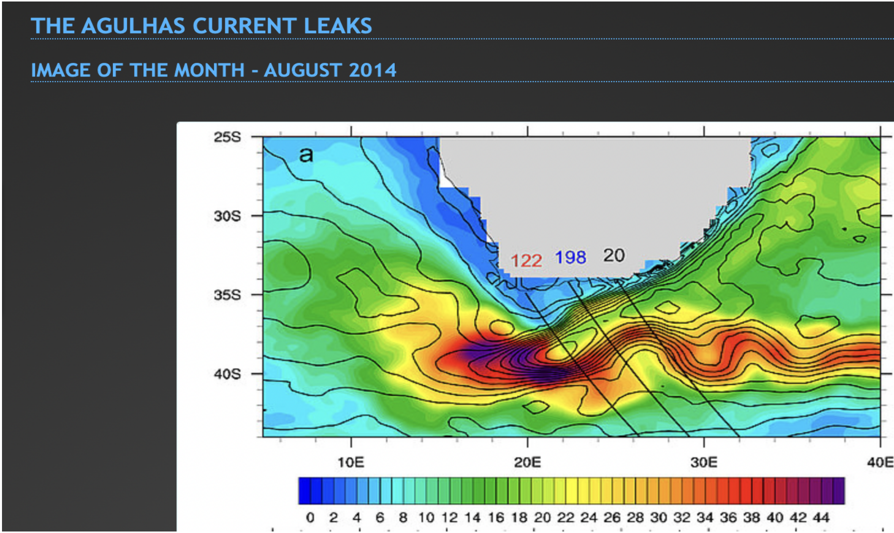

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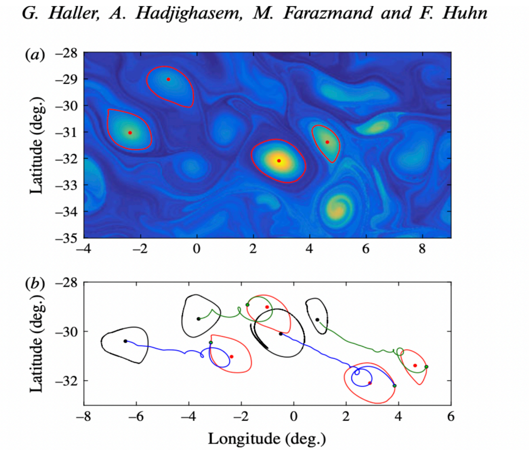
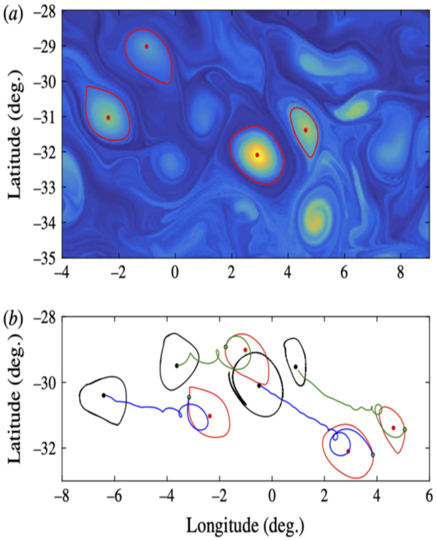


Figure: Heavy particles (blue) converge to the centre of anticyclonic eddies. Heavy particles (green) converge to the centre of cyclonic eddies.

Vortex identification

G. Haller, A. Hadjighasem, M. Farazmand and F. Huhn



Helmholtz and beyond

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Vorticity should be further decomposed into: 1) vortical vorticity contributed to rotation and 2) non-vortical vorticity (e.g. vorticity without rotation in laminar boundary layer flow)

$$\omega = R + \omega - R$$

R is vorticity as part of a vortex

$$\omega = \omega \cdot \frac{R}{|R|}\frac{R}{|R|} + \text{remainder} = \sqrt{\mathbf{\Omega}}|\omega|\frac{R}{|R|} + \text{remainder}$$

with

$$\mathbf{\Omega} := \frac{(\omega \cdot R)^2}{|\omega|^2|R|^2}$$

Pure deformation $\mathbf{\Omega} = 0$ while the rigidly rotational flow $\mathbf{\Omega} = 1$. Note that $0 \leq \mathbf{\Omega} \leq 1$

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When vorticity is aligned with vortex axial, the deformation will become very small.

Omega vortex identification

New methods have been proposed to estimate $\mathbf{\Omega}$ (Liu et al 2016)

$$\mathbf{\Omega} = \frac{(\omega \cdot R)^2}{|\omega|^2|R|^2} \sim \frac{|\omega|}{|Du| + |\omega|} \in [0, 1]$$

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Dissipation of vortical flow is lower than the corresponding non-vortical flow, like laminar boundary layer. The rotation state is more stable: transition laminar $\rightarrow$ turbulent is a process toward to a more "stable state".

Vortex Omega and LES

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Some argues to have a better identification with rotation

$$\nu_T = \nu_T\Big(\frac{|\omega - ⨍\,\omega|}{|Du| + |\omega - ⨍\,\omega|}\Big)$$

This because in an accelerating rotating frame

$$\omega^* = Q^T(t)\omega Q(t) - Q^T(t)\dot{Q}(t) \quad \omega^* - ⨍\,\omega^* = Q^T(t)(\omega - ⨍\,\omega)Q(t)$$

Thank you for your attention!