Minimal Solutions of Master Equations for Extended Mean Field Games

Chenchen Mou

a joint work with Jianfeng Zhang(USC)

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Outline

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Extended MFG

- Given $\{\nu_\cdot\}$, solve the standard optimization problem

$$X_t^{0,\alpha} = \xi + \int_0^t b_0(X_s^{0,\alpha}, \nu_s, \alpha_s)ds + B_t;$$
$$\inf_\alpha \mathbb{E}\Big[g(X_T^{0,\alpha}, \nu_T) + \int_0^T f(X_s^{0,\alpha}, \nu_s, \alpha_s)ds\Big].$$

Assume $\exists!$ minimizer $\alpha_t^* = I(t, X_t^{0,\alpha^*}, \nu_t)$.

- Mean field equilibrium (fixed point) :

$$\mathcal{L}_{X_t^{\nu_\cdot}} = \nu_t, \quad \forall t \in [0,T], \quad \text{where}$$
$$X_t^{\nu_\cdot} = \xi + \int_0^t b(X_s^{\nu_\cdot}, \nu_s, I(s, X_s^{\nu_\cdot}, \nu_s))ds + B_t.$$

- Standard MFG: $b = b_0$.

A toy model

- The toy model:

$$b_0(x,\mu,\alpha) = -\alpha, \quad f(x,\mu,\alpha) = \frac{1}{2}\alpha^2, \quad b(x,\mu,\alpha) = b(x,\alpha).$$

- Given $\nu = \nu_T$, solve the HJB equation $u(\nu;t,x)$:

$$\partial_t u + \frac{1}{2}\partial_{xx}u - \frac{1}{2}|\partial_x u|^2 = 0, \quad u(\nu;T,x) = g(x,\nu).$$

- The target mapping: given $\mu = \mathcal{L}_\xi$,

$$\Phi(\mu,\nu) := \mathcal{L}_{X_T^{\mu,\nu}}, \quad \text{where}$$
$$X_t^{\mu,\nu} = \xi + \int_0^t b(X_s^{\mu,\nu}, \partial_x u(\nu;s,X_s^{\mu,\nu}))ds + B_t.$$

- Mean field equilibrium (fixed point): given μ,

$$\Phi(\mu,\nu^*(\mu)) = \nu^*(\mu).$$

The master equation

- If $\nu^*(\mu)$ is unique for any μ, we have the value function

$$V(0,x,\mu) := u(\nu^*(\mu);0,x).$$

Similarly we define the dynamic value function $V(t,x,\mu)$.

- The dynamic value function $V(t,x,\mu)$ induces the master equation:

$$\begin{aligned}&\partial_t V + \frac{1}{2}\partial_{xx}V - \frac{1}{2}|\partial_x V|^2\\&+\mathbb{E}\Big[\frac{1}{2}\partial_{\tilde{x}\mu}V(t,x,\mu,\xi) + \partial_\mu V(t,x,\mu,\xi)b(\xi,\partial_x V(t,\xi,\mu))\Big] = 0;\\&V(T,x,\mu) = g(x,\mu).\end{aligned}$$

- Our goal: global (in time) wellposedness of the master equation.

Existing monotonicity conditions

- Lasry-Lions monotonicity condition (Lasry-Lions, $\cdots\cdots$, M.-Zhang (2020), $\cdots\cdots$.)

$$\mathbb{E}\Big[\partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\eta\tilde{\eta}\Big] \geq 0, \ \forall \xi, \eta.$$

- Semi-displacement monotonicity condition (Gangbo-Meszaros-M.-Zhang (2022))

$$\mathbb{E}\Big[\partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\eta\tilde{\eta} + \partial_{xx} U(\xi, \mathcal{L}_\xi)\eta^2 + \lambda\eta^2\Big] \geq 0, \ \forall \xi, \eta.$$

- Anti-monotoncity condition (M.-Zhang (2022))

$$\mathbb{E}\Big[\partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\eta\tilde{\eta} + \partial_{xx} U(\xi, \mathcal{L}_\xi)\eta^2 + \lambda_1 \Big|\tilde{\mathbb{E}}[\partial_{x\mu} U(\xi, \mathcal{L}_\xi, \tilde{\xi})\tilde{\eta}]\Big|^2 + \lambda_2 \Big|\partial_{xx} U(\xi, \mathcal{L}_\xi)\eta\Big|^2 - \lambda_3\eta^2\Big] \leq 0.$$

Some existing works

- Extended MFG:
  ◇ Lions-Souganidis (2020): local extended MFG system under Lasry-Lions type of monotonicity.
  ◇ Munoz (2021) : classical solution for the first order local system.
  ◇ ......

Outline

The monotonicity

- Recall

$$\partial_t u + \frac{1}{2}\partial_{xx} u - \frac{1}{2}|\partial_x u|^2 = 0,\ \ u(\nu; T, x) = g(x, \nu);$$
$$X_t^{\mu,\nu} = \xi + \int_0^t b(X_s^{\mu,\nu}, \partial_x u(\nu; s, X_s^{\mu,\nu}))ds + B_t.$$

- Assumptions:
  - ◇ b, g are sufficiently smooth, with bounded x-derivatives;
  - ◇ $-C_0 \le b \le C_0$;
  - ◇ $\partial_x g \searrow$ in ν, $b \searrow$ in a; or $\partial_x g \nearrow$ in ν, $b \nearrow$ in a;
- Lemma. $\partial_x u \searrow$, $X \nearrow$, and $\Phi \nearrow$ in ν,
  - ◇ Dianetti-Ferrari-Fischer-Nendel (2021), $\cdots$: submodular MFG in the standard MFG framework.

The Picard iteration

- The Picard iteration

$$\underline{X}_t^0 := \xi - C_0 t + B_t,\ \underline{\nu}_n := \mathcal{L}_{\underline{X}_T^n},$$
$$\underline{X}_t^{n+1} = \xi + \int_0^t b(X_s^{n+1}, \partial_x u(\underline{\nu}_n; s, X_s^{n+1}))ds + B_t.$$

- Lemma. $\underline{X}^n \nearrow$ and $\underline{\nu}_n \nearrow$ in n.
- Similarly, by denoting $\overline{X}_t^0 := \xi + C_0 t + B_t$, one can show

$$\overline{X}^n \searrow \text{ and } \overline{\nu}_n \searrow \text{ in } n.$$

- Define

$$\underline{X}_t := \lim_{n\to+\infty} \underline{X}_t^n,\ \overline{X}_t := \lim_{n\to+\infty} \overline{X}_t^n,\ \underline{\nu} := \mathcal{L}_{\underline{X}_T},\ \overline{\nu} := \mathcal{L}_{\overline{X}_T}.$$

The minimum/maximum MFE

Theorem

(i) For each μ, the above $\underline{\nu}$ (resp. $\overline{\nu}$) is the minimum (resp. maximum) MFE. That is, for any MFE ν^*,

$$\underline{\nu} \leq \nu^* \leq \overline{\nu}.$$

Consequently, MFE is unique if and only if $\underline{\nu} = \overline{\nu}$.
(ii) $\underline{\nu}$ and $\overline{\nu}$ are time consistent:

$$\underline{\nu}(0, \mu) = \underline{\nu}(t, \mathcal{L}_{\underline{X}_t}), \ \overline{\nu}(0, \mu) = \overline{\nu}(t, \mathcal{L}_{\overline{X}_t}).$$

- We can then define the dynamic value functions:

$$\underline{V}(t, x, \mu) := u(\underline{\nu}(t, \mu); t, x), \ \overline{V}(t, x, \mu) := u(\overline{\nu}(t, \mu); t, x).$$

The regularity issue

- The function u is smooth in (t, x, ν). Consequently, $\underline{V}$ and $\overline{V}$ are smooth in x.
- However, unless MFE is unique, typically $\underline{\nu}(t, \mu)$ is discontinuous, and thus $\underline{V}(t, x, \mu) := u(\underline{\nu}(t, \mu); t, x)$ is discontinuous in (t, μ).
- Lemma. $\partial_x \underline{V}$ is decreasing in μ, and is upper semi-continuous in μ in the sense that :

$$\overline{\lim_{n \to +\infty}} \partial_x \underline{V}(t, x, \mu_n) \le \partial_x \underline{V}(t, x, \mu) \text{ whenever } \lim_{n \to +\infty} \mathcal{W}_1(\mu_n, \mu) = 0.$$

- Lemma. If $\partial_x V$ is unif. Lipschitz in x and upper semi-continuous in μ, then the following SDE has a (strong) solution :

$$X_t = \xi + \int_0^t b(X_s, \partial_x V(s, X_s, \mathcal{L}_{X_s}))ds + B_t. \tag{1}$$

Weak-viscosity solution of the master equation

Definition

Let $V : [0, T] \times \mathbb{R} \times \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ be smooth in x with bounded $\partial_x V, \partial_{xx} V$. We say V is a weak-viscosity subsolution of the master equation if, for any $\xi \in L^2(\mathcal{F}_0)$,
(i) the SDE (1) has a strong solution X;
(ii) denote $v(t,x) := \partial_x V(t, x, \mathcal{L}_{X_t})$. Then its upper semi-cont. envelope v^* is a viscosity subsolution of the PDE:

$$\partial_t v + \frac{1}{2}\partial_{xx} v - v\partial_x v = 0, \;\; v(T,x) = \partial_x g(x, \mathcal{L}_{X_T}). \tag{2}$$

- (i) means the Fokker-Planck equation has a weak solution.
- (ii) PDE (2) is obtained by differentiating the HJB equation in x.

Main results

- Define weak-viscosity supersol. (resp. solution) in obvious sense.

Theorem

(i) Both $\underline{V}$ and $\overline{V}$ are weak-viscosity sol. of the master equ.
(ii) $\underline{V}$ is the largest weak-viscosity subsolution in the sense: $\partial_x V \le \partial_x \underline{V}$ for any weak-viscosity subsolution V.
(iii) $\overline{V}$ is the smallest weak-viscosity supersolution in the sense: $\partial_x V \ge \partial_x \overline{V}$ for any weak-viscosity supersolution V.

- The results hold true for standard MFG: $b = b_0$.
- The results hold true for more general models, including multi-dim. case by considering viscosity solutions for PDE systems.

Thank you for your attention!