Weak solutions in mean-field game systems: applications to optimal transport and congestion models

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Plan of the talk

- Brief recap: the standard PDE system in MFG theory
- Local couplings $\rightsquigarrow$ weak solutions

  (i) Second order system: weak setting for Fokker-Planck equations

  (ii) Vanishing viscosity & first order: a relaxed notion of weak solutions
- Links with optimal transport problems
- Congestion models in crowd dynamics

  $\rightsquigarrow$ Mean field games Vs Mean field control

  (i) Diffusive models

  (ii) Vanishing viscosity and first order systems

  $\rightsquigarrow$ the technical box: weak duality, convexity, convolution, renormalization, etc....

Mean field game theory: what is about ?

MFG theory was introduced since 2006 by J-M Lasry and P-L Lions.
A similar model developed independently by [Huang-Caines-Malhamé].

Goal: study Nash equilibria in large populations of rational agents with weak interaction

large population $\rightsquigarrow$ infinite number (a continuum) of similar players

rational agents $\rightsquigarrow$ each agent is controlling his own dynamical state

weak interaction $\rightsquigarrow$ each single agent has no influence on the others'.
But everyone takes into account the collective behavior through the distribution law (empirical density) of the states.

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Basic idea: Limit of Nash equilibria of symmetric N-players games will satisfy, as $N \to \infty$, a system of PDEs coupling the equation for the individual strategies with the equation for the distribution law

$\rightsquigarrow$ equilibria between individual optimization and collective behavior

Macroscopic (mean-field) description

$$\begin{cases} dX_s = \alpha(X_s)ds + \sqrt{2}\; dB_s\,, \\ X_t = x \end{cases}$$

dynamics of each agent

$$u(t,x) = \inf_{\{\alpha(\cdot)\}} \mathbb{E}\int_t^T L(X_s, \alpha(X_s), \mu_s) + G(X_T, \mu_T)$$

where $\{\mu_t\}$ is an exogenous family of measures

HJB

$$-\partial_t u - \Delta u + H(x, \mu, Du) = 0$$

$$\underbrace{\alpha^*(\cdot) = -H_p(\cdot, \mu_t, Du(t,\cdot))}_{\text{optimal policy}},$$

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The MFG system of PDEs

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Usually complemented with initial-terminal conditions:

-$m(0) = m_0$ (initial distribution of the agents, $m_0 \geq 0\,, \int m_0 = 1$)

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Two kind of settings:

1. Smoothing coupling (nonlocal case):
   $m \mapsto H(x, p, m)$ and $G(x, m)$ are continuous from $\mathcal{P}(\Omega)$ to $C^2(\Omega)$
   ($\rightsquigarrow$ regularity works as if equations were decoupled)

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2. Local coupling: $m \in L^1$ and $H(x, p, m)$, $G(x, m)$ depend on the local Lebesgue density of m

Towards a theory of weak solutions

New PDE questions arise for *local couplings*. Model case:

$$\begin{cases} -\partial_t u - \varepsilon\Delta u + H(x, Du) = F(x, m) & \text{in } (0, T) \times \Omega \\ \partial_t m - \varepsilon\Delta m - \mathrm{div}(m\, H_p(x, Du)) = 0 & \text{in } (0, T) \times \Omega\,, \\ m(0) = m_0\,, \qquad u(T) = G(x, m(T)) & \end{cases}$$

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- the existence of smooth solutions is known only in few cases:

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  Otherwise, regularity of solutions is not known.

- But it is not difficult to construct weak (distributional) solutions as soon as $F(x, m)$ is bounded below.

However: Weak solutions are in general unbounded !

A priori estimates of the system (valid with & without diffusion !):

$$\left.\begin{array}{ll}(1) & \int_0^T\!\int_\Omega F(x,m)m \\ (2) & \int_0^T\!\int_\Omega H(x,Du) \\ (3) & \int_0^T\!\int_\Omega m\,L(x,H_p(x,Du))\end{array}\right\} \le C(\|m_0\|_\infty)$$

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Typical growth conditions

- $F(m) \simeq m^{p-1}$, $p > 1$:

$$(1) \Rightarrow \; m \in L^p \quad \Rightarrow \quad F(m) \in L^{p/p-1}$$

p large $\rightsquigarrow$ Hamilton-Jacobi with (nearly) L^1-data

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- $L(x,\alpha)$, $H(x,p)$ with coercive quadratic growths:

$$(2)-(3) \Rightarrow \; Du \in L^2\,, \quad m\,|Du|^2 \in L^1$$

$\rightsquigarrow$ Fokker-Planck with L^2- drift

Main difficulties of a weak theory:

(i) Uniqueness may fail for unbounded solutions of HJB:

$$\exists \;\; u \in L^2(0,T;H^1_0)\,, \;\; u \neq 0 \text{ sol. of } \quad \begin{cases} u_t - \Delta u + |Du|^2 = 0 \\ u(0) = 0 \end{cases}$$

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(ii) The typical setting of well-posedness of the Fokker-Planck

$$(FP) \qquad\qquad m_t - \Delta m + \operatorname{div}(m\,b) = 0$$

requires much more than L^2 drifts, usual theory needs $b \in L^\infty(0,T;L^d(\Omega))$, or $b \in L^{d+2}((0,T)\times\Omega)$ ([Aronson-Serrin], [Ladysenskaya-Solonnikov-Uraltseva])

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But......a full theory is still possible entirely relying on the estimate

$$m|b|^2 \in L^1$$

In mean field games this is indeed the estimate $m|Du|^2 \in L^1$ which comes from optimization !!

The typical statement is the following (adapted to Dirichlet, Neumann, or to entire space $\mathbb{R}^N$ under suitable modifications)

Theorem (P. ARMA '15)

Let $b \in L^2(Q_T)^N$ and $m_0 \in L^1$. Then the problem

$$\begin{cases} m_t - \Delta m - \mathrm{div}(m\, b) = 0\,, & \textit{in } (0,T)\times\Omega, \\ m(0) = m_0 & \textit{in } \Omega. \\ + BC & \end{cases} \qquad (1)$$

admits at most one weak sol. $m \in L^1(Q_T)_+$: $m|b|^2 \in L^1(Q_T)$.

Moreover, in this case any weak solution is a renormalized solution, belongs to $C^0([0,T];L^1)$ and satisfies (for a suitable truncation $T_k(\cdot)$):

$$(T_k(m))_t - \Delta T_k(m) - \mathrm{div}(T_k'(m) m\, b) = \omega_k\,, \qquad \textit{in } Q_T \qquad (2)$$

where $\omega_k \in L^1(Q_T)$, and $\omega_k \overset{k\to\infty}{\to} 0$ in $L^1(Q_T)$.

Main idea: *a nonlinear look at a linear equation*

- for general convection-diffusion problems (possibly nonlinear)

$$\begin{cases} m^\varepsilon_t + A m^\varepsilon = \operatorname{div}\left(\phi(t,x,m^\varepsilon)\right) & \text{in } Q_T \\ m^\varepsilon(0) = m^\varepsilon_0 \,, \ +\text{BC} \end{cases}$$

we have that if

$$|\phi(t,x,m)| \leq c(1+\sqrt{m})\, k(t,x)\,, \qquad k \in L^2(Q_T) \tag{3}$$

then

$$m^\varepsilon \to m \quad \text{in } C^0([0,T];L^1)$$

and m is renormalized solution relative to m_0.

- One can apply this idea even in the Di Perna-Lions approach, regularizing m by convolution:

$$m_t - \Delta m - \mathrm{div}(m\,b) = 0 \qquad \star \rho_\varepsilon$$

$$\Rightarrow m^\varepsilon := m \star \rho_\varepsilon \quad \text{solves}$$

$$m^\varepsilon_t - \Delta m_\varepsilon - \mathrm{div}((mb) \star \rho_\varepsilon) = 0$$

where Schwartz's inequality + $m \geq 0$ imply

$$|(mb) \star \rho_\varepsilon| \leq \underbrace{(m \star \rho_\varepsilon)^{\frac{1}{2}}}_{\sqrt{m^\varepsilon}} \underbrace{\left((m|b|^2) \star \rho_\varepsilon\right)^{\frac{1}{2}}}_{B^\varepsilon}$$

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$\to$ for purely second order operators, no need of commutators

Weak solutions to Mean Field Games systems

$$\begin{cases} -u_t - \Delta u + H(x, Du) = F(x, m)\,, \\ m_t - \Delta m - \text{ div } (m\, H_p(x, Du)) = 0\,, \\ u(T) = G(x, m(T))\,, \ m(0) = m_0 \end{cases}$$

- $F, G \in C^0(\overline{\Omega} \times \mathbb{R})$
- $p \mapsto H(x, p)$ is convex and satisfies structure conditions

Ex: $H \simeq \gamma(t, x)|\nabla u|^2 + b(t, x) \cdot \nabla u$.

Def. of weak solutions:

- $u, m \in C^0([0, T]; L^1)$, $m\,|Du|^2 \in L^1$

-$G(x, m(T)) \in L^1$, $H(x, Du) \in L^1$, $F(x, m) \in L^1$,

- the equations hold in the sense of distributions.

Theorem (P. '15)

Assume that $m \mapsto G(x,m)$ *is nondecreasing, and let* $m_0 \in L^\infty_+$.
(i) If F, G *are bounded below, then there exists a weak solution.*
(ii) If in addition $m \mapsto F(x,m)$ *is nondecreasing,* $p \mapsto H(x,p)$ *is strictly convex (at infinity), then there is at most one weak solution* (u,m) *such that* $m > 0$.

Rmk: The coupling functions F, G have no growth restriction from above

• The case $F = F(x)$ is included !! $\rightsquigarrow$ new results for HJ equations with L^1-data

$$\begin{cases} u_t - \Delta u + H(x, Du) = F(x) \\ u_{\partial\Omega} = 0\,, \;\; u(0) = u_0 \end{cases}$$

Uniqueness $\iff$ $m_t - \Delta m - \operatorname{div}\,(H_p(x,Du)\,m) = 0$ admits a sol. m with $H_p(x,Du) \in L^2(m)$.

$\rightsquigarrow$ *uniqueness holds if the adjoint of the linearized admits nice solutions ....a Fredholm-type result !*

- Numerical schemes converge towards weak solutions [Achdou-P. '16]

We use finite differences approximations of the mean field games system as in [Achdou-Capuzzo Dolcetta & Camilli]:

$$\begin{cases} \frac{u_{i,j}^{k+1}-u_{i,j}^{k}}{\Delta t} - (\Delta_h u^k)_{i,j} + g(x_{i,j}, \left[\nabla_h u^k\right]_{i,j}) = F(m_{i,j}^{k+1}), \\ \frac{m_{i,j}^{k+1}-m_{i,j}^{k}}{\Delta t} - (\Delta_h m^{k+1})_{i,j} + \mathcal{T}_{i,j}(u^k, m^{k+1}) = 0, \end{cases}$$

where g is a monotone approximation of the Hamiltonian H as in upwind schemes:
Ex (1-d): $g = g\left(\frac{u_{i+1}-u_i}{h}, \frac{u_i-u_{i-1}}{h}\right)$ with $g(p_1, p_2)$ increasing in p_2 and decreasing in p_1, $g(q, q) = H(q)$.

while $\mathcal{T}$ is the discrete adjoint of the associated linearized transport:

$$\mathcal{T}(v, m) \cdot w = m\, g_p([\nabla_h v]) \cdot [\nabla_h w]$$

Similar structure allows to have discrete estimates and compactness as in the continuous model.

Vanishing viscosity & first order MFG systems

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Assume some coercivity on the coupling terms:

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Assume some coercivity on the coupling terms:

- $F, G \simeq m^{p-1}$, with $p > 1$.

$\Rightarrow$ As $\varepsilon \to 0$, weak solutions converge towards a relaxed formulation of the first order system [Cardaliaguet-Graber-P.-Tonon '15]

(i) u is a distributional subsolution: $-u_t + H(x, \nabla u) \le F(x, m)$

(ii) m is a distributional solution: $m_t - \text{ div } (m\, H_p(x, \nabla u)) = 0$

(iii) the energy equality holds

$$\int_0^T\!\!\int_\Omega m\, F(x,m)dxdt + \int_0^T\!\!\int_\Omega m\left\{H_p(x,Du)Du - H(x,Du)\right\}dxdt$$
$$= \int_\Omega m_0\, u(0) - \int_\Omega G(x, m(T))m(T)$$

Theorem (CGPT)

Assume in addition that $p \mapsto H(x,p)$ is strictly convex and $m \mapsto F(x,m)$ is increasing. Then the first order system

$$\begin{cases} -u_t + H(x, Du) = F(x,m)\,, \\ m_t - \ div\,(m\, H_p(x, Du)) = 0\,, \\ m(0) = m_0\,,\ u(T) = G(x, m(T)) \end{cases}$$

admits a unique weak (relaxed) solution (u,m) in the sense that m is unique and Du is unique in $\{m > 0\}$.

- Existence is proved through vanishing viscosity limit. Ingredients: coercivity + weak limits + Minty's argument (convex Hamiltonian and monotone couplings...)
- Key point: duality between sub solutions of Hamilton-Jacobi and solutions of the continuity equation

Theorem (CGPT)

Assume in addition that $p \mapsto H(x,p)$ is strictly convex and $m \mapsto F(x,m)$ is increasing. Then the first order system

$$\begin{cases} -u_t + H(x, Du) = F(x, m)\,, \\ m_t - \ div\,(m\, H_p(x, Du)) = 0\,, \\ m(0) = m_0\,,\ \ u(T) = G(x, m(T)) \end{cases}$$

admits a unique weak (relaxed) solution (u, m) in the sense that m is unique and Du is unique in $\{m > 0\}$.

- Existence is proved through vanishing viscosity limit. Ingredients: coercivity + weak limits + Minty's argument (convex Hamiltonian and monotone couplings...)
- Key point: duality between sub solutions of Hamilton-Jacobi and solutions of the continuity equation
- Weak solutions are equivalent to relaxed minima of corresponding optimal control problems [Cardaliaguet-Graber '15]

Link with optimal control systems

MFG as optimality system (optimal control with Fokker-Planck state eq.).

Ex: Optimize in terms of the field α

$$\partial_t m = \Delta m + \text{ div } (\alpha\, m)\,, \qquad m(0) = m_0$$

$$\longrightarrow \quad \inf_\alpha \int_0^T \int_\Omega \{L(x,\alpha)m + \Phi(m(s))\}\, dt + \mathcal{G}(m(T))$$

where $\Phi'(m) = F(m)$ and $\mathcal{G}'(m) = G(m)$.

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First order optimality conditions give the adjoint state u:

$$\begin{cases} Du + L_\alpha(x,\alpha) = 0 \quad (m - q.o.) \\ -\partial_t u - \Delta u - \alpha \cdot Du - L(x,\alpha) = F(m) \end{cases} \Leftrightarrow \begin{array}{l} \alpha_{opt} = -H_p(x, Du(t,x)) \\ -\partial_t u - \Delta u + H(x, Du) = F(m) \end{array}$$

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Rmk: $F(m), G(m)$ nondecreasing $\Rightarrow$ convexity of the functional

Mean field games $\to$ optimal transport

Planning problem in MFG: prescribe a final distribution law $m(T) = m_1$

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Here, no condition is assumed on u at time T.

$\rightsquigarrow$ this is a singular limit of MFG systems with terminal condition $u_\varepsilon(T) = \frac{m_\varepsilon(T) - m_1}{\varepsilon}$, $\varepsilon \to 0$.

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• This is an optimal transport problem: recall the dynamic version of Wasserstein distance

$$W_2^2(m_0, m_1) = \inf \left[\int_0^1 \int_{\mathbb{R}^d} \frac{1}{2} |v|^2 dm(t,x) \;:\quad \begin{cases} \partial_t m + \operatorname{div}\,(vm) = 0, \\ m(0) = m_0, m(1) = m_1 \end{cases} \right]$$

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Key point [Benamou-Brenier]:

if $w = mv$, then $\frac{1}{2}|v|^2 m = \frac{1}{2}\frac{|w|^2}{m}$ (which is convex in (m, w) !!)

The mean field planning problem can be characterized in terms of optimal transport [Orrieri- P.- Savaré '19], [Graber-Meszaros-Silva-Tonon '19]

$$\mathcal{B}(m,v) := \inf \left[\int_0^T \int_{\mathbb{R}^d} L(x,v)\, m\, dxdt + \int_0^T \int_{\mathbb{R}^d} \Phi(x,m) \ : \ \begin{cases} \partial_t m + \text{ div } (vm) = 0, \\ m(0) = m_0, m(T) = m_1 \end{cases} \right].$$

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Assume as before: $F(x,m) \simeq m^{p-1}$ and increasing $L(x,v) \simeq |v|^2$ and smooth. Marginal measures $m_0, m_1 \in L^p$.

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$\rightsquigarrow$ $\exists\,!$ minimizer (m,v) of the optimal transport problem, and $v = -H_p(x,Du)$, where (u,m) is a weak solution of MFG-planning:

$$\begin{cases} -u_t + H(x,Du) \leq F(x,m) \\ m_t - \operatorname{div}(m\, H_p(x,Du)) = 0\,, \quad m(0) = m_0\,, m(T) = m_1 \\ \int_0^T\!\int_\Omega m\, F(x,m) dxdt + \int_0^T\!\int_\Omega m\, \{H_p(x,Du)Du - H(x,Du)\}\, dxdt \\ \qquad = \int_\Omega m_0\, u(0)\, dx - \int_\Omega m_1\, u(T)\, dx \end{cases}$$

MFG modeling crowd dynamics

Mean field games provide a natural environment for the study of crowd dynamics (see also [LaChapelle-Wolfram '11], [Burger-Di Francesco-Markowich-Wolfram '13].

- One population of identical agents: the pedestrians
- Each pedestrian may be affected by a random (individual) noise
- The impact of a single agent on the mass is negligible. But the strategy of a single pedestrian depends on the density $m(x, t)$ of the crowd at space-time point (x, t)

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Rmk: Previous models of crowd motion mainly rely on fluid dynamics models or ideas of statistical mechanics. The strategic (rational) component is often neglected.

Conversely, mean field games lead to crowd motion models including rational anticipation.

Congestion models in crowd dynamics

[P.L. Lions '10]: cost of motion is proportional to the density of crowd

$$L = L(x, v, m) = h(m)\,\frac{|v|^2}{2}\,, \qquad h \uparrow$$

where h increases as m increases. Model case: $h(m) = m^{\alpha}$, $\alpha > 0$.

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Rmk: in the deterministic case, v is the velocity of the agent $\rightsquigarrow$ the kinetic energy is weighted by the distribution density

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The corresponding Hamiltonian becomes

$$H = H(x, Du, m) = \frac{1}{2}\frac{|Du|^2}{m^{\alpha}}$$

- coercivity of the Hamiltonian degenerates as $m \to \infty$
- this is an example of non separate Hamiltonian $H = H(m, Du)$
- Degeneracy, singularity in m $\rightsquigarrow$ weak solutions

MFG system with congestion (model problem, local coupling):

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + \frac{1}{2} \frac{|Du|^2}{m^\alpha} = f(x,m)\,, & (t,x) \in (0,T) \times \Omega \\ \partial_t m - \varepsilon \Delta m - \mathrm{div}(m \frac{Du}{m^\alpha}) = 0\,, & (t,x) \in (0,T) \times \Omega \\ m(0,x) = m_0\,,\ u(T,x) = g(x, m(T))\,, & x \in \Omega\,, \end{cases} \tag{4}$$

- f, g represent additional terms in the individual preferences (running cost, final pay-off) depending on the mass distribution (may simulate other inhomogenous aversion effects)

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Two main features were already addressed by P-L. Lions:

1. the system has not a variational structure
2. uniqueness of solutions holds under some restriction on α

MFG systems with general Hamiltonians $H(m, Du)$:

$$\begin{cases} -\partial_t u - \varepsilon \Delta u + H(m, Du) = 0\,, \\ \partial_t m - \varepsilon \Delta m - \mathrm{div}(m H_p(m, Du)) = 0\,, \end{cases}$$

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$\rightsquigarrow$ performing an optimal control over the Fokker-Planck equation is different from the MFG paradygm !

Ex: Optimizing the FP equation

$$\partial_t m = \Delta m + \text{ div } (\mathrm{v}\, m)\,, \qquad m(0) = m_0$$

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Minimization leads (formally) to the optimality system

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which is NOT a MFG system !!! Notice the threshold $\alpha < 1$

Variational systems arise from Mean Field type Control [Bensoussan-Frehse-Yam]:

every agent is using the same feedback control and optimizes using the corresponding law:

$$
\begin{cases} dX_s = \beta(X_s)ds + \sqrt{2}\, dB_s \\ X_t = x \end{cases}
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\rightsquigarrow \inf_{\beta} \mathbb{E}_{t,x} \left\{ \int_t^T [L(X_s, \beta(X_s), \mathcal{L}(X_s))]ds + G(X_T, \mathcal{L}(X_T)) \right\}
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In congestion models, *mean field control* and *mean field games* equilibria are truly different

Mean field type control systems with congestion $\rightsquigarrow$ [Achdou-Lauriere '16]

Uniqueness for MFG systems with general Hamiltonians $H(m, Du)$?

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P.L. Lions proved that uniqueness of smooth solutions holds whenever

$$\begin{pmatrix} -H_m & \frac{1}{2}\, m\, H_{mp} \\ \frac{1}{2}\, m\, H_{mp} & m\, H_{pp} \end{pmatrix} > 0 \tag{5}$$

Indeed, (5) implies that

$\frac{d}{dt} \int (u_1 - u_2)(m_1 - m_2) < 0$ for any couple of solutions $(u_1, m_1), (u_2, m_2)$

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Rmk: in the variational models of congestion uniqueness holds if $\alpha < 1$

Back to the model problem:

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Goal: prove existence and uniqueness of solutions under the Lions' condition $\alpha \in (0,2)$.

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- Extends to general Hamiltonians $H(x, m, Du)$ under structure conditions

$$H(t,x,m,p) \simeq c_0 \frac{|p|^q}{(m+\mu)^\alpha} - c_1\left(1 + m^{\frac{\alpha}{q-1}}\right), \qquad \mu \geq 0$$

Vanishing viscosity & first order system

Pb: What happens as $\varepsilon \to 0$....

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Extra motivations:

- Yves' simulations look stable for very small viscosity
- Most models of congestion in pedestrian dynamics are often deterministic (e.g. [Hughes '02], [Piccoli-Tosin '11]) and there is an increasing literature on deterministic mean field games with other congestion effects (e.g. minimal time exit problems [Mazanti-Santambrogio '18], density constraints [Cardaliaguet-Santambrogio]).

- There is an extensive and important literature on *variational* mean field congestion problems, specifically related to optimal transport

  $\rightsquigarrow$ Wassernstein's distance with congestion

$$W_\gamma(m_0, m_1)^2 := \inf \left\{ \int_0^1 \int_\Omega m^\gamma |v|^2 \, , \partial_t m - \text{ div } (m^\gamma v) = 0 \right.$$
$$\left. m(0) = m_0 \, , \; m(1) = m_1 \right\}$$

  introduced in [Dolbeault-Nazaret-Savaré '09] and studied e.g. in [Carrilllo-Lisini-Savaré-Slepcev '10], [Cardaliaguet-Carlier-Nazaret '08].

  For displacement convexity and links to MFG with congestion, see also the recent paper [Gomes-Seneci '19]

Setting of the problem

$$\begin{cases} -\partial_t u + \frac{1}{2}\frac{|Du|^2}{m^\alpha} = f(x,m)\,, & (t,x) \in (0,T) \times \Omega \\ \partial_t m - \mathrm{div}(m^{1-\alpha} Du) = 0\,, & (t,x) \in (0,T) \times \Omega \\ m(0) = m_0\,,\ \ u(T) = g(x, m(T))\,, & x \in \Omega\,, \end{cases}$$

- We assume a bit of coercivity on f, g:

$$c_0\, m^{q-1} - c_1 \leq f(x,m), g(x,m) \leq C_0(m^{q-1} + 1) \qquad q > 1$$

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- f, g monotone
- $0 < \alpha < 2$
- Weak solutions are defined in a relaxed sense - as before

Weak solutions: $u \in L^1(0,T;W^{1,1})$, $m \in L^q(Q_T)$,

$Du = 0$ a.e. in $\{m=0\}$, $\quad \frac{|Du|^2}{m^\alpha}\mathbb{1}_{\{m>0\}}(1+m) \in L^1(Q_T)$

(i) u is a distributional subsolution:

$$\begin{cases} -\partial_t u + \frac{1}{2}\frac{|Du|^2}{m^\alpha}\mathbb{1}_{\{m>0\}} \leq f(x,m) \\ u(T) \leq g(m(T)) \end{cases}$$

(ii) $m \in C^0([0,T];\mathcal{M}(\Omega))$ solves the continuity equation

$$\begin{cases} \partial_t m - \operatorname{div}\,(m^{1-\alpha} Du \mathbb{1}_{\{m>0\}}) = 0 \\ m(0) = m_0 \end{cases}$$

(iii) the energy equality holds (u is a sol. in the support of m)

$$\int_0^T\!\!\int_\Omega m\, f(x,m) + \frac{1}{2}\int_0^T\!\!\int_\Omega m^{1-\alpha}\,|Du|^2 = \int_\Omega m_0\, u(0) - \int_\Omega g(x,m(T))\, m(T)$$

- the traces of u are defined by time monotonicity, as in [Orrieri-P.-Savaré]

Quite surprisingly, we are able to extend the existence and uniqueness to the full range $0 < \alpha < 2$:

Theorem (Achdou-P.)

Let $0 < \alpha < 2$. Under the above conditions, there exists a unique weak solution (u, m) to the deterministic MFG system

$$\begin{cases} -\partial_t u + \dfrac{1}{2}\dfrac{|Du|^2}{m^\alpha} = f(x, m)\,, & (t, x) \in (0, T) \times \Omega \\ \partial_t m - \mathrm{div}(m^{1-\alpha} Du) = 0\,, & (t, x) \in (0, T) \times \Omega \\ m(0) = m_0\,,\; u(T) = g(x, m(T))\,, & x \in \Omega\,, \end{cases}$$

Moreover, (u, m) is the limit, as $\varepsilon \to 0$, of the unique solution $(u_\varepsilon, m_\varepsilon)$ with ε- viscosity.

The technical box: a recipe for vanishing viscosity MFG

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(ii) The weak limits u, m satisfy some relaxed form of the limiting system, namely

$$-\partial_t u + \bar{H} \leq \bar{f}\,; \qquad \partial_t m - \operatorname{div}(w) = 0$$

where $\bar{H}, \bar{f}, w$ are weak limits of, respectively,
$H(x, m_\varepsilon, Du_\varepsilon), f(x, m_\varepsilon)$ and $m_\varepsilon H_p(t, x, m_\varepsilon, Du_\varepsilon)$.

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(iii) Show that *any couple* (u, m) satisfying the conditions obtained verifies the duality estimate, namely one can "multiply" the (in)equation of u by m and the equation of m by u.

(iv) Identify the weak limits and verify that the duality is preserved

$$\int_\Omega m_0\, u(0) = \int_\Omega g(m(T))\, m(T) + \int_0^T\!\!\int_\Omega f(x, m)m + \int_0^T\!\!\int_\Omega m\, L(m, H_p(m, Du))$$

where $L(m, q) = \sup_a [a \cdot q - H(m, a)]$

$\to [H_p(m, Du) \cdot Du - H(m, Du)] = L(m, H_p(m, Du))$.

1. Estimates. Natural estimates of the system (independent of ε):

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- The transport field $w := m\, H_p(x, m, Du)$ satisfies

$$w \text{ bounded in } L^{1+\sigma}(Q_T), \text{ for some } \sigma > 0$$

$$\int_0^T\int_\Omega |w|^{1+\theta}\, m \leq C, \text{ for some } \theta > 0$$

$\rightsquigarrow$ w weakly compact in L^1, $m(t)$ equi-Hölder in the Wasserstein distance.

2. The relaxed limit system. Estimates $\Rightarrow$ weak compactness:

$$
\begin{aligned}
&u_\varepsilon \to u && \text{weakly in } L^1(0,T;W^{1,1})\\
&m_\varepsilon \to m && \text{weakly in } L^q(Q_T)\\
&w_\varepsilon := m_\varepsilon\, H_p(m_\varepsilon, Du_\varepsilon) \to w && \text{weakly in } L^1(Q_T)\\
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Pb: identify the weak limits. Ingredients: monotonicity, convexity.

Key point: the lower semi-continuous function

$$\Psi_\gamma(m,p) = \begin{cases} \frac{|p|^2}{m^\gamma} & \text{if} \quad m > 0, \\ 0 & \text{if} \quad m = 0 \text{ and } p = 0, \\ +\infty & \text{otherwise.} \end{cases}$$

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is convex on $\mathbb{R}^N \times \mathbb{R}$ whenever $\gamma \in (0,1]$.

But things go (very) differently according to $\alpha \leq 1$ and $1 < \alpha < 2$!!

Case $0 < \alpha \leq 1$. Here we have

$$H(m, Du) = \frac{1}{2}\frac{|Du|^2}{m^\alpha} \quad \text{is convex in } (m, Du)$$

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$\rightsquigarrow$ passing to the limit we find a relaxed HJ equation:

$$\begin{cases} -\partial_t u + H(x, m, Du) \leq \bar{f} & \text{in } (0, T) \times \Omega \\ u(T) \leq \bar{g} \end{cases}$$

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Notice: we also have that

$$m[H_p(m, Du) \cdot Du - H(m, Du)] = m\, L(m, H_p(m, Du)) = \frac{1}{2}\,|Du|^2\, m^{1-\alpha}$$

Since $w = m^{1-\alpha} Du$, we rephrase for (m, w) (the [Benamou-Brenier] trick)

$$m\, L(m, H_p(m, Du)) = m\, L\left(m, \frac{w}{m}\right) = \frac{1}{2}\frac{|w|^2}{m^{1-\alpha}} \quad \text{is convex in } (m, w)$$

3. Weak duality in the limit system. Let (u, m) satisfy

$$\begin{cases} u \in L^2(0,T; H^1_0(\Omega)) \\ -\partial_t u + H(m, Du) \le \bar f \\ u(T) \le \bar g \end{cases} \qquad \begin{cases} m \in L^q(Q_T) \cap C^0([0,T]; \mathcal{M}(\Omega)), \\ \partial_t m - \operatorname{div}(w) = 0 \\ m(0) = m_0 \end{cases}$$

for some $\bar f, \bar g \in L^{q'}$, some $w \in L^1$ such that $w = 0$ a.e. in $\{m = 0\}$ and $\frac{|w|^2}{m^{1-\alpha}} \mathbb{1}_{\{m>0\}} \in L^1(Q_T)$. Then

$$\int u(0) m_0 \le \int_0^T\!\!\int_\Omega [w \cdot Du - H(m, Du)\, m] + \int_0^T\!\!\int_\Omega \bar f\, m + \int_\Omega \bar g\, m(T)$$

Key-point: we smoothen the weak solutions by convolution

$H(m, Du)$ is convex in (m, Du) $\Rightarrow H(m, Du) \star \rho_\delta \ge H(m_\delta, Du_\delta)$

$\to$ the convoluted u_δ is still a sub solution $\to (m_\delta, u_\delta)$ replaces the initial couple satisfying the same conditions !

4. Identifying the weak limits & recovering the energy identity

At ε fixed the energy identity holds:

$$\int_0^T \int_\Omega f(x, m_\varepsilon) m_\varepsilon + \int_\Omega g(x, m_\varepsilon(T)) m_\varepsilon(T)$$

$$= \int_\Omega u_\varepsilon(0) m_0 - \int_0^T \int_\Omega m_\varepsilon \overbrace{H_p(m_\varepsilon, Du_\varepsilon) Du_\varepsilon - H(m_\varepsilon, Du_\varepsilon)}^{L(m_\varepsilon, H_p(m_\varepsilon, Du_\varepsilon))}$$

$$= \int_\Omega u_\varepsilon(0) m_0 \, dx - \int_0^T \int_\Omega m_\varepsilon \, L\left(m_\varepsilon, \frac{w_\varepsilon}{m_\varepsilon}\right) dxdt \, .$$

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$$= \int_\Omega u_\varepsilon(0)m_0\, dx - \int_0^T\int_\Omega m_\varepsilon\, L\left(m_\varepsilon, \frac{w_\varepsilon}{m_\varepsilon}\right) dxdt\,.$$

Another key-help from convexity: $mL(m, \frac{w}{m})$ is convex in (m, w) !

$$\rightsquigarrow \quad \int_0^T\int_\Omega m_\varepsilon\, L\left(m_\varepsilon, \frac{w_\varepsilon}{m_\varepsilon}\right) \quad \text{is weakly lower-semicontinuous}$$

Hence

$$\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega f(x, m_\varepsilon) m_\varepsilon + \int_\Omega g(x, m_\varepsilon(T)) m_\varepsilon(T)$$

$$\leq \int_\Omega u(0) m_0 \, dx - \int_0^T \int_\Omega m \, L\left(m, \frac{w}{m}\right)$$

$$\leq \quad \text{(by weak duality)} \int_0^T \int_\Omega \bar{f} \, m + \int_\Omega \bar{g} \, m(T)$$

$$+ \underbrace{\int_0^T \int_\Omega m \left[\frac{w}{m} \cdot Du - H(m, Du)\right] - \int_0^T \int_\Omega m \, L\left(m, \frac{w}{m}\right)}_{\leq 0} .$$

Hence

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We conclude that

$$
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$\underline{f, g \text{ monotone \& Minty's argument}} \Rightarrow \bar f = f(m), \bar g = g(m)$

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f,g monotone & Minty's argument $\Rightarrow$ $\bar f = f(m), \bar g = g(m)$

All inequalities now become equalities... $\rightsquigarrow$ $w = mH_p(m,Du)$

Summary: in the case $0 < \alpha \le 1$ we identified the weak limits and we obtained the existence of a weak solution by using:

- coercivity and monotonicity of the couplings f, g.
- convexity of Lagrangian and Hamiltonian, precisely:

$$(m, p) \mapsto H(m, p) \quad \text{and} \quad (m, w) \mapsto m\, L\left(m, \frac{w}{m}\right) \quad \text{are convex functions.}$$

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What about the case $1 < \alpha < 2$?

This is a very special range, which exceeds all results based on variational approach...in particular $mL\left(\frac{w}{m}\right)$ is no longer convex....

But...we can change the roles played before by Hamiltonian and Lagrangian !.

$\rightsquigarrow$ $H(m, Du)$ is no longer convex in (m, Du) but it is convex in (m, w) !

$$H(m, Du) = \frac{1}{2}\frac{|Du|^2}{m^\alpha} = \frac{1}{2}\frac{|w|^2}{m^{2-\alpha}} \qquad \text{is convex in } (w, m)$$

Case $1 < \alpha < 2$ As $\varepsilon \to 0$, we have $(u_\varepsilon, m_\varepsilon)$ weakly converging to (u, m) which solve the relaxed system

$$\begin{cases} -\partial_t u + \frac{|w|^2}{m^{2-\alpha}} \mathbb{1}_{\{m>0\}} \leq \bar{f} \\ u(T) \leq \bar{g} \end{cases} \qquad \begin{cases} \partial_t m - \operatorname{div}(w) = 0 \\ m(0) = m_0 \end{cases}$$

for some $\bar{f}, \bar{g} \in L^{q'}$, some $w \in L^1$ such that $w, Du = 0$ a.e. in $\{m = 0\}$, $\frac{|w|^2}{m^{2-\alpha}} \mathbb{1}_{\{m>0\}} \in L^1(Q_T)$ and $\frac{|Du|^2}{m^{\alpha-1}} \mathbb{1}_{\{m>0\}} \in L^1(Q_T)$.

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Weak duality: any (u, m, w) satisfying the above conditions verify

$$\begin{aligned} \int u(0) m_0 &\le \iint_\Omega [w \cdot Du - \frac{1}{2} \frac{|w|^2}{m^{1-\alpha}}] \mathbb{1}_{\{m>0\}} + \iint_\Omega \bar f\, m + \int_\Omega \bar g\, m(T) \\ &\le \iint_\Omega \frac{1}{2} \frac{|Du|^2}{m^{\alpha-1}} \mathbb{1}_{\{m>0\}} + \int_0^T \int_\Omega \bar f\, m + \int_\Omega \bar g\, m(T) \end{aligned}$$

Then we follow the above strategy: use the weak duality to identify the weak limits... $\rightsquigarrow$ $w = m H_p(m, Du) \mathbb{1}_{\{m>0\}}$, $\bar f = f(m)$, $\bar g = g(m)$.

Conclusion:

- In the whole range $0 < \alpha < 2$ - using differently the convexity ingredients according to the range of α - we prove that the vanishing viscosity is a weak solution (u, m)

Conclusion:

- In the whole range $0 < \alpha < 2$ - using differently the convexity ingredients according to the range of α - we prove that the vanishing viscosity is a weak solution (u, m)
- The weak solution is also proved to be unique.

  Crucial point: extend the weak duality argument to different couples (u, m), $(\hat{u}, \hat{m})$:

  cross multiply $-\partial_t u + \frac{1}{2}\frac{|Du|^2}{m^\alpha}\mathbb{1}_{\{m>0\}} \leq f(m)$ and $\partial_t \hat{m} - \operatorname{div}(\hat{w}) = 0$

  or

  cross multiply $-\partial_t u + \frac{|w|^2}{m^{2-\alpha}}\mathbb{1}_{\{m>0\}} \leq f(m)$ and $\partial_t \hat{m} - \operatorname{div}(\hat{w}) = 0$

Work in progress...

- **Boundary conditions**. Of course, the realistic model must include boundary conditions (possibly mixed...)

  By now, the result extends to Dirichlet condition: $u = 0$ on $\partial\Omega$

  Under some restriction (hopefully to be removed), the approach described before works. Highlights:

  (i) the Dirichlet condition is preserved for u from the estimates in Sobolev spaces: $u \in L^1(0, T; W_0^{1,1}(\Omega))$

  (ii) using that costs are nonnegative, we also have $u \geq 0$ and so $\frac{\partial u}{\partial \nu} \leq 0$ on the boundary. This allows us to *relax the HJ equation up to the boundary*

  (ii) smoothing the solutions is much more delicate: one needs suitable localized convolutions

  Indeed: the weak duality argument is a major technical issue for any kind of generalization (either going through convolution or renormalization)

- convergence of numerical schemes
- first order MFGs & degenerate quasilinear elliptic equations
  Ex: (model case)

$$\begin{cases} -\partial_t u + \frac{1}{2}\frac{|Du|^2}{m^\alpha} = 0\,, & (t,x) \in (0,T)\times\Omega \\ \partial_t m - \mathrm{div}(m^{1-\alpha} Du) = 0\,, & (t,x) \in (0,T)\times\Omega \end{cases}$$

$$m = \left(\frac{1}{2}\frac{|Du|^2}{u_t}\right)^{\frac{1}{\alpha}} \rightsquigarrow \mathrm{tr}\left(A(\mathcal{D}u)\mathcal{D}^2 u\right) = 0 \qquad \mathcal{D} = (\partial_t(\cdot), D(\cdot))$$

Notice: the operator is elliptic iff $\alpha < 2$!!
(see also [Lions '12, Munoz '22])

Further interesting directions: planning problems, non-coercive couplings, etc.....

Thanks for the attention !