Self-similar spirals for $(\mathrm{SQG})_\gamma$

In collaboration with Javier Gómez-Serrano

Claudia García
Universidad de Granada

New York University Abu Dhabi

Generalized surface quasi-geostrophic equation

For $\gamma \in [0,1]$, the generalized surface quasi-geostrophic equation reads as

$$\begin{cases} \partial_t \theta + (u \cdot \nabla)\theta = 0, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ u = \nabla^\perp \psi, \quad \psi = -(-\Delta)^{-1+\frac{\gamma}{2}}\theta, & \text{in } [0,+\infty) \times \mathbb{R}^2, \\ \theta(t=0,\cdot) = \theta_0, & \text{in } \mathbb{R}^2, \end{cases}$$

- $\gamma = 0$: 2D Euler; $\gamma = 1$: SQG
- Charney (1940s), Blumen (1978), Held-Pierrehumbert-Garner-Swanson (1995): geophysical origin ($\gamma = 1$).
- Constantin-Majda-Tabak (1994): analogy with 3D incompressible Euler ($\gamma = 1$).
- Córdoba-Fontelos-Mancho-Rodrigo (2005): $\gamma \in (0,1)$.

Local well-posedness?

- Wu (2005), Chae-Constantin-Wu (2011): local well-posedness for in $C^{k,\alpha} \cap L^2$ ($k \geq 1,\ 0 < \alpha < 1$).
- Chae-Constantin-Wu (2011): local well-posedness in H^s with $s > \gamma + 1$, $\gamma \in [0, 1]$.
- Córdoba, Martínez-Zoroa (2021): strong ill-posedness in C^k ($k \geq 2$) for $\gamma = 1$
- Resnick (1995), Chae-Constantin-Córdoba-Gancedo-Wu (2012): global existence in L^2.
- Buckmaster-Shkoller-Vicol (2019): non uniqueness for $\gamma = 1$ for lower regularity than L^2 ($\Lambda^{-1}\theta \in C_t^\sigma C_x^\alpha$, with $\frac{1}{2} < \alpha < \frac{4}{5}$, $\sigma < \frac{\alpha}{2-\alpha}$).
- Global in time rigid solutions: Castro, Córdoba, Dristchel, G., Gómez-Serrano, Hassainia, Hmidi, Masmoudi, Mateu,..

Self-similar solutions

- Scaling for $(SQG)_\gamma$:

For any $\alpha > -1$ we get that

$$\theta(t,x) \mapsto \theta_\lambda(t,x) = \lambda^{1+\alpha-\gamma}\theta(\lambda^{1+\alpha}t, \lambda x),$$
$$u(t,x) \mapsto u_\lambda(t,x) = \lambda^{\alpha}u(\lambda^{1+\alpha}t, \lambda x),$$

for any $\lambda > 0$, is a solution.

- Self-similar solution:

$$\theta = \theta_\lambda, \quad \forall \lambda > 0.$$

- Self-similar profile $(\hat{\theta}, \hat{u})$:

$$\theta(t,x) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\hat{\theta}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right),$$
$$u(t,x) = \frac{1}{t^{\frac{\alpha}{1+\alpha}}}\hat{u}\left(\frac{x}{t^{\frac{1}{1+\alpha}}}\right),$$

where $\hat{u} = -\nabla^\perp(-\Delta)^{-1+\frac{\gamma}{2}}\hat{\theta}$.

Self-similar solutions

- Self-similar equation for $(\hat{\theta}, \hat{u})$:

$$\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) - \left\{\hat{u}(z) - \frac{z}{1+\alpha}\right\} \cdot \nabla\hat{\theta}(z) = 0,$$

where $z = \frac{x}{t^{\frac{1}{1+\alpha}}}$.

- We can write it as a system for $(\hat{\theta}, \hat{\psi})$:

$$\text{(vorticity equation)} \quad \frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) - \left\{\nabla^{\perp}\hat{\psi}(z) - \frac{z}{1+\alpha}\right\} \cdot \nabla\hat{\theta}(z) = 0,$$

$$\text{(elliptic equation)} \quad \hat{\psi} = -(-\Delta)^{-1+\frac{\gamma}{2}}\hat{\theta}(z).$$

- Trivial self-similar solution:

$$\hat{\theta}_{\text{trivial}}(z) = |z|^{-\frac{1}{\mu}}, \quad \frac{1}{2} < \mu < \frac{1}{2-\gamma}, \quad \frac{1}{\mu} = 1+\alpha-\gamma.$$

Self-similar solutions for 2D Euler

• 2D Euler equations: existence of non trivial self-similar solutions, which are perturbation of θ_{trivial}, with *large* m-fold symmetry

$$\omega_0(re^{i\vartheta}) = r^{-\frac{1}{\mu}}\left\{1+\tilde{\Omega}(\vartheta)\right\}$$

⇝ Elling (2016), Bressan-Murray (2020)

⇝ They strongly use that the elliptic equation is local!

⇝ Motivation: non uniqueness for Euler

Self-similar solutions for 2D Euler

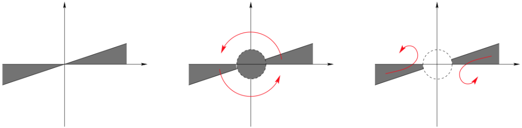


Figure 1: The supports of the initial vorticity $\overline{\omega}$, $\overline{\omega}_1^\varepsilon$, and $\overline{\omega}_2^\varepsilon$, considered at (1.3)-(1.6).

We then consider two sequences of initial data, with vorticity $\overline{\omega}_1^\varepsilon, \overline{\omega}_2^\varepsilon \in \mathbf{L}^\infty(\mathbb{R}^2)$, as shown in Figure 1, center and right. Namely

$$\overline{\omega}_1^\varepsilon(x) \doteq \begin{cases} \overline{\omega}(x) & \text{if} \quad |x| > \varepsilon, \\ \varepsilon^{-\frac{1}{\mu}} & \text{if} \quad |x| \leq \varepsilon, \end{cases} \tag{1.5}$$

$$\overline{\omega}_2^\varepsilon(x) \doteq \begin{cases} \overline{\omega}(x) & \text{if} \quad |x| > \varepsilon, \\ 0 & \text{if} \quad |x| \leq \varepsilon. \end{cases} \tag{1.6}$$

By Yudovich' theorem [24], for every $\varepsilon > 0$ the Cauchy problem for the incompressible Euler equation (1.1) with initial data (1.5) or (1.6) has a unique solution. Numerical simulations of these solutions, performed by Wen Shen [23], are shown in Figures 2 and 3, respectively. As $\varepsilon \to 0$, the initial data $\overline{\omega}_1^\varepsilon, \overline{\omega}_2^\varepsilon$ converge to the same limit $\overline{\omega}$ in $\mathbf{L}^p_{loc}$. However, at times $t > 0$ the corresponding solutions converge to two different limits. Both of these limits yield solutions to the Euler equations (1.1) with the same initial data (1.3).

Figure: Bressan, Murray (2020)

Self-similar solutions for 2D Euler

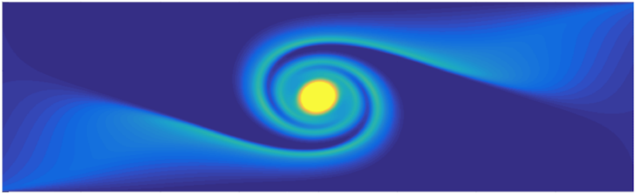


Figure 2: The vorticity distribution at time $t = 1$, for a solution to (1.1) with initial vorticity (1.5).

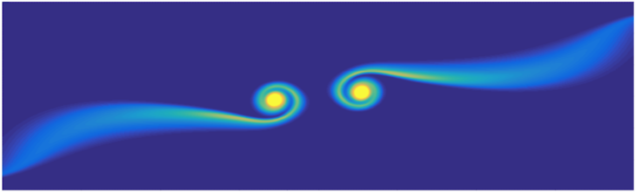


Figure 3: The vorticity distribution at time $t = 1$, for a solution to (1.1) with initial vorticity (1.6).

Figure: Bressan, Murray (2020)

Negative results for $(\mathrm{SQG})_\gamma$

• $(\mathrm{SQG})_\gamma$ equations: non existence of non trivial self-similar solutions for *smooth* self-similar profiles.

$\rightsquigarrow$ Chae (2007):

$$\widehat{\theta} \in L^{p_1} \cap L^{p_2},\ \widehat{u} \in C^1, \quad 0 < p_1 < p_2 \leq +\infty.$$

$\rightsquigarrow$ Cannone-Xue (2015):

$$\widehat{\theta} \in C^1_{\mathrm{loc}} \cap L^{\frac{2p+2}{3-\gamma}} \cap L^{\frac{2(2p+5-\gamma)}{(3-\gamma)^2}}, \quad p \in (1, \frac{2}{1-\gamma}).$$

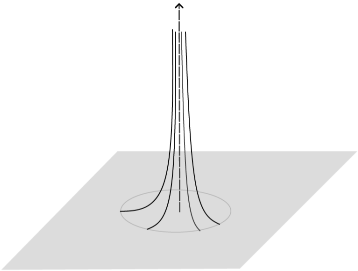
Our goal

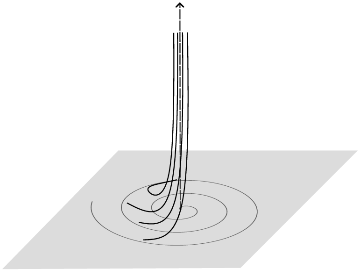
G., Gómez-Serrano (2022)

Let $\gamma \in (0,1)$ and $\alpha \in (1, 1+\gamma)$. Then, there exists $\varepsilon > 0$ such that for any $\frac{2\pi}{m}$-periodic $\tilde{\Omega} \in B_{L^p([0,2\pi])}(1,\varepsilon)$ and $p > \frac{1}{1-\gamma}$, the initial condition

$$\theta_0(re^{i\vartheta}) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta),$$

describes a self-similar solution for $(\text{SQG})_\gamma$, for any $m \geq 1$.





Our goal

G., Gómez-Serrano (2022)

Let $\gamma \in (0,1)$ and $\alpha \in (1, 1+\gamma)$. Then, there exists $\varepsilon > 0$ such that for any $\frac{2\pi}{m}$-periodic $\tilde{\Omega} \in B_{L^p([0,2\pi])}(1,\varepsilon)$ and $p > \frac{1}{1-\gamma}$, the initial condition

$$\theta_0(re^{i\vartheta}) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta),$$

describes a self-similar solution for $(\mathrm{SQG})_\gamma$, for any $m \geq 1$.

$\rightsquigarrow$ Idea of the proof:

1. Use the **adapted coordinates** to solve the vorticity equation.
2. Solve the **elliptic equation** in the adapted coordinates close to the trivial solution.
3. Come back to the **physical variables**.

Vorticity equation: adapted coordinates

$$\textit{(vorticity equation)} \quad \frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) - \left\{\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right\}\cdot\nabla\hat{\theta} = 0$$

We propose a change of coordinate $(r, \vartheta) \mapsto (\beta, \phi)$ with

$$\vartheta = \beta + \phi, \quad \phi \in \mathbb{T},$$

such that β **is chosen in order to have vanishing** ϕ**-component of the pseudo velocity** $\rightsquigarrow$ and then solve the vorticity equation!
In general, denoting

$$\Theta(\beta,\phi) = \hat{\theta}(z(\beta,\phi)), \quad \Psi(\beta,\phi) = \hat{\psi}(z(\beta,\phi)),$$

we get

$$\left\{\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right\}\cdot\nabla_z\hat{\theta} = \Big((\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2}),$$
$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2})\Big)\cdot\nabla_{\beta,\phi}\Theta$$

Vorticity equation: adapted coordinates

(vorticity equation) $$\frac{1+\alpha-\gamma}{1+\alpha}\hat{\theta}(z) - \left\{\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right\} \cdot \nabla\hat{\theta} = 0$$

We propose a change of coordinate $(r, \vartheta) \mapsto (\beta, \phi)$ with

$$\vartheta = \beta + \phi, \quad \phi \in \mathbb{T},$$

such that β **is chosen in order to have vanishing ϕ-component of the pseudo velocity** $\rightsquigarrow$ and then solve the vorticity equation!
In general, denoting

$$\Theta(\beta, \phi) = \hat{\theta}(z(\beta, \phi)), \quad \Psi(\beta, \phi) = \hat{\psi}(z(\beta, \phi)),$$

we get

$$\left\{\nabla^{\perp}\hat{\psi} - \frac{z}{1+\alpha}\right\} \cdot \nabla_z\hat{\theta} = \Big((\beta_{z_2}\phi_{z_1} - \beta_{z_1}\phi_{z_2})\Psi_\phi - \frac{1}{1+\alpha}(z_1\beta_{z_1} + z_2\beta_{z_2}),$$
$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2})\Big) \cdot \nabla_{\beta,\phi}\Theta$$

Vorticity equation: adapted coordinates

- Adapted coordinates:

$$\vartheta = \beta + \phi,$$
$$(\beta_{z_1}\phi_{z_2} - \beta_{z_2}\phi_{z_1})\Psi_\beta - \frac{1}{1+\alpha}(z_1\phi_{z_1} + z_2\phi_{z_2}) = 0,$$

which equals to

$$\vartheta = \beta + \phi,$$
$$r = (-(1+\alpha)\Psi_\beta)^{\frac{1}{2}}.$$

$\rightsquigarrow$ $(r, \vartheta) \mapsto (\beta, \phi)$ is an **implicit nonlinear change of variables that depend on the solution itself!**

- Jacobian of the adapted coordinates:

$$|J| = \frac{1+\alpha}{2}\Psi_{\beta\varphi},$$

where $\partial_\varphi := \partial_\phi - \partial_\beta$.

Vorticity equation: adapted coordinates

- Vorticity equation in the adapted coordinates:

$$(1+\alpha-\gamma)\Theta(\beta,\phi)\Psi_{\beta\varphi}(\beta,\phi)+2\Psi_{\varphi}(\beta,\phi)\Theta_{\beta}(\beta,\phi)=0,$$

which has the following trivial solution

$$\Theta(\beta,\phi)=(\Psi_{\varphi})^{-\frac{1/\mu}{2}}(\beta,\phi)\Omega(\phi),$$

for any Ω!

Elliptic equation

$$\textit{(elliptic equation)} \quad \hat{\psi} = -(-\Delta)^{-1+\frac{\gamma}{2}}\hat{\theta}(z) = -\frac{C_\gamma}{2\pi}\int_{\mathbb{R}^2}\frac{\hat{\theta}(y)}{|x-y|^\gamma}dA(y),$$

which, in terms of the *adapted coordinates* reads as

$$F(\Psi, \Omega)(\beta, \phi) = 0$$

with

$$F(\Psi,\Omega)(\beta,\phi) := \Psi(\beta,\phi) - \frac{C_\gamma(1+\alpha)^{1-\gamma/2}}{4\pi}\int_0^{2\pi}\int_0^\infty \frac{(\Psi_\varphi(b,\Phi))^{-\frac{1}{2\mu}}\Omega(\Phi)\Psi_{b\varphi}(b,\Phi)dbd\Phi}{\{-\Psi_\beta - \Psi_b + 2\Psi_\beta^{\frac{1}{2}}\Psi_b^{\frac{1}{2}}\cos(\phi+\beta-\Phi-b)\}^{\frac{\gamma}{2}}}$$

Elliptic equation

- Trivial solution to F:

Note that $\theta(x) = |x|^{-1/\mu}$ is a stationary solution with $\psi = -C_0|x|^{1-\alpha}$. Then:

$$\Omega^0 := \left(\frac{1}{1+\alpha}\right)^{\frac{1}{2\mu}}, \quad \Psi^0(\beta) := -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}},$$

is a root of F:

$$F(\Psi^0, \Omega^0)(\beta, \phi) = 0, \quad \forall (\beta, \phi) \in [0, \infty) \times \mathbb{T}.$$

- Adapted coordinates at the trivial solution:

$$r = (C_0(\alpha-1))^{\frac{1}{1+\alpha}}\beta^{\frac{-1}{1+\alpha}}, \quad \vartheta = \phi + C_0(\alpha-1)r^{-1-\alpha}$$

$\rightsquigarrow$ $\beta = 0$ correspond to $r \to +\infty$ and $\vartheta = \phi$
$\rightsquigarrow$ $\beta \to +\infty$ correspond to $r = 0$, and $\vartheta = \phi + C_0(\alpha-1)r^{-1-\alpha}$, which is the **center of the spiral**

Elliptic equation: formulation

Using

$$\Psi(\beta,\phi)=-C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}\left(1+f(\beta,\phi)\right),\quad \Omega(\phi)=(1+\alpha)^{-\frac{1}{2\mu}}\tilde{\Omega}(\phi).$$

we reescale F as

$$\tilde{F}(f,\tilde{\Omega})(\beta,\phi)=\beta^{-\frac{\alpha-1}{1+\alpha}}\frac{F(\Psi,\Omega)(\beta,\phi)}{-C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}}=1+f(\beta,\phi)$$

$$-C_\gamma\beta^{-\frac{\alpha-1}{1+\alpha}}\int_0^{2\pi}\int_0^\infty\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\left\{1+\partial_{\overline{\varphi}}f(b,\Phi)\right\}^{-\frac{1}{2\mu}}\left\{1+\partial_{\overline{\beta\varphi}}f(b,\Phi)\right\}\tilde{\Omega}(\Phi)dbd\Phi}{D(f)(\beta,\phi,b,\Phi)^{\frac{\gamma}{2}}},$$

where

$$\begin{aligned}D(f)(\beta,\phi,b,\Phi):=&\left(\beta^{\frac{-1}{1+\alpha}}\left\{1+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{\frac{1}{2}}-b^{\frac{-1}{1+\alpha}}\left\{1+\partial_{\overline{b}}f(b,\Phi)\right\}^{\frac{1}{2}}\right)^2\\&+2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\left\{1+\partial_{\overline{\beta}}f(\beta,\phi)\right\}^{\frac{1}{2}}\left\{1+\partial_{\overline{b}}f(b,\Phi)\right\}^{\frac{1}{2}}\\&\times(1-\cos(\phi+\beta-\Phi-b)).\end{aligned}$$

Elliptic equation: formulation

Using

$$\Psi(\beta,\phi) = -C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}\beta^{\frac{\alpha-1}{1+\alpha}}\left(1+f(\beta,\phi)\right), \quad \Omega(\phi) = (1+\alpha)^{-\frac{1}{2\mu}}\tilde{\Omega}(\phi).$$

we reescale F as

$$\tilde{F}(f,\tilde{\Omega})(\beta,\phi) = \beta^{-\frac{\alpha-1}{1+\alpha}}\frac{F(\Psi,\Omega)(\beta,\phi)}{-C_0^{\frac{2}{1+\alpha}}(\alpha-1)^{\frac{1-\alpha}{1+\alpha}}} = 1+f(\beta,\phi)$$

$$-\ C_\gamma\beta^{-\frac{\alpha-1}{1+\alpha}}\int_0^{2\pi}\int_0^{\infty}\frac{b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}\left\{1+\partial_{\overline{\varphi}}f(b,\Phi)\right\}^{-\frac{1}{2\mu}}\left\{1+\partial_{\overline{\beta\varphi}}f(b,\Phi)\right\}\tilde{\Omega}(\Phi)dbd\Phi}{D(f)(\beta,\phi,b,\Phi)^{\frac{\gamma}{2}}},$$

where

$$\begin{aligned}
\partial_{\overline{\beta}}f(\beta,\phi) &:= \frac{\alpha-1}{1+\alpha}f(\beta,\phi)+\beta f_\beta(\beta,\phi),\\
\partial_{\overline{\varphi}}f(\beta,\phi) &:= -\frac{\alpha-1}{1+\alpha}f(\beta,\phi)+\beta f_\varphi(\beta,\phi),\\
\partial_{\overline{\beta\varphi}}f &:= \frac{2(\alpha-1)}{(1+\alpha)^2}f+\frac{\alpha-1}{1+\alpha}\beta(f_\varphi-f_\beta)+\beta^2 f_{\beta\varphi}.
\end{aligned}$$

Elliptic equation: function spaces

Define:

$$||f||_{L^\infty_\sigma} := \sup_{\beta\in[0,+\infty),\phi\in[0,2\pi]} \frac{(1+\beta)^{2\sigma}}{\beta^\sigma}|f(\beta,\phi)|,$$

$$||f||_{C^\delta_\beta} := \sup_{\phi\in[0,2\pi]} \sup_{|\beta-b|<1} |\beta+b|^\delta \frac{|f(\beta,\phi)-f(b,\phi)|}{|\beta-b|^\delta},$$

$$||f||_{\text{Lip}_\phi} := \sup_{\beta\in[0,+\infty)} \sup_{|\phi-\Phi|<1} \frac{|f(\beta,\phi)-f(\beta,\Phi)|}{\sin((\Phi-\phi)/2)}.$$

and

$$\mathcal{X}^{\sigma,\delta} := \left\{ f : (\beta,\phi)\in[0,+\infty)\times[0,2\pi]\to\mathbb{C}, \quad ||f||_{\mathcal{X}^{\sigma,\delta}} := ||f||_{L^\infty_\sigma} + ||f||_{C^\delta_\beta} < +\infty \right\}.$$

Now, we can define the following function spaces:

$$X_1^{\sigma,\delta} := \{f, \quad f, \beta f_\beta, \beta f_\varphi, \beta^2 f_{\beta\varphi} \in \mathcal{X}^{\sigma,\delta}, ||f||_{\text{Lip}_\phi} + ||\beta f_\beta||_{\text{Lip}_\phi} < +\infty, f = \sum_{k\in\mathbb{Z}} f_k(\beta) e^{ik\phi}\}$$

$$X_2^p := \{f \in L^p(0,2\pi), \quad f(\phi) = 1 + \sum_{0\neq k\in\mathbb{Z}} f_k e^{ik\phi}\}.$$

$$Y^{\sigma,\delta} := \{f, \quad f, \beta f_\varphi, \in \mathcal{X}^{\sigma,\delta}, ||f||_{\text{Lip}_\phi} < +\infty\}.$$

IFT: kernel study

- How to solve the elliptic equation close to the trivial solution?

$\rightsquigarrow$ Infinite dimensional Implicit Function Theorem!

$$\partial_f \tilde{F}(0,1)h(\beta,\phi) = \sum_{k\in\mathbb{Z}} e^{ik\phi} \left\{ \frac{(1+\alpha)}{2}\partial_\beta(\beta h_k(\beta)) \right.$$

$$\left. -ik\frac{1}{\mu}\frac{C_\gamma}{4\pi C_0}\left(\frac{1}{\alpha-1}\right)e^{ik\beta}\beta^{-\frac{\alpha-1}{1+\alpha}}\int_0^{2\pi}\int_0^{\infty}\frac{e^{-ikb}b^{-\frac{\gamma}{1+\alpha}}b^{\frac{-2}{1+\alpha}}bh_k(b)\cos(k\eta)dbd\eta}{\{\beta^{\frac{-2}{1+\alpha}}+b^{\frac{-2}{1+\alpha}}-2\beta^{\frac{-1}{1+\alpha}}b^{\frac{-1}{1+\alpha}}\cos(\eta)\}^{\frac{\gamma}{2}}} \right\}.$$

$\rightsquigarrow$ $\partial_f \tilde{F}(0,1)$ is Fredholm of zero index, and hence one can reduce to the **kernel study**

IFT:kernel study

Denoting $g_k = e^{-ik\beta}h_k$, the kernel equation reduces to

$$\frac{1+\alpha}{2}g_k + ik\frac{1+\alpha}{2}\beta g_k + \frac{1+\alpha}{2}\beta g_k' - ikJ_k(g_k) = 0,$$

where

$$J_k(g_k)(\beta) := \int_0^\infty g_k(b)K_k(\beta/b)db,$$

with

$$K_k(z) := \frac{C_\gamma}{\mu 4\pi C_0(\alpha-1)}\int_0^{2\pi}\frac{z^{-\frac{\alpha-1}{1+\alpha}}\cos(k\eta)d\eta}{\{z^{\frac{-2}{1+\alpha}}+1-2z^{\frac{-1}{1+\alpha}}\cos\eta\}^{\frac{\gamma}{2}}}.$$

$\rightsquigarrow$ Mellin transforms are very useful to solve this kind of equations:

$$M[f](s) = \int_0^\infty x^{s-1}f(x)dx,$$

since

$$M[f\wedge g] = M[f]M[g], \quad f\wedge g = \int_0^\infty f(y)g(x/y)\frac{dy}{y}$$

IFT:kernel study

In terms of Mellin transform, the kernel equation reduces to

$$(1-s)G_k(s) - ik\left(\frac{2}{1+\alpha}\tilde{K}_k(s) - 1\right)G_k(s+1) = 0.$$

- G_k is well-defined in $\{s \in \mathbb{C}, -\sigma < \mathrm{Re}(s) < \sigma\}$.
- $\frac{2}{1+\alpha}\tilde{K}_k(0) < 1$ and

$$\lim_{s \to \frac{\alpha-1+k}{1+\alpha}} \tilde{K}_k(s) = +\infty$$

- For $s \in [0, \frac{\alpha-1+k}{1+\alpha})$, $k \in \mathbb{N} \mapsto \tilde{K}_k(s)$ strictly decreases, $s \in [0, \frac{\alpha-1+k}{1+\alpha}) \mapsto \tilde{K}_k(s)$ increases.
- For any $k \in \mathbb{N}^*$, there exists a unique $s_0 \in [0, \frac{\alpha-1+k}{1+\alpha})$ such that $\tilde{K}_k(s_0) = 1$.

IFT:kernel study

If G_k solves the kernel equation, with $g_k = e^{-ik\beta} h_k$ with $h \in X_1$, hence $h = 0$.

$\rightsquigarrow$ We understand the kernel equation as a **recurrence equation:**

$$G_k(M) = G_k(0) \prod_{n=1}^{M-1} \frac{F_k(n)}{k}, \quad F_k(n) = i \frac{1-n}{1 - \frac{2}{1+\alpha} \tilde{K}_k(n)}$$

$\rightsquigarrow$ **Recall:** $\exists s_0$ s.t. $\frac{2}{1+\alpha} \tilde{K}_k(s_0) = 1$, and $\frac{2}{1+\alpha} \tilde{K}_k(n) \to +\infty$, $n \geq n_0$.

- $s_0 \notin \mathbb{N}$: then $F_k(1) = 0$ and hence $G_k(M) = 0$, $\forall M \geq 2$.

Note that G_k is related to the **moment problem:**

$$\mathcal{L}[g_k](t) = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} \int_0^{\infty} z^n g_k(z) dz = \sum_{n=0}^{\infty} \frac{(-t)^n}{n!} G_k(n+1) = G_k(1).$$

However, the inverse of the Laplace transform of a constant is a dirac mass getting a contradiction since g_k is continuous!

IFT:kernel study

If G_k solves the kernel equation, with $g_k = e^{-ik\beta} h_k$ with $h \in X_1$, hence $h = 0$.

$\rightsquigarrow$ We understand the kernel equation as a **recurrence equation:**

$$G_k(M) = G_k(0) \prod_{n=1}^{M-1} \frac{F_k(n)}{k}, \quad F_k(n) = i\frac{1-n}{1-\frac{2}{1+\alpha}\tilde{K}_k(n)}$$

$\rightsquigarrow$ **Recall:** $\exists s_0$ s.t. $\frac{2}{1+\alpha}\tilde{K}_k(s_0) = 1$, and $\frac{2}{1+\alpha}\tilde{K}_k(n) \to +\infty$, $n \geq n_0$.

- $s_0 \in \mathbb{N}$:
  Since s_0 is a simple root, we get that $\mathcal{L}[g_k]$ is a polynomial, which grows at $+\infty$. However, since $g_k \in L^\infty$ one has that $\mathcal{L}[g_k](t) \to 0$, as $t \to +\infty$, getting again a contradiction.

Main theorem in the adapted coordinates

$$X^{\sigma,\delta}_{1,m} := \left\{ f \in X^{\sigma,\delta}_1, \quad f\left(\beta, \phi + \frac{2\pi}{m}\right) = f(\beta,\phi), \, (\beta,\phi) \in [0,+\infty) \times [0,2\pi] \right\},$$
$$X^{p}_{2,m} := \left\{ f \in X^{p}_2, \quad f\left(\phi + \frac{2\pi}{m}\right) = f(\phi), \, \phi \in [0,2\pi] \right\}.$$

For each $m \geq 1$, there exists $\varepsilon > 0$ and a C^1 function

$$\tilde{\Omega} \in B_{X^p_{2,m}}(1,\varepsilon) \mapsto \tilde{f}(\tilde{\Omega}) \in X^{\sigma,\delta}_{1,m},$$

such that

$$\tilde{F}(f,\tilde{\Omega}) = 0, \, (f,\tilde{\Omega}) \in B_{X^{\sigma,\delta}_{1,m}}(0,\varepsilon) \times B_{X^p_{2,m}}(1,\varepsilon) \iff f = \tilde{f}(\Omega),$$

$\rightsquigarrow$ We can come back to the **physical variables** since the jacobian is non vanishing for *small* perturbations:

$$|J| = C\left\{1 + \partial_{\overline{\beta\varphi}} f\right\}$$

Then, for $\tilde{\theta}(r,\vartheta) = \hat{\theta}(re^{i\vartheta})$ we have

$$\begin{aligned}
&\tilde{\theta}(r,\vartheta) = \Theta(\beta(r,\vartheta), \phi(r,\vartheta)) \\
=&\Theta\Big(C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta)), \vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\Big) \\
=&(1+\alpha)^{-\frac{1}{2\mu}}(\alpha-1)^{\frac{1+\alpha-\gamma}{2}} r^{-(1+\alpha-\gamma)} \\
&\times \left(\frac{\alpha-1}{1+\alpha} - \tilde{f}_{\overline{\varphi}}(\tilde{\Omega})\Big(C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta)), \vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\Big)\right) \\
&\times \tilde{\Omega}\Big(\vartheta - C_0(\alpha-1)r^{-1-\alpha}(1+\tilde{\beta}(r,\vartheta))\Big).
\end{aligned}$$

Since $\theta(t,x) = \frac{1}{t^{\frac{1+\alpha-\gamma}{1+\alpha}}}\tilde{\theta}(\frac{r}{t^{\frac{1}{1+\alpha}}}, \eta)$, the initial data is got at $+\infty$:

$$\theta_0(z) = r^{-(1+\alpha-\gamma)}\tilde{\Omega}(\vartheta).$$

Thank you!!