MFG with aggregation: existence and nonexistence of equilibria

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Describe interacting “rational” particles / players

- players have controlled dynamical state $\leftrightarrow$ (stochastic) diff. equation

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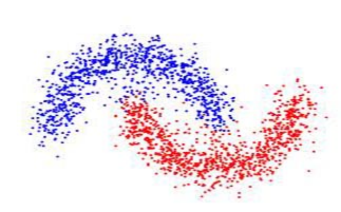
Study equilibria in the large population limit, i.e. as the number N of players $\to \infty$

Possible applications

vehicular traffic

crowd motion

engineering



clustering (big data)

The limit problem

Reduce the asymptotic analysis to one typical player
that interacts with a population with density $\mathbf{m(x,t)} : \mathbb{R}^d \times [0,T] \to \mathbb{R}$.

State of a typical player: $X_0 \sim m_0$,

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with cost:

$$J_{\mathbf{m}}(v) = \mathbb{E}\int_0^T |v_s|^{\gamma'} + V(X_s) + f\big(\mathbf{m}(X_s)\big)\, ds + u_T(X_T),$$

- $V : \mathbb{R}^d \to \mathbb{R}$ encodes "spatial preferences"
- $f : \mathbb{R} \to \mathbb{R}$ is the "coupling". Today,

$$f(m) = -\sigma m^\alpha(x), \qquad \sigma, \alpha > 0.$$

- $u_T : \mathbb{R}^d \to \mathbb{R}$ is the final cost

Suppose that, given m, the typical player behaves optimally. Denote by μ his density. In an equilibrium / consensus regime,

$$\mathsf{m} \equiv \mu,$$

described by solutions (u, m) of

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{\gamma}|Du|^\gamma = -\sigma m^\alpha(x,t) + V(x) & \text{in } \Omega \times (0,T) \\ \partial_t m - \Delta m - \mathrm{div}(m|Du|^{\gamma-2}Du) = 0 & \\ m(0) = m_0, \quad u(T) = u_T & \text{in } \Omega \end{cases}$$

where $\Omega = \mathbb{T}^d$ or $\mathbb{R}^d$, $\alpha, \sigma > 0$, $\gamma > 1$.

MFG systems

Stationary equilibria are given by $(\bar{u}, \bar{m}, \lambda)$ solving

$$\begin{cases} -\Delta \bar{u} + \frac{1}{\gamma}|D\bar{u}|^\gamma + \lambda = -\sigma \bar{m}^\alpha(x) + V(x) \quad \text{in } \Omega \\ -\Delta \bar{m} - \mathrm{div}(\bar{m}|D\bar{u}|^{\gamma-2}D\bar{u}) = 0 \\ \int_\Omega \bar{m} = 1. \end{cases}$$

where $\quad \Omega = \mathbb{T}^d \quad \text{or} \quad \mathbb{R}^d \quad , \alpha, \sigma > 0, \gamma > 1.$

Ergodic problems: known results

Uniqueness: false in general.

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In our case:

Theorem (C. 2016)

the ergodic problem with Hamiltonian $\sim |Du|^\gamma$ *and coupling* $\sim -\sigma m^\alpha$

$0 < \alpha < \frac{\gamma'}{d}$	$\frac{\gamma'}{d} \leq \alpha < \frac{\gamma'}{d-\gamma'}$	$\alpha > \frac{\gamma'}{d-\gamma'}$
has solutions $\Omega = \mathbb{T}^d$ *or* $\mathbb{R}^d$	*has solutions* *provided that* σ *is small* $\Omega = \mathbb{T}^d$	*does not have solutions* $\Omega = \mathbb{R}^d$

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Why two critical constant $\frac{\gamma'}{d}, \quad \frac{\gamma'}{d-\gamma'}$?

The variational viewpoint

Solutions are related with critical points of

$$\mathcal{E}(v, m) = \int_\Omega \frac{1}{\gamma'}|v|^{\gamma'} m - \frac{\sigma}{\alpha+1} m^{\alpha+1} + Vm \, dx$$

subject to

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how to compare

$$\int_\Omega \frac{1}{\gamma'}|v|^{\gamma'} m \quad \text{and} \quad -\int_\Omega m^{\alpha+1} \quad ?$$

$\int |v|^{\gamma'} m$ vs $-\int m^{\alpha+1}$

1. by the Kolmogorov equation,

$$\|Dm\|_{L^r} \leq C\left(\int |v|^{\gamma'} m + 1\right), \quad r = r(\gamma, d)$$

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2. Interpolation:

$$\int_\Omega m^{\alpha+1} = \|m\|_{L^{\alpha+1}}^{\alpha+1} \leq C\|Dm\|_{L^r}^{\theta(\alpha+1)} \cdot \|m\|_{L^1}^{(1-\theta)(\alpha+1)}$$

$$\leq C\left(\int \frac{1}{\gamma}' |v|^{\gamma'} m + 1\right)^{\theta(\alpha+1)} \cdot 1,$$

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and $\theta(\alpha+1) < 1$ whenever

$$\alpha < \frac{\gamma'}{d} \qquad \mathcal{E} \text{ is bounded below!}$$

hence, the variational problem is

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- Need estimates to have Du bounded.
- Nonexistence when $\alpha > \frac{\gamma'}{d-\gamma'}$: Pohozaev like identities.

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$2 - \gamma' > 0 \;\; \rightarrow$ superquadratic : BAD !!!!

Theorem (C., Tonon 2019)

the parabolic problem w. Hamiltonian $\sim |Du|^\gamma$ *and coupling* $\sim -\sigma m^\alpha$ *has solutions if*

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What happens when

$$\alpha \geq \min\left\{\frac{\gamma'}{d+2-\gamma'}, \frac{\gamma'}{d}\right\} \qquad ?$$

In the quadratic case,

$$\boxed{\alpha \geq \frac{2}{d}}$$

Case $\alpha \geq \frac{2}{d}$: joint work w. Daria Ghilli

$$
\begin{cases}
-\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = -\sigma m^\alpha + V & \text{in } \mathbb{R}^d \times (0,T) \\
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V be a bounded potential ("not too much radially decreasing").

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Theorem (nonexistence)

Let $\alpha \geq \frac{2}{d}$. Under some integrability assumptions on m_0, u_T, there exist σ^*, T^* such that if

$$\sigma > \sigma^* \qquad \text{and} \qquad T > T^*,$$

then the MFG system has no (classical) solutions.

Theorem (existence)

Let $\frac{2}{d} \leq \alpha < \frac{2}{d-2}$*. Under some integrability assumptions on* m_0, u_T*, there exists* $\sigma_0 = \sigma_0(d, \alpha, m_0, u_T, T\|\Delta V\|_\infty)$ *such that if*

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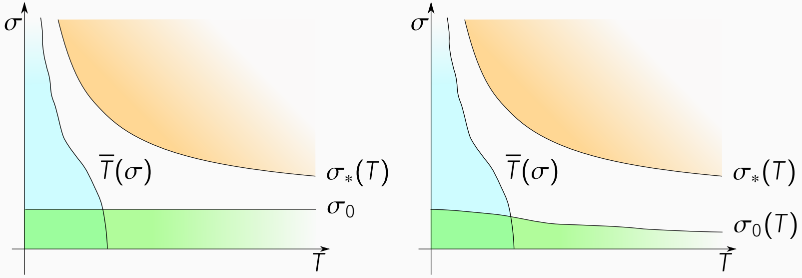

$V \equiv 0$, $V \neq 0$.

short-time existence, small-σ existence, non-existence

On the Planning problem

Consider the problem

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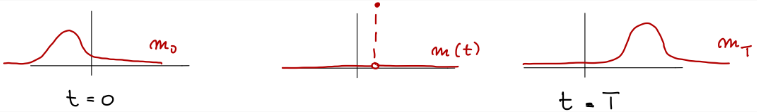

Nonexistence

Based on three ingredients:

$$\left.\begin{array}{ll} \textit{(i)} & \text{conservation of energy} \\ \textit{(ii)} & \text{momentum} \\ \textit{(iii)} & \text{“virial identity”} \end{array}\right\} \quad \text{involve} \quad \int_{\mathbb{R}^d} |x|^2 \, m(x,t)$$

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(i) the MFG system can be seen as an Hamiltonian system:

$$\begin{cases} u_t = \frac{\delta \mathcal{H}}{\delta m}(m,u) \\ m_t = -\frac{\delta \mathcal{H}}{\delta u}(m,u) \end{cases}$$

where

$$\mathcal{H}(m,u) = \int_{\mathbb{R}^d} \nabla m \nabla u + \frac{1}{2}|\nabla u|^2 m - Vm + \sigma m^{\alpha+1}\, dx \equiv E.$$

obtained by $\quad HJ \cdot \partial_t m + FP \cdot \partial_t u$

(ii) testing FP by x^2,

$$\frac{d}{dt}\int_{\mathbb{R}^d} m(t)x^2\,dx = 2d - \int_{\mathbb{R}^d} m(t)\nabla u(t)\cdot x\,dx$$

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Then,

$$\frac{d^2}{dt^2}\int_{\mathbb{R}^d} m(t)x^2\,dx = 4E + \overbrace{\frac{2d\sigma}{\alpha+1}\left(\alpha-\frac{2}{d}\right)\int_{\mathbb{R}^d} m^{\alpha+1}}^{\geq 0} + \overbrace{2\int_{\mathbb{R}^d}[2(V-\inf V)+\nabla V\cdot x]m\,dx}^{\geq 0} \geq 4E$$

$$\Rightarrow \quad \frac{d^2}{dt^2}\int_{\mathbb{R}^d} m(t)x^2\,dx \geq \int_{\mathbb{R}^d} -\frac{|\nabla m_0|^2}{m_0} - Vm_0 + \sigma m_0^{\alpha+1}\,dx > 0$$

whenever σ is large (or m_0 is “large”).

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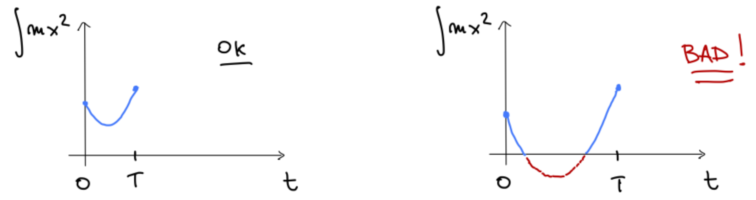

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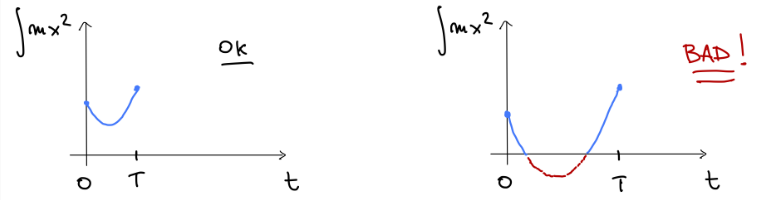

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MFG problem: $\int_{\mathbb{R}^d} m_0 x^2 dx$ is given, $\frac{d}{dt}\int_{\mathbb{R}^d} m(T)x^2 dx \leq 2d$.

Existence

As usual, solutions can be obtained as fixed points of the operator $m \mapsto \mu$,

$$\begin{cases} -\partial_t u - \Delta u + \frac{1}{2}|\nabla u|^2 = -\sigma m^\alpha + V & \text{in } \mathbb{R}^d \times (0,T) \\ \partial_t \mu - \Delta \mu - \mathrm{div}(\mu \nabla u) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ \mu(0) = m_0, \quad u(T) = u_T & \text{in } \mathbb{R}^d \end{cases}$$

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To use fixed point theorems, one needs compactness and suitable estimates, i.e.

$$\|m\|_{L^{\alpha+1}} + \|m\|_{\mathcal{H}^{1,r}} \leq C.$$

Unfortunately, the nonexistence step does not help here.

A priori estimates in $L^{2\alpha+1}$

Let $\frac{2}{d} \leq \alpha \leq \frac{2}{d-2}$ and (u, m) be a solution to the MFG system. Then,

$$\frac{Y^\delta}{C} - \sigma^2 Y \leq 1, \quad \delta < 1, \quad Y = \left(\int_0^T \int_{\mathbb{R}^d} \mu^{2\alpha+1}(t)\, dxdt \right)^2,$$

where C depends on d, α, m_0, u_T and δ depends on α.

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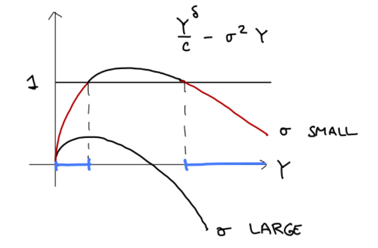

Enough to set up a fixed point procedure (σ small).

(combination of conservation of energy, “Second order estimates” and parabolic regularity)

Open problems

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- stability of stationary solutions ?? almost open for any $\alpha > 0$
  σ is small "ok", large σ ?? .

Open problems

- Large α, non-quadratic problems $|Du|^{\gamma}$, $\gamma \neq 2$
  - non-existence:
    (i), (ii), (iii) hold, but it's not clear how to combine them
  - existence: same issues
- stability of stationary solutions ?? almost open for any $\alpha > 0$
  σ is small "ok", large σ ?? .
- back to the monotone case $+m^{\alpha}$: existence for any α ??

Thank you for your attention.