Time periodic solutions for the 3D Quasigeostrophic model

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joint work with C. García and T. Hmidi

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$$\begin{cases} \partial_t q + u\partial_1 q + v\partial_2 q = 0, \quad (t,x) \in [0,+\infty) \times \mathbb{R}^3, \\ \Delta\psi = q, \\ u = -\partial_2\psi, \ v = \partial_1\psi, \\ q(0,x) = q_0(x). \end{cases} \tag{1}$$

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The second equation can be inverted obtaining a representation of the stream function ψ,

$$\psi(t,x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{q(t,y)}{|x-y|} dA(y),$$

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This system is a model commonly used in the ocean and atmosphere circulations to describe the vortices and to track the emergence of long–lived structures.

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This latter context allows to deal with discontinuous vortices of the patch form, meaning a characteristic function of a bounded domain.

Since we have a transport equation, this structure is preserved in time and the vortex patch problem consists of studying the regularity of the boundary and to analyze whether singularities can be formed in finite time on the boundary.

Known Results for 2D Euler

For the 2D Euler equations, the $C^{1,\alpha}$ regularity of the boundary of the patch, with $\alpha \in (0,1)$, is preserved in time. (Chemin and Bertozzi-Constantin)

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Kiselev and Luo proved recently that the regularity of the patch for 2D-Euler is not preserverd in time when the initial condition is in C^2. More precisely they prove that the regularity C^2 of the boundary is lost instantly.

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Rankine vortex (1858)
If $D = D(0,1)$ is the unit disc, then

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Annulus
If $D = D(0,2) \setminus D(0,1)$ is an annulus, then $D_t = D, \quad 0 < t$ and $\chi_D(z)$ is a steady solution to the vorticity equation.

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Having this kind of V-state solutions in mind, Burbea designed a rigorous approach to generate them close to Rankine vortices through complex analysis tools and bifurcation theory.

This idea was fruitfully improved and extended in various directions, generating a lot of contributions dealing, for instance, with interesting topics like the regularity problem of the relative equilibria, their existence with different topological structure or for different active scala equations. (Castro, Córdoba, García, Gómez-Serrano, Hassainia, Hmidi, M., Verdera.)

3D QG Model. Revolution shapes

The analogues to Kirchhoff ellipses still surprisingly survive in the 3D QG case. It is shown that a standing ellipsoid of arbitrary semi-axis rotates steadily about the $z-$axis with constant angular velocity depending on the semi-axis (Meacham).

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Lemma 1

Let $r : [-1, 1] \to \mathbb{R}_+$ be a continuous function with $r(-1) = r(1) = 0$ and let D be the domain enclosed by the surface $\{(r(z)e^{i\theta}, z),\ \theta \in [0, 2\pi], z \in [-1, 1]\}$, then $q(t, x) = \mathbf{1}_D(x)$ defines a stationary solution for (1).

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where the shape is sufficiently close to a revolution shape domain, meaning that

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$$\forall\, \phi \in [0,\pi], \quad C^{-1} \sin\phi \le r_0(\phi) \le C \sin(\phi).$$

(H3) r_0 is symmetric with respect to $\phi = \frac{\pi}{2}$, i.e., $r_0\left(\frac{\pi}{2} - \phi\right) = r_0\left(\frac{\pi}{2} + \phi\right)$, for any $\phi \in [0, \frac{\pi}{2}]$.

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Theorem 2

Assume that r_0 satisfies the assumptions **(H)**. *Then for any $m \geq 2$, there exists a curve of non trivial rotating solutions with m-fold symmetry to the equation* (1) *bifurcating from the trivial revolution shape associated to r_0 at the angular velocity Ω_m.*

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A rotating solution of the equation means is a time–dependent solution taking the form,

$$q(t,x) = q_0(e^{-i\Omega t}x_h, x_3), \quad q_0 = \mathbf{1}_D,\, x_h = (x_1, x_2).$$

Doubly connected case

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This is equivalent to find two bounded domains D_1 and D_2 of $\mathbb{R}^3$, with D_2 embedded in D_1 ($D_2 \Subset D_1$) such that

$$q(t,\cdot) = \mathbf{1}_{D(t)}, \quad D(t) = \mathcal{R}_{\Omega t} D, \quad D = D_1 \backslash D_2, \tag{2}$$

and $R_{\Omega t}(x_1, x_2, x_3) = (e^{i\Omega t}(x_1, x_2), x_3)$, in a small neighborhood of the following stationary solution

$$q_0 = \mathbf{1}_{D_0}, \quad D_0 = D_{0,1} \backslash D_{0,2}, \quad D_{0,2} \Subset D_{0,1}, \tag{3}$$

where $D_{0,j}$ is a revolution shape domain with a particular structure.

Doubly connected case

As for the simply connected case the way we parametrize the domain D is a key point for the analysis and spherical coordinates type are privileged, that is for each $j \in \{1, 2\}$,

$$\partial D_j = \Big\{ \big(r_j(\phi,\theta) e^{i\theta}, d_j \cos(\phi) \big), \quad (\phi,\theta) \in [0,\pi] \times \mathbb{T} \Big\}, \tag{4}$$

for $d_1 > d_2 > 0$ and with

$$r_j(\phi,\theta) = r_{0,j}(\phi) + f_j(\phi,\theta) \quad \text{with} \quad f_j(\phi,\theta) = \sum_{n \geqslant 1} f_{j,n}(\phi) \cos(n\theta). \tag{5}$$

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Note that for $f_j = 0$ we recover the parametrization of the revolution shape domain $D_{0,j}$ covering by this way a large class of stationary solutions of the type (3).

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(H4) *(Interfaces separation)* There exists $\delta > 0$ such that

$$\forall \phi, \varphi \in [0, \pi], \quad (r_{0,1}(\phi) - r_{0,2}(\varphi))^2 + (d_1 \cos(\phi) - d_2 \cos(\varphi))^2 \geqslant \delta,$$

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supplemented with the strong condition

$$\sup_{\phi\in[0,\pi]} \frac{U_0^2\Big(r_{0,2}(\phi),0,d_2\cos(\phi)\Big)}{r_{0,2}(\phi)} := \overline{\Omega}_2 < \inf_{\phi\in[0,\pi]} \frac{U_0^2\Big(r_{0,1}(\phi),0,d_1\cos(\phi)\Big)}{r_{0,1}(\phi)} := \overline{\Omega}_1, \tag{6}$$

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2) We can show the validity of this assumption with specific elementary shapes combining sphere and ellipsoid and by default any of their small perturbation,

3) We can also check the condition when the shapes are very well-separated.

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Theorem 3

Let $r_{0,j}$ satisfy $(\mathbf{H})$ *and (6), for* $j = 1, 2$*. Then there is* $m_0 \in \mathbb{N}$ *such that for any* $m \geq m_0$ *there exists a curve of nontrivial rotating doubly connected solutions to* (1)*. That is* $q(x,t) = \chi_{D(t)}(x)$, $D(t)$ *is a rotation of* D_0*. The horizontal sections of the domains are m-fold symmetric.*

On the proof for the simply connected case

We shall look for a rotating solution close to a stationary one described by a given revolution shape $(\theta, \phi) \mapsto (r_0(\phi)e^{i\theta}, \cos(\phi))$.

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This means that we are looking for a parametrization in the form

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We have assumed that the domain D is symmetric with respect to the plane $x_2 = 0$. In addition, we ask the following boundary conditions,

$$r_0(0) = r_0(\pi) = f(0,\theta) = f(\pi,\theta) = 0,$$

meaning that the domain D intersects the vertical axis at the points $(0,0,-1)$ and $(0,0,1)$.

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$$(\partial_t \gamma_t - U(t, \gamma_t)) \cdot n(\gamma_t) = 0.$$

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Now using that $(u, v) = (-\partial_2 \Psi, \partial_1 \Psi)$ and integrating, we get an

equation.

On the proof for the simply connected case

Then, finding a rotating solution amounts to solving in f, for some specific angular velocity constant Ω, the equation

$$F(\Omega, f)(\phi, \theta) = 0, \quad \forall (\phi, \theta) \in [0, \pi] \times [0, 2\pi].$$

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$$F(\Omega,f)(\phi,\theta)=0, \quad \forall(\phi,\theta)\in[0,\pi]\times[0,2\pi].$$

where

$$F(\Omega,f)(\phi,\theta):=\frac{1}{r_0(\phi)}\Big[\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi))-\frac{\Omega}{2}r^2(\phi,\theta)-m(\Omega,f)(\phi)\Big],$$

for any $(\phi,\theta)\in[0,\pi]\times[0,2\pi]$

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$$m(\Omega,f)(\phi) := \frac{1}{2\pi}\int_0^{2\pi}\left\{\psi_0(r(\phi,\theta)e^{i\theta},\cos(\phi)) - \frac{\Omega}{2}r^2(\phi,\theta)\right\}d\theta,$$

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and the stream function is given by

$$\psi_0(r(\theta,\phi)e^{i\theta},\cos(\phi)) = -\frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi}\int_0^{r(\varphi,\eta)}\frac{\sin(\varphi)rdrd\eta d\varphi}{|(re^{i\eta},\cos(\varphi)) - (r(\phi,\theta)e^{i\theta},\cos(\phi))|}.$$

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Remark that one may check directly from this reformulation that any revolution shape is a solution for any angular velocity Ω, meaning that, $F(\Omega, 0) = 0$, for any Ω.

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How to prove the existence of solutions for this equation ?

On the proof for the simply connected case

Theorem 4 (Crandall-Rabinowitz Theorem)

Let X, Y be two Banach spaces, V be a neighborhood of 0 in X and $F : \mathbb{R} \times V \to Y$ be a function with the properties,

1. $F(\lambda, 0) = 0$ *for all* $\lambda \in \mathbb{R}$.
2. *The partial derivatives $\partial_\lambda F$, $\partial_f F$ and $\partial_\lambda \partial_f F$ exist and are continuous.*
3. *The operator $\partial_f F(0,0)$ is Fredholm of zero index and* $\mathrm{Ker}(F_f(0,0)) = \langle f_0 \rangle$ *is one-dimensional.*
4. *Transversality assumption:* $\partial_\lambda \partial_f F(0,0) f_0 \notin \mathrm{Im}(\partial_f F(0,0))$.

If Z is any complement of $\mathrm{Ker}(\partial_f F(0,0))$ *in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R} \times X$, an interval $(-a, a)$, and two continuous functions $\Phi : (-a,a) \to \mathbb{R}$, $\beta : (-a,a) \to Z$ such that* $\Phi(0) = \beta(0) = 0$ *and*

$$F^{-1}(0) \cap U = \{(\Phi(s), sf_0 + s\beta(s)) : |s| < a\} \cup \{(\lambda, 0) : (\lambda, 0) \in U\}.$$

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The right space to work with is for $\alpha \in (0,1)$ and $m \in \mathbb{N}$, set

$$X_m^\alpha := \Big\{ f \in C^{1,\alpha}\big((0,\pi)\times\mathbb{T}\big), f(\phi,\theta) = \sum_{n\geq 1} f_n(\phi)\cos(nm\theta) \Big\},$$

supplemented with the conditions

$$f(0,\theta) = f(\pi,\theta) = 0 \quad \text{and} \quad f(\pi-\phi,\theta) = f(\phi,\theta).$$

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The study of the kernel agrees with the eigenvalue problem of a Hilbert–Schmidt operator,we achieve that the dimension is one.

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More precisely with an argument of symmetrization of an operator we reduce the problem of solving the kernel equation to a problem of studying a compact, self adjoint and Hilbert-Schmidt operator on L^2 of a measure.

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One can see that the kernel equation can be reduced to study the linear operator

$$L_n(h)(\phi) := \nu_\Omega(\phi)\Big\{h(\phi) - \frac{1}{\nu_\Omega}\int_0^\pi H_n(\phi,\varphi)h(\varphi)d\varphi\Big\} = 0$$

If we symmetrize the operator, the above equation can be written as

$$T_n^\Omega(h)(\phi) := \int_0^\pi K_n^\Omega(\phi,\varphi)h(\varphi)d\mu_\Omega(\varphi) = h(\phi)$$

and this operator $T_n^\Omega : L^2_{\mu_\Omega} \to L^2_{\mu_\Omega}$ is compact, self-adjoint and Hilbet-Schmidt.

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If $M = \sup_{||h||_{\mu_\Omega}} < T_n^\Omega h, h >_{\mu_\Omega}$ and we define $\lambda_n(\Omega) = M$. Then one can see that $\lambda_n(\Omega)$ are positive, simple, strictly decreasing (as function of n) and strictly increasing (as function of Ω).

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Moreover one can see that for any $n \geq 2$, there exists a unique Ω_n such that $\lambda_n(\Omega_n) = 1$.

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The kernel is one dimensional.

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The spherical change of coordinates used to recover both the velocity and the stream function from the surface geometry of the patch yields a deformation of the Green function.

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To deal with these defects one needs refined treatments in the behavior of the kernel or also the adaptation of the function spaces which are of Dirichlet type.

Thank you for your attention