Some optimal control & game theoretical problems in spatial ecology: The tragedy of the commons

I. Mazari-Fouquer

Collaboration with D. Ruiz-Balet & Z. Kobeissi
CEREMADE, Paris Dauphine Université

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What I will be discussing

[KMFRB23] Z. Kobeissi, I. Mazari-Fouquer and D. Ruiz-Balet.
The tragedy of the commons: A Mean-Field Game approach to the reversal of travelling waves, *Preprint available online, 2023.*

[MBR22] I. Mazari,and D. Ruiz-Balet.
Spatial ecology, optimal control and game theoretical fishing problems
Journal of Mathematical Biology, 2022.

Overview

Main question

(General version)

What is the influence of *harvesting/fishing* on *population dynamics*?

Main question

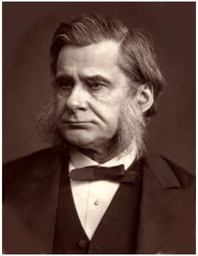
Formulations under scrutiny

How should different fishermen **fish** to optimise their fishing output?
What are the **consequences** on the fishes' population of **competition** between fishermen?
Can **coordination** remedy some of the worst of these consequences?

London, 139 years ago...

I believe then that the cod fishery, the herring fishery, pilchard fishery, the mackerel fishery, and probably all the great sea **fisheries are inexhaustible***: that is to say that* **nothing we do seriously affects the number of fish***. And any attempt to regulate these fisheries seems consequently from the nature of the case to be useless*

T.H.Huxley (1884) Inaugural Address, Fisheries Exhibition Lit., 4,1-22, London



The world, today...

SCIENCE | NEWS

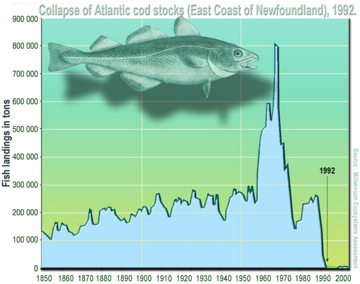
The sea is running out of fish, despite nations' pledges to stop it

Major countries that are promising to curtail funding for fisheries are nevertheless increasing handouts for their seafood industries.

BY TODD WOODY

PUBLISHED OCTOBER 8, 2019 • 7 MIN READ

As global fish stocks that feed hundreds of millions of people dwindle, nations are scrambling to finalize by year's end an international agreement to ban government subsidies that fuel overfishing.



The sea is running out of fish, despite nations' pledges to stop it, The National Geographic, 2019

The Tragedy of the Commons

Definition (≈ Wikipedia)

The **tragedy of the commons** is a situation in which individual users, who have **open access to a resource**, act independently according to their **own self-interest** and **cause depletion of the resource** through their uncoordinated action.

Effects on unregulating land in Ireland

William Forster Lloyd

The tragedy of the commons

Garret Hardin

What we want to discuss

1. **Simple, paradigmatic models** amenable to mathematical analysis $\rightsquigarrow$ **qualitative understanding of fishing phenomena**.
2. Bounded domains, finite number of players: basic model, geometric properties & first approach to the Trag. o. Com.
3. Unbounded domains, infinite number of players: travelling waves, and an invasion/extinction approach to the Trag. o. Com. Additionally **very telling instance of how coordination can remedy a dramatic situation**.
4. Disclaimer: simple mathematical models with limited applicability but known to capture accurate qualitative behaviours.

Reaction-diffusion equations

$u \rightsquigarrow$ population density

$$(\partial_t u) = \underbrace{\mu\Delta u}_{\text{random diffusion}} + \underbrace{f(u)}_{\text{intrinsic reaction: death, birth...}} - \underbrace{m \cdot u}_{m\text{: density of fishermen}}.$$

Bounded domain, finite number of players

I. Mazari, D. Ruiz-Balet, Spatial ecology, optimal control and game theoretical fishing problems. *Journal of Mathematical Biology*, 2022.

The setting

1. Reaction term of monostable type:

$$f(u) = u(K(x) - u) \rightsquigarrow \text{linear growth+malthusian death term}$$

2. Fishes living inside an inescapable domain $\rightsquigarrow$ Neumann b.c.
3. One or two fishing companies (1 & 2): the i-th fishes with a spatial stragegy $\alpha_i = \alpha_i(x)$.
4. We are interested in static situations.

Overall

$$-\mu\Delta u = u(K - u) - \alpha_1 u - \alpha_2 u.$$

The harvesting functional

$$-\mu\Delta u_{\alpha_1,\alpha_2} = u_{\alpha_1,\alpha_2}(K - u_{\alpha_1,\alpha_2}) - \alpha_1 u_{\alpha_1,\alpha_2} - \alpha_2 u_{\alpha_1,\alpha_2}.$$

Amount of fish harvested by the i-th company:

$$J_1(\alpha_1,\alpha_2) = \int \alpha_1 u_{\alpha_1,\alpha_2}\, , J_2(\alpha_1,\alpha_2) = \int \alpha_2 u_{\alpha_1,\alpha_2}$$

Goal: each player has a limited fishing ability $(0 \leq \alpha_i \leq 1\, , \int \alpha_i = V_i)$

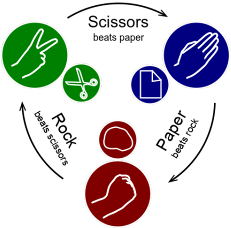
Is there a Nash equilibrium? What does it look like? What happens if the fishermen coordinate?

Nash equilibria

$$J_1(\alpha_1^*, \alpha_2^*) \geq J_1(\alpha_1, \alpha_2^*)\,, J_2(\alpha_1^*, \alpha_2^*) \geq J_2(\alpha_1^*, \alpha_2).$$

There is no way to increase your payoff by changing your strategy knowing other players do not change theirs.

In general: N.E. DO NOT EXIST



Plan

To explain our results:

1. First, the case of a single player to **underline the qualitative difficulties**.
2. Second, some **qualitative results about Nash equilibria**.
3. Third, the **tragedy of the commons**.

What about a single fishing companies?

In this case, the control is the fishing strategy $\alpha : \Omega \to \mathbb{R}_+$. The population accesses natural resources $K : \Omega \to \mathbb{R}_+$ and satisfies

$$\begin{cases} -\mu\Delta\theta_{\alpha,\mu} - \theta_{\alpha,\mu}(K - \alpha - \theta_{\alpha,\mu}) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_{\alpha,\mu} = 0 & \text{on } \partial\Omega\,, \\ \theta_{\alpha,\mu} \geq 0\,, \neq 0. \end{cases} \tag{1}$$

Constraints on the fishing strategy α:

$$\alpha \in \mathcal{M} = \left\{ 0 \leq \alpha \leq \kappa\,, \int_\Omega \alpha \leq V_0 < \int_\Omega K \right\}.$$

Optimisation problem

$$\boxed{\max_{\alpha \in \mathcal{M}} J(\alpha) = \int_\Omega \alpha \theta_{\alpha,\mu}.}$$

First remarks on the integral constraint

First remark on $\int_\Omega \alpha \leq V_0$: a player should **not always** play a strategy α^* that satisfies $\int_\Omega \alpha^* = V_0$.
Intuitive explanation: If I fish too much, I am killing too many fishes.

First remarks on the integral constraint

First remark on $\int_\Omega \alpha \leq V_0$: a player should **not always** play a strategy α^* that satisfies $\int_\Omega \alpha^* = V_0$.
Intuitive explanation: If I fish too much, I am killing too many fishes.
How is this formalised?

Theorem (M.,Ruiz-Balet, JOMB, 2022)

For any K, μ, there exists $V_\pm$ such that:

1. *If $V_0 \leq V_-$, J is increasing and any optimal fishing strategy α^* satisfies $\int_\Omega \alpha^* = V_0$ (saturated L^1 constraint).*
2. *If $V_0 \geq V_+$, then any optimal fishing strategy α^* satisfies $\int_\Omega \alpha^* < V_0$ (non-saturated L^1 constraint).*

A consequence of this result

1. For **bilinear control problems**,

   Monotonicity of the functional $\rightsquigarrow$ Concavity/convexity properties

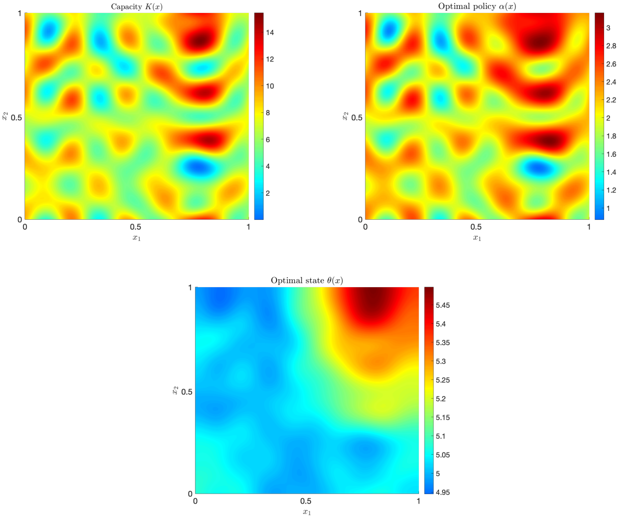
   1. M., Nadin, Privat, Communications in PDEs, 2021
   2. M., Journal of Functional Analysis, 2023.

   More precisely: we can show that if we further impose that K is "almost constant" then J is concave if V_0 is small enough. The influence of κ is hard to quantify.

2. Concavity is important for the study of Nash equilibria.

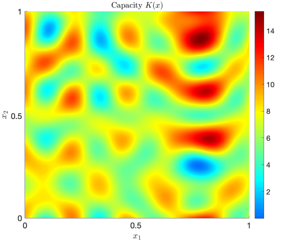
Simulation of the optimal single player

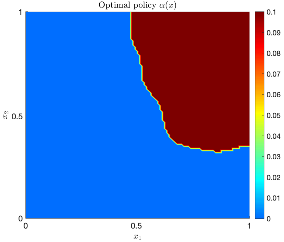
Same V_0, $\kappa = 7$

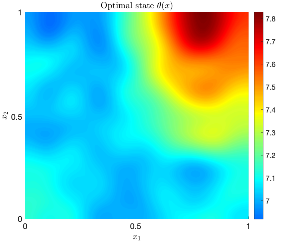


Simulation of the optimal single player

Same V_0, $\kappa = 0.1$







What about two fishing companies?

In this case, the controls are two fishing strategies $\alpha_1, \alpha_2 : \Omega \to \mathbb{R}_+$. The population access natural resources $K : \Omega \to \mathbb{R}_+$ and satisfies

$$\begin{cases} -\mu\Delta\theta_{\alpha_1,\alpha_2,\mu} - \theta_{\alpha_1,\alpha_2,\mu}(K - \alpha_1 - \alpha_2 - \theta_{\alpha_1,\alpha_2,\mu}) = 0 & \text{in } \Omega\,, \\ \partial_\nu \theta_{\alpha_1,\alpha_2,\mu} = 0 & \text{on } \partial\Omega\,, \\ \theta_{\alpha_1,\alpha_2,\mu} \geq 0\,, \neq 0. \end{cases} \tag{2}$$

Constraints on the fishing strategies α_i:

$$\alpha_i \in \mathcal{M}_i = \left\{ 0 \leq \alpha \leq \kappa_i\,, \int_\Omega \alpha \leq V_i \right\}\,, V_1 + V_2 < \int_\Omega K$$

Optimisation problem

$$\boxed{\text{Each player tries to solve } \max_{\alpha_i \in \mathcal{M}_i} J_i(\alpha_1, \alpha_2) = \int_\Omega \alpha_i \theta_{\alpha_1,\alpha_2,\mu}.}$$

Nash equilibria

The question is: **does there exist a Nash equilibrium?**
Recall that a Nash equilibrium is (α_1^*, α_2^*) such that

$$J_1(\alpha_1^*, \alpha_2^*) = \max_{\alpha_1 \in \mathcal{M}_1} J_1(\alpha_1, \alpha_2^*)$$

$$J_2(\alpha_1^*, \alpha_2^*) = \max_{\alpha_2 \in \mathcal{M}_2} J_2(\alpha_1^*, \alpha_2).$$

An existence result

Theorem (M. Ruiz-Balet, JOMB, 2022)

In one dimension, or in several dimensions if K is almost constant, if $V_1 + V_2 \ll 1$, there exists a Nash equilibrium.

Related to the aforementioned concavity properties in the single player case.

The tragedy of the commons

Do there exist Nash equilibria that show a **depletion** of the fishery? We take the case of $N \gg 1$ players, with the same admissible controls.
Consider

$$\begin{cases} -\mu\Delta u_{\vec{\alpha}} = u_{\vec{\alpha}}(1 - u_{\vec{\alpha}}) - \left(\frac{1}{N}\sum_{i=1}^{N}\alpha_i(x)\right)u_{\vec{\alpha}} & x \in \Omega \\ \partial_\nu u_{\vec{\alpha}} = 0 & x \in \partial\Omega \end{cases}$$

Where each player is optimizing

$$J_i(\vec{\alpha}) = \int_\Omega \alpha_i u_{\vec{\alpha}} dx$$

Tragedy of the commons

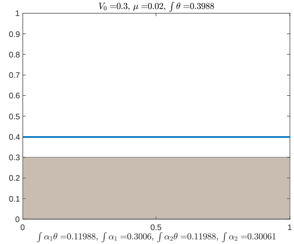
There exist a sequence of Nash equilibria $\vec{\alpha}_N^* \in \mathcal{M}^N$, $N \in \mathbb{N}$ such that

$$\frac{1}{4} = \max_{\vec{\alpha}\in\mathcal{M}^N}\left(\sum_{i=1}^{N} J_i(\vec{\alpha})\right) > \sum_{i=1}^{N} J_i(\vec{\alpha}_N^*) \underbrace{\longrightarrow}_{N\to+\infty} 0$$

Furthermore, in this equilibrium sequence, each players gets $\approx 1/N^2$ fishes, and the fishes' population goes to zero. If they **collaborated and split the fish each would get $\approx 1/N$ and the fishes' population would have a lower bound.**

Some numerical simulations in one-d

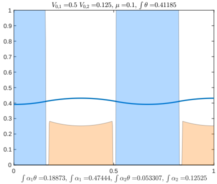
Here $K \equiv 1$ (homogeneous environment), with a bi-level optimisation scheme.

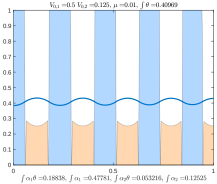


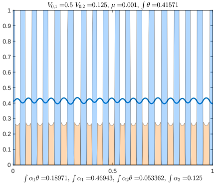
If the players have exactly the same integral constraints, the strategies coincide.

Some numerical simulations in one-d

Here $K \equiv 1$ (homogeneous environment), with a bi-level optimisation scheme.







With different integral bounds, the strategies no longer coincide, but neither do their supports! As $\mu \to 0$, We observe a fragmentation property.

Unbounded domains, infinite number of players

The tragedy of the commons: A Mean-Field Game approach to the reversal of travelling waves

Z. Kobeissi, I. Mazari-Fouquer, D. Ruiz-Balet, *Submitted*, 2023.

The goal

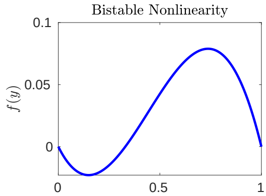
1. Further our understanding of overfishing phenomena.
2. We want a **drastic** illustration of the tragedy of commons: can invasive species go extinct due to the action of fishermen?
3. Ill-posed: it suffices to fish a lot. However, not always in the best interest of fishermen.
4. Better formulation: can invasive species go extinct due to the action of fishermen that are competing and acting in their best interest? Can coordination remedy this?

The framework

1. Travelling wave formalism (Fisher, KPP, Skellam).
2. Bistable non-linearity.

$$\partial_t \theta = \partial_{xx}\theta + f(\theta) \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+$$

$$\theta(x,0) = \theta_0(x)$$



Allee Effect

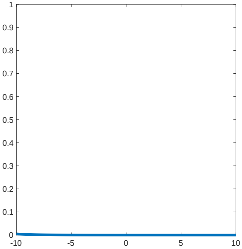
When the density of individuals is too low the population decreases

$$f(0) = f(\eta) = f(1) = 0, \quad f'(0), f'(1) < 0, f'(\eta) > 0, \int_0^1 f > 0.$$

Traveling waves

Particular feature of bistable equations: uniqueness of a travelling wave solution $u = U(x - c^* t)$, (U, c^*) with $c^* < 0$:

$$-U'' - c^* U' = f(U)\,, U(-\infty) = 0, U(+\infty) = 1.$$



Moreover, these solutions are "generic": they are dynamically attractive (Fife, 1979)

The MFG model

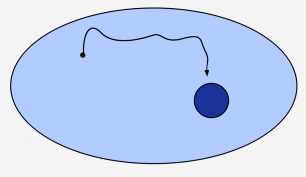
1. Context of Mean Field Games: infinitely many players driven by their self-interest.
2. Lasry & Lions, Caines, Huang & Malhamé.

Assumption

Infinitely many players.
The **influence of an individual** on the population is **negligible**.

$$\max_{\alpha \in L^\infty} J(x_0, \alpha) = \int_0^{+\infty} \underbrace{e^{-\lambda t}}_{\text{discount factor}} \left(\underbrace{\theta(t, x_\alpha(t))}_{\text{Harvested Fish}} - \underbrace{L(\alpha(t))}_{\text{control cost}} \right) dt$$

$$\dot{x}_\alpha(t) = \alpha(t), \quad x(0) = x_0$$



Mean-field games

Assumption

An **individual fisherman** has a **negligible effect** on the population.

Important

The cumulated action of fishermen has an impact on the population.

$$\partial_t\theta - \partial_{xx}\theta = f(\theta) \quad \underbrace{-m(x,t)\theta}_{\text{Total Harvested Fish}}$$

where m is the fishermen density. It satisfies a continuity equation

$$\partial_t m + \partial_x\big(\alpha(x,t)m\big) = 0$$

where $\alpha(x,t) = \alpha_x(t)$ is optimal in

$$J(t,\boldsymbol{\alpha}_x) = \int_0^{+\infty} e^{-\lambda t}\left(\theta_{\boldsymbol{\alpha}}(y(t),t) - L(\boldsymbol{\alpha}_x(\boldsymbol{t}))\right)dt \quad \dot{y}_{\boldsymbol{\alpha}}(t) = \boldsymbol{\alpha}_x(\boldsymbol{t}), \quad y(0) = x$$

The MFG system

$$\begin{cases} \lambda V - \partial_t V - H(\partial_x V) = \theta & \text{in } (0;T)\times\mathbb{R}\,, \\ V(+\infty,\cdot) = 0, & \\ \partial_t m + \partial_x \left(H'(\partial_x V)m\right) = 0 & \text{in } (0;T)\times\mathbb{R}\,, \\ m(0,\cdot) = m_0, & \\ \partial_t\theta - \partial^2_{xx}\theta = f(\theta) - m\theta & \text{in } (0;T)\times\mathbb{R}\,, \\ \theta(0,\cdot) = \theta_0. & \end{cases}$$

Goal:travelling wave solutions of this system. Related to [Porretta, Rossi, 2021], but here we have a reaction-diffusion equation which complexifies this query.

Extinction and Reversed Traveling waves

There are **many possible ways** in which a population can **go extinct**.

Reversed Traveling Wave

- If $m = 0$ then the "fishes invade" (this is the case here since $\int_0^1 f > 0$)
- There exist $m(x,t) = \mathcal{M}(x - ct), u(t,x) = U(x - ct)$ with $c > 0$ and $U(-\infty) = 0\,, U(+\infty) = 1 \rightsquigarrow$ The fishes go extinct!

Related works

1. Takes place in the wider context of **controlling travelling waves**.
2. Bressan, Chiri, Salehi (2022): existence of reversed travelling waves, but different (what is the best way to kill the population?)
3. Almeida, Leculier, Nadin, Privat (2023): optimal control for pest eradication.

Reversed MFG Traveling Wave

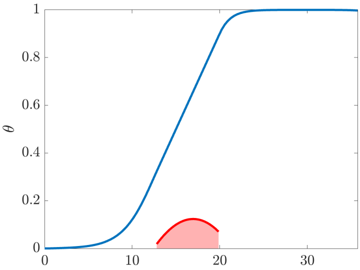
Reversed MFG Traveling Wave

- If $m = 0$ then the "fishes invade"
- There exist $m(x,t) = \mathcal{M}(x-ct), u(t,x) = U(x-ct)$ with $c > 0$ and $U(-\infty) = 0\,, U(+\infty) = 1 \rightsquigarrow$ The fishes go extinct!
- Each fisherman is playing an optimal strategy.

Theorem K,M,R-B, 2023

There exist reversed traveling waves in which every fisherman is playing an optimal strategy.

The fishermen are acting optimally and yet killing the fishes.



Coordination and the tragedy of the commons revisited

Theorem (Kobeissi, M-F, Ruiz-Balet, 2023)

There exist a family of Lagrangians L and a coordinated strategy such that

1. *The fishes' population remains invading,*

Coordination and the tragedy of the commons revisited

Theorem (Kobeissi, M-F, Ruiz-Balet, 2023)

There exist a family of Lagrangians L and a coordinated strategy such that

1. *The fishes' population remains invading,*
2. *The total amount of harvested fishes is higher.*

Coordination and the tragedy of the commons revisited

Theorem (Kobeissi, M-F, Ruiz-Balet, 2023)

There exist a family of Lagrangians L and a coordinated strategy such that

1. *The fishes' population remains invading,*
2. *The total amount of harvested fishes is higher.*
3. *Each fisherman obtains a higher harvest.*

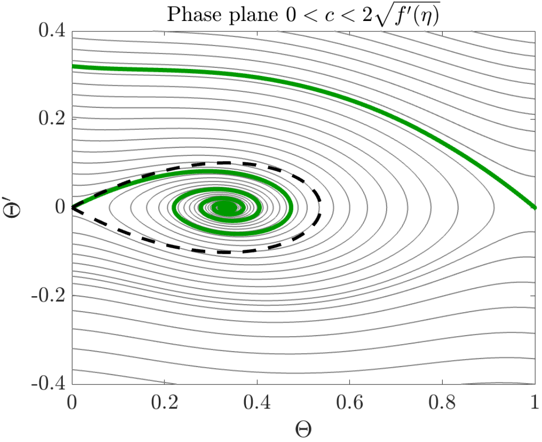
1. Prisoner's dilemna-type situation.
2. No more travelling wave structure.
3. The Lagrangians L look a lot like L^∞ constraints...

Proof

1. Relies on a phase portrait-analysis.
2. Explicit construction of a reversed Travelling Wave profile.
3. Optimality investigated at the level of each individual player.
4. For the coordination, explicit construction of a "spreading out" density of players.

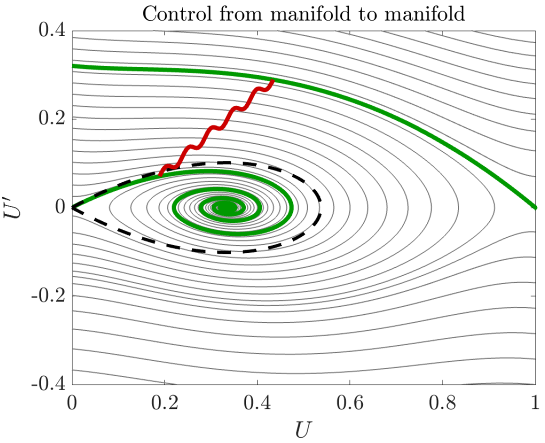
Elements of proof 1

1. Basic idea: phase portrait for a given (target) speed $c > 0$ for the system
$$-U'' - cU = f(U) \text{ without fishermen.}$$
2. In green: (un)stable manifolds associated with the equilibria $(0, 0)$ and $(0, 1)$.



Elements of proof 2

1. The idea is to use the control m to go from the unstable manifold associated with (0,0) to the stable one associated with (1,0).
2. However, there are many ways to do this:



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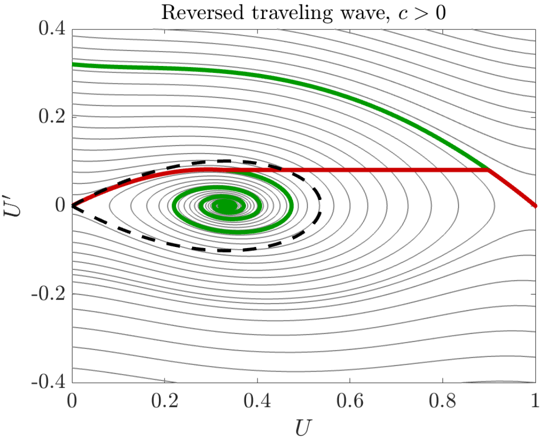


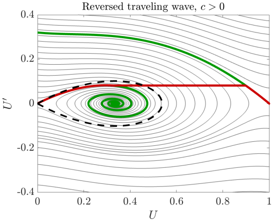
3. Which one do we choose?

Elements of proof 3

1. We look for a reversed travelling wave: $V = V(x-ct)\,, \theta = U(x-ct)\,, m = M(x-ct)$, that solves the MFG system.
2. We thus have the system of ODE

$$\begin{cases} \lambda V + CV' - H(V') = U\,, \\ cM' + (H'(V')M)' = 0\,, \\ -U'' - cU' = f(U) - MU \end{cases}$$

   and we deduce $H'(V') = \mathit{constant}$ in $\mathrm{supp}(M) \Rightarrow V' = \mathit{constant}$ in $\mathrm{supp}(M)$.
3. Thus: U' is constant in $\mathrm{supp}(M)$.
4. A good strategy is then:



Elements of proof 4

Thus we have constructed a travelling wave solution

$$-U'' - cU' = f(U) - MU.$$

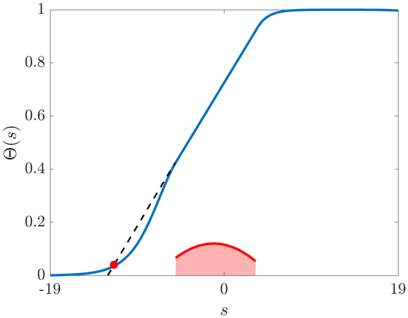
M is a macroscopic density of fishermen. is there any chance this is optimal at the level of every individual fisherman?

1. Let us consider the case
$$L(\alpha) = D\frac{||\alpha||^2}{2}.$$
2. In this case, for any $x_0 \in \mathrm{supp}(M)$ a fisherman starting from x_0 tries to solve
$$\max J(x_0, \alpha) = \max \int e^{-\lambda t}(U(x_0 - \alpha t) - L(\alpha)).$$
Goal: check that $\alpha = c$ is optimal.
3. We first check (easy) that $\alpha = c$ is a critical point.
4. Second, we check that if D is big enough then $J(x_0, \cdot)$ is concave; this concludes the proof.

Elements of proof 5

We now consider a strictly convex Lagrangian L.
The goal is once again to prove that for any $x_0 \in \operatorname{supp}(M)$ $\overline{\alpha} \equiv c$ is optimal.

1. Recall: U is linear in $\operatorname{supp}(M)$.
2. We compare with the linear extension of U outside of $\operatorname{supp}(M)$.



3. First we prove that there is no point in going backwards too quickly ($\alpha \geq -c$)

Elements of proof 6

We now consider a strictly convex Lagrangian L.
The goal is once again to prove that for any $x_0 \in \mathrm{supp}(M)$ $\overline{\alpha} \equiv c$ is optimal.

1. Recall: U is linear in $\mathrm{supp}(M)$.
2. We compare with the linear extension of U outside of $\mathrm{supp}(M)$.
3. First we prove that there is no point in going backwards too quickly ($\alpha \geq -c$).
4. Second, we prove that there is no point in reaching the crossing point between the two functions: Bellman type arguments.

Elements of proof 6

We now consider the influence of adding coordination.

1. Start at $t = 0$ from (U, M), a reversed travelling wave (no reference to optimality at this point).
2. We construct an explicit coordinated strategy m as a "spreading" of M:

$$m(t,x) \sim \frac{1}{1+t} M((1+t)x) \Rightarrow \begin{cases} \operatorname{supp}(m(t,\cdot)) \subset [0;+\infty), \\ \|m(t)\|_{L^\infty} \leq \frac{1}{t} \sim \epsilon \ll 1. \end{cases}$$

3. For this coordinated strategy, in large times:

$$-u'' - cu' \sim f(u) - \epsilon u$$

   similar to a bistable equation with an invasive travelling front: the population remains invading.

4. To conclude: we show that if $q \gg 1$, this is a better strategy than the constant speed for $L_q \sim |\cdot|^q$.

Perspectives

- Many properties of the fishing game not presented here: large diffusivity limits, precised behaviour etc
- Can we get **geometric properties of Nash** equilibria?
- Can we get convergence of the numerical simulations?
- For MFG, study the **stability of travelling waves** and the impact of **governmental regulations**.

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