Dynamics of vortex cap solutions on the rotating unit sphere

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November 16th 2023, NYUAD

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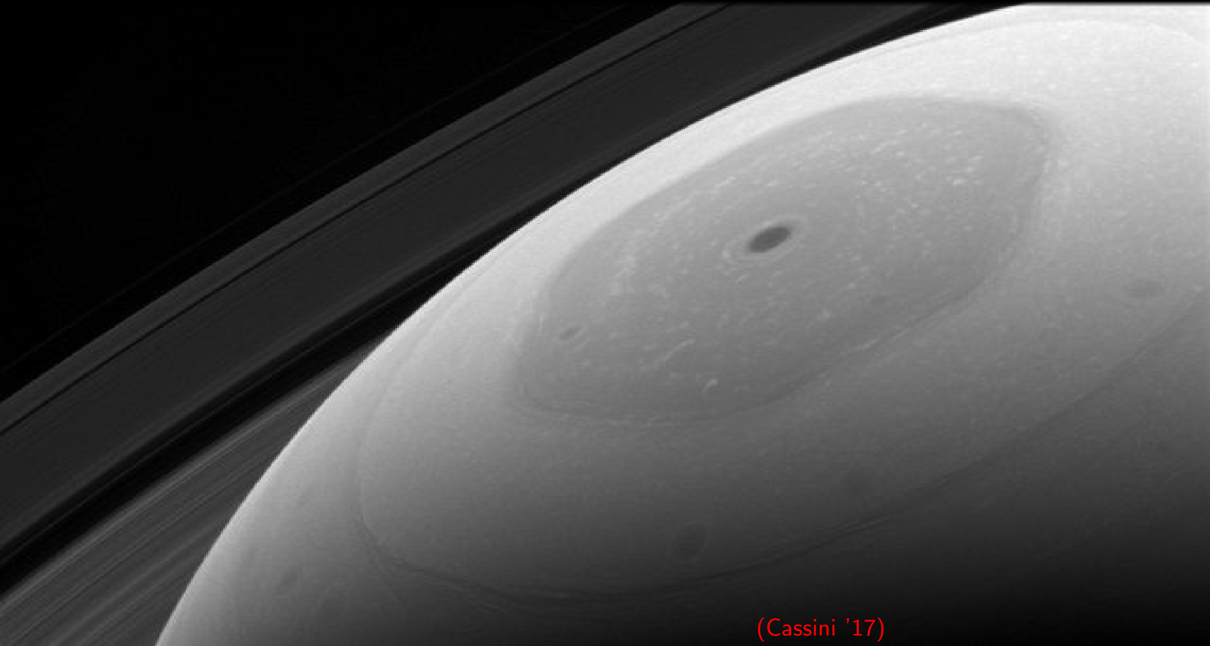


(Cassini '17)

1. The planar case : vortex patches

2. The case of the sphere
   - Introduction to vortex cap solutions
   - One interface
   - Two interfaces

2D Euler equations

Active scalar equation :

$$\begin{cases} \partial_t \omega + v \cdot \nabla \omega = 0 & \text{in } \mathbb{R}_+ \times \mathbb{R}^2 \\ v = \nabla^\perp \psi & \\ \omega(0, \cdot) = \omega_0 & \text{in } \mathbb{R}^2 \end{cases}$$

Vorticity (unknown) :

$$\omega = \nabla^\perp \cdot v, \qquad \nabla^\perp = \begin{pmatrix} -\partial_{x_2} \\ \partial_{x_1} \end{pmatrix}$$

Velocity potential / Green function :

$$\psi(t, z) = \int_{\mathbb{R}^2} G(z, \xi) \omega(t, \xi) d\xi, \qquad G(z, \xi) = \frac{1}{2\pi} \log |z - \xi|$$

Vortex patch equation

Votex patch :

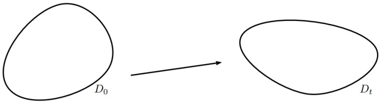
$$\omega(0,\cdot) = 1_{D_0} \in L^1(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \quad \text{Yudovich}$$

with D_0 bounded domain.

$$\partial_t \omega + v \cdot \nabla \omega = 0 \quad \Rightarrow \quad \omega(t,\cdot) = 1_{D_t}$$

where

$$D_t = \Phi_t(D_0), \qquad \Phi_t(z) = z + \int_0^t v\big(s, \Phi_s(z)\big)\, ds, \qquad |D_t| = |D_0|$$



Vortex patch equation (contour dynamics approach) : For $z(t,\cdot)$ a parametrization of ∂D_t

$$\Big\langle \partial_t z(t,x) - v\big(t, z(t,x)\big)\,,\; n\big(t, z(t,x)\big) \Big\rangle_{\mathbb{R}^2} = 0$$

or equivalently

$$\mathrm{Im}\Big\{ \partial_t z(t,x) \overline{\partial_x z(t,x)} \Big\} = \partial_x \Big(\psi\big(t, z(t,x)\big) \Big)$$

Stationary / radial solutions

Stationary solutions

$$\nabla^{\perp}\Psi(z)\cdot\nabla\Delta\Psi(z)=0$$

Radial :

$$\Psi(z)=\widetilde{\Psi}(|z|)\qquad \text{i.e.}\qquad \forall\eta\in\mathbb{R},\quad \Psi\left(e^{i\eta}z\right)=\Psi(z)$$

Lemma

$\Omega(z)=\widetilde{\Omega}(|z|)\Rightarrow\Psi(z)=\widetilde{\Psi}(|z|)$

1. Discs [Rankine, 1858]

$$\mathbb{D}=\{z\in\mathbb{C}\quad \text{s.t.}\quad |z|<1\}$$

2. Annuli

$$A_b=\{z\in\mathbb{C}\quad \text{s.t.}\quad b<|z|<1\}$$

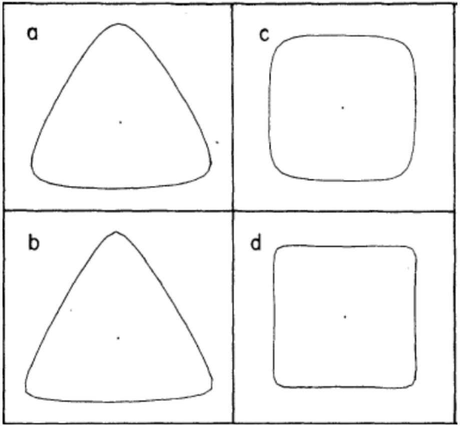
"V-states"

V-states :

$$\omega(t,\cdot) = 1_{D_t}, \qquad D_t = e^{i\Omega t} D_0$$

[Kirchhoff, 1876] : Ellipses with semi-axis a and b rotate uniformly iff $\Omega = \frac{ab}{(a+b)^2}$

Numerical simulations : [Deem, Zabusky '78] m-fold "V-states" ($m \in \{3, 4\}$)



V-states close to disc and annuli

Theorem [Burbea '82]

For all $m \in \mathbb{N}^*$, there exist m-fold V-states bifurcating from the unit disc at angular velocity

$$\Omega_m = \frac{m-1}{2m}$$

The case $m = 1$ is a translation of the trivial solution.

Theorem [de la Hoz, Hmidi, Mateu, Verdera '14]

Let $b \in (0,1)$ For any $m \in \mathbb{N}^*$ st

$$1 + b^m - \frac{1-b^2}{2} m < 0$$

there exist two branches of m-fold doubly-connected V-states bifurcating from the annulus A_b at the angular velocities

$$\Omega_m^{\pm}(b) = \frac{1-b^2}{4} \pm \frac{1}{2m} \sqrt{\left(\frac{m(1-b^2)}{2} - 1 \right)^2 - b^{2m}}$$

Some additional literature

For Euler

1. Regularity : [Hmidi, Mateu, Verdera '12] [Castro, Córdoba, Gómez-Serrano '15,'18]
2. Triviality constraints : [Fraenkel '00], [Hmidi '14], [Gómez-Serrano, Park, Shi, Yao '20]
3. Global bifurcation : [Hassainia, Masmoudi, Wheeler '17]
4. Degenerate bifurcation : [Hmidi-Mateu '16] [Wang, Xu, Zhou '22]
5. Second bifurcation from the ellipse branch : [Hmidi, Mateu '15]

For other models

1. Euler disc [de la Hoz-Hassainia-Hmidi-Mateu '16]
2. $(SQG)_\alpha$, $\alpha \in (0,2)$ [Hassainia, Hmidi '14] [de la Hoz, Hassainia, Hmidi '15] [Castro, Córdoba, Gómez-Serrano '16] [Renault '17] [Hmidi, Xue, Xue '22]
3. $(QGSW)_\lambda$, $\lambda > 0$ [Dritschel, Hmidi, Renault '18] [R. '22]
4. Euler-α, $\alpha > 0$ [R. '22]
5. 3D quasi-geostrophic model [García, Hmidi, Mateu '20, '21]
6. Lake [Hmidi, Houamed, R., Zerguine '23]

Differential calculus/geometry on the sphere

$$\mathbb{S}^2 = \{(x,y,z) \in \mathbb{R}^3 \quad \text{s.t.} \quad x^2 + y^2 + z^2 = 1\}$$

Local colatitude-longitude chart

$$\begin{array}{rcl} C_1 : (0,\pi) \times (0,2\pi) & \to & \mathbb{R}^3 \\ (\theta,\varphi) & \mapsto & \big(\sin(\theta)\cos(\varphi)\,,\ \sin(\theta)\sin(\varphi)\,,\ \cos(\theta)\big) \end{array}$$

Volume

$$d\sigma(\xi) = \sin(\theta) d\theta d\varphi$$

Orthonormal basis of the tangent space

$$\mathrm{e}_\theta = \partial_\theta, \qquad \mathrm{e}_\varphi = \frac{1}{\sin(\theta)}\partial_\varphi$$

For $\mathtt{f} : \mathbb{S}^2 \to \mathbb{R}$, we denote $f(\theta,\varphi) = \mathtt{f}\big(C_1(\theta,\varphi)\big)$. Gradient / Laplace-Beltrami operators

$$\nabla f(\theta,\varphi) = \partial_\theta f(\theta,\varphi)\mathrm{e}_\theta + \frac{1}{\sin(\theta)}\partial_\varphi f(\theta,\varphi)\mathrm{e}_\varphi$$

$$\nabla^\perp f(\theta,\varphi) = J\nabla f(\theta,\varphi), \qquad \underset{(\mathrm{e}_\theta,\mathrm{e}_\varphi)}{\mathrm{Mat}}(J) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Delta f(\theta,\varphi) = \frac{1}{\sin(\theta)}\partial_\theta\big[\sin(\theta)\partial_\theta f(\theta,\varphi)\big] + \frac{1}{\sin^2(\theta)}\partial_\varphi^2 f(\theta,\varphi)$$

Euler equations on the unit sphere in rotation (barotropic model)

The model ($\widetilde{\gamma} \in \mathbb{R}$)

$$\begin{cases} \partial_t \overline{\Omega}(t,\theta,\varphi) + U(t,\theta,\varphi) \cdot \nabla \overline{\Omega}(t,\theta,\varphi) = 0, \\ \overline{\Omega}(t,\theta,\varphi) = \Omega(t,\theta,\varphi) - 2\widetilde{\gamma}\cos(\theta) \\ U(t,\theta,\varphi) = \nabla^{\perp}\Psi(t,\theta,\varphi), \\ \Delta\Psi(t,\theta,\varphi) = \Omega(t,\theta,\varphi). \end{cases}$$

Integral representation of the stream function

$$\Psi(t,\xi) = \int_{\mathbb{S}^2} G(\xi,\xi')\Omega(t,\xi')d\sigma(\xi'), \qquad G(\xi,\xi') = \frac{1}{2\pi}\log\left(\frac{|\xi-\xi'|_{\mathbb{R}^3}}{2}\right)$$

Green kernel in colatitude/longitude variables

$$G(\theta,\theta',\varphi,\varphi') = \tfrac{1}{4\pi}\log\left(D(\theta,\theta',\varphi,\varphi')\right)$$

$$D(\theta,\theta',\varphi,\varphi') = \sin^2\left(\tfrac{\theta-\theta'}{2}\right) + \sin(\theta)\sin(\theta')\sin^2\left(\tfrac{\varphi-\varphi'}{2}\right)$$

Gauss constraint

$$\int_{\mathbb{S}^2} \overline{\Omega}(t,\xi)d\sigma(\xi) = \int_{\mathbb{S}^2} \Omega(t,\xi)d\sigma(\xi) = 0$$

Stationary / Zonal solutions

Stationary solutions

$$\nabla^{\perp}\Psi(\theta,\varphi)\cdot\nabla\Big(\Delta\Psi(\theta,\varphi)-2\widetilde{\gamma}\cos(\theta)\Big)=0$$

Zonal : longitude independent solutions

$$\Psi(\theta,\varphi)=\Psi(\theta)\qquad \text{i.e.}\qquad \forall\eta\in\mathbb{R},\quad \Psi\big(\mathcal{R}(\eta)\xi\big)=\Psi(\xi),\qquad \mathcal{R}(\eta)=\begin{pmatrix}\cos(\eta) & -\sin(\eta) & 0\\ \sin(\eta) & \cos(\eta) & 0\\ 0 & 0 & 1\end{pmatrix}$$

Lemma

$\Omega(\theta,\varphi)=\Omega(\theta)\Rightarrow\Psi(\theta,\varphi)=\Psi(\theta)$

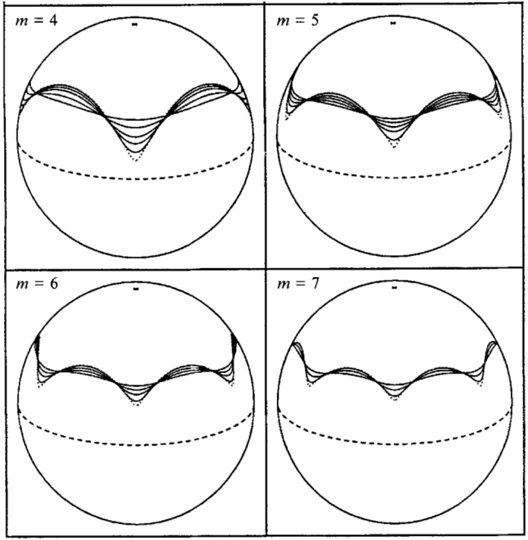
Other way to generate stationary solutions

$$\Delta\Psi(\theta,\varphi)-2\widetilde{\gamma}\cos(\theta)=F\big(\Psi(\theta,\varphi)\big),\qquad F\in C^1(\mathbb{R},\mathbb{R})$$

1. Stability of Rossby-Haurwitz solutions : [Constantin, Germain '22] [Cao, Wang, Zuo '23]
2. Non-zonal stationary solutions close to degree 2 Rossby-Haurwitz zonal solutions : [Nualart '22]

Numerical simulations of vortex cap solutions

[Dritschel, Polvani '92-'93] [Kim, Sakajo, Sohn '18] [Kim, Sohn '21]



Vortex cap solutions (1/2)

Fix $M \in \mathbb{N} \setminus \{0,1\}$ and $(\omega_k)_{1\leqslant k\leqslant M} \in \mathbb{R}^M$ such that

$$\forall k \in [\![1, M-1]\!], \quad \omega_k \neq \omega_{k+1}$$

Consider a partition of the unit sphere in the form

$$\mathbb{S}^2 = \bigsqcup_{k=1}^{M} \mathscr{C}_k(0)$$

where for any $k \in [\![1, M-1]\!]$, the boundary $\Gamma_k(0) = \partial\mathscr{C}_k(0) \cap \partial\mathscr{C}_{k+1}(0)$ is diffeomorphic to a circle. Take an initial condition in the form

$$\overline{\Omega}(0,\cdot) = \sum_{k=1}^{M} \omega_k \mathbf{1}_{\mathscr{C}_k(0)}.$$

The Gauss constraint requires the following additional condition

$$\sum_{k=1}^{M} \omega_k \sigma\big(\mathscr{C}_k(0)\big) = 0$$

Vortex cap solutions (2/2)

The transport equation $\partial_t\overline{\Omega} + U\cdot\nabla\overline{\Omega} = 0$ implies

$$\forall t\geqslant 0,\quad \forall \xi\in\mathbb{S}^2,\quad \overline{\Omega}(t,\xi) = \overline{\Omega}\big(0,\Phi_t^{-1}(\xi)\big)$$

where

$$\forall \xi\in\mathbb{S}^2,\quad \partial_t\Phi_t(\xi) = U\big(t,\Phi_t(\xi)\big),\qquad \Phi_0(\xi)=\xi$$

Hence,

$$\overline{\Omega}(t,\cdot) = \sum_{k=1}^{M}\omega_k \mathbf{1}_{\mathscr{C}_k(t)},\qquad \text{with}\qquad \forall k\in[\![1,M]\!],\quad \mathscr{C}_k(t) = \Phi_t\big(\mathscr{C}_k(0)\big)$$

Since U is solenoidal, then the flow $t\mapsto\Phi_t$ is measure preserving

$$\forall k\in[\![1,M]\!],\quad \sigma\big(\mathscr{C}_k(t)\big) = \sigma\big(\mathscr{C}_k(0)\big)$$

Contour dynamics equation for vortex cap solutions : for any $k\in[\![1,M-1]\!]$ and $(t,x)\in\mathbb{R}_+\times\mathbb{T}$

$$\Big\langle \partial_t z_k(t,x)\,,\, J\partial_x z_k(t,x)\Big\rangle_{\mathbb{R}^3} = \partial_x\Big(\Psi\big(t,z_k(t,x)\big)\Big)$$

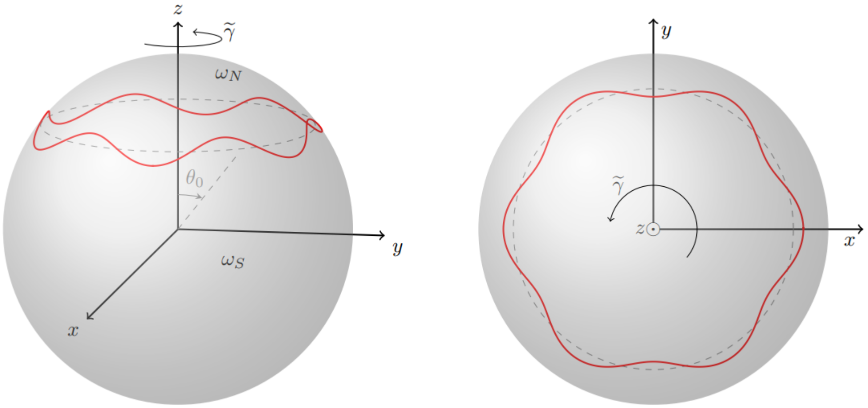


Figure – Representation of one interface (in red) vortex cap solutions with 6-fold symmetry

Theorem (García, Hassainia, R. 23')

Let $\widetilde{\gamma} \in \mathbb{R}$, $m \in \mathbb{N}^*$ and $\theta_0 \in (0, \pi)$. Consider $\omega_N, \omega_S \in \mathbb{R}$ such that

$$\frac{\omega_N + \omega_S}{\omega_N - \omega_S} = \cos(\theta_0).$$

There exists a branch of m-fold uniformly rotating vortex cap solutions with one interface bifurcating from

$$\overline{\Omega}_{\mathsf{FC}}(\theta) = \omega_N 1_{0<\theta<\theta_0} + \omega_S 1_{\theta_0 \leqslant \theta < \pi}$$

at the velocity

$$c_m(\widetilde{\gamma}) = \widetilde{\gamma} - (\omega_N - \omega_S)\frac{m-1}{2m}$$

Proof (1/4) : Ansatz

$$\overline{\Omega}(t,\theta,\varphi) = \omega_N\big(f(t,\varphi)\big)\mathbf{1}_{0<\theta<\theta_0+f(t,\varphi)} + \omega_S\mathbf{1}_{\theta_0+f(t,\varphi)\leqslant\theta<\pi}, \qquad |f(t,\varphi)| \ll 1.$$

$\omega_N(f)$ chosen to satisfy the Gauss constraint

$$0 = \int_0^{2\pi} \omega_N\big(f(t,\varphi)\big)\big(1-\cos(\theta_0+f(t,\varphi)\big) + \omega_S\big(1+\cos(\theta_0+f(t,\varphi)\big)d\varphi$$

Properties

$$\omega_N(0) = \omega_N, \qquad d_f\omega_N(0) \equiv 0$$

Parametrization of the boundary

$$z(t,\varphi) = C_1\big(\theta_0+f(t,\varphi),\varphi\big)$$

Vortex cap equation

$$\partial_t f(t,\varphi) = \frac{\partial_\varphi\Big(\Psi\big(t,z(t,\varphi)\big)\Big)}{\sin\big(\theta_0+f(t,\varphi)\big)}$$

Proof (2/4) : Reformulation

Looking for traveling solutions with speed $c \in \mathbb{R}$

$$f(t,\varphi) = f(\varphi - ct)$$

Insert

$$\mathscr{F}(c,f)(\varphi) = c\partial_\varphi f(\varphi) + \frac{\partial_\varphi\Big(\Psi\{f\}(\theta_0 + f(\varphi),\varphi)\Big)}{\sin\left(\theta_0 + f(\varphi)\right)} = 0$$

with

$$\Psi\{f\}(\theta,\varphi) = \Psi_{\mathbf{FC}}(\theta) + \tfrac{\omega_N - \omega_S}{2\pi}\int_0^{2\pi}\int_{\theta_0}^{\theta_0 + f(\varphi')} \log\left(D(\theta,\theta',\varphi,\varphi')\right)\sin(\theta')d\theta' d\varphi'$$
$$+ \frac{1}{4\pi}\int_0^{2\pi}\Big(\omega_N\big(f(t,\varphi')\big) - \omega_N\Big)\int_0^{\theta_0 + f(t,\varphi')}\log\Big(D\big(\theta,\theta',\varphi,\varphi'\big)\Big)\sin(\theta')d\theta' d\varphi'$$

Proof (3/4) : Function spaces

Hölder spaces $\alpha \in (0,1)$

$$\|f\|_{C^\alpha(\mathbb{T})} = \|f\|_{L^\infty(\mathbb{T})} + \sup_{\substack{(\varphi,\varphi')\in\mathbb{T}^2\\ \varphi\neq\varphi'}} \frac{|f(\varphi)-f(\varphi')|}{|\varphi-\varphi'|^\alpha}, \qquad \|f\|_{C^{1+\alpha}(\mathbb{T})} = \|f\|_{L^\infty(\mathbb{T})} + \|f'\|_{C^\alpha(\mathbb{T})}$$

Parity and symmetries : for $\mathbf{m} \in \mathbb{N}^*$

$$X^{1+\alpha}_{\mathbf{m}} = \left\{ f \in C^{1+\alpha}(\mathbb{T}) \quad \text{s.t.} \quad \forall \varphi \in \mathbb{T}, \quad f(\varphi) = \sum_{n=1}^{\infty} a_n \cos(\mathbf{m} n\varphi), \quad a_n \in \mathbb{R} \right\},$$

$$Y^{\alpha}_{\mathbf{m}} = \left\{ g \in C^{\alpha}(\mathbb{T}) \quad \text{s.t.} \quad \forall \varphi \in \mathbb{T}, \quad g(\varphi) = \sum_{n=1}^{\infty} b_n \sin(\mathbf{m} n\varphi), \quad b_n \in \mathbb{R} \right\},$$

Proposition (Regularity)

For any $\alpha \in (0,1)$, there exists $r > 0$ such that

$$\mathscr{F} : \mathbb{R} \times B_{X^{1+\alpha}_{\mathbf{m}}}(0,r) \to Y^{\alpha}_{\mathbf{m}} \text{ is of class } C^1$$

Proof (4/4) : Bifurcation study

Linearized operator Fredholm from $X_{\mathrm{m}}^{1+\alpha}$ into Y_{m}^{α} $(\alpha \in (0,1))$

$$d_f\mathscr{F}(c,0)[h](\varphi) = \left(c + \tfrac{\omega_N - \omega_S}{2} - \widetilde{\gamma}\right)\partial_\varphi + \tfrac{\omega_N - \omega_S}{2}\mathcal{H}$$

In Fourier

$$d_f\mathscr{F}(c,0)\left[\sum_{n=1}^{\infty} h_n\cos(\mathrm{m}n\varphi)\right](\varphi) = \sum_{n=1}^{\infty}\mathrm{m}n\left(c_{\mathrm{m}n}(\widetilde{\gamma}) - c\right)\sin(\mathrm{m}n\varphi)$$

The strict monotonicity of $n \mapsto c_{\mathrm{m}n}(\widetilde{\gamma})$ implies

$$\ker\Big(d_f\mathscr{F}\left(c_{\mathrm{m}}(\widetilde{\gamma}),0\right)\Big) = \mathrm{span}\left(\varphi \mapsto \cos(\mathrm{m}\varphi)\right)$$

The Fredholmness property implies

$$\mathrm{Im}\Big(d_f\mathscr{F}\left(c_{\mathrm{m}}(\widetilde{\gamma}),0\right)\Big) = \mathrm{span}^{\perp(\cdot|\cdot)}\left(\varphi \mapsto \sin(\mathrm{m}\varphi)\right)$$

where

$$\left(\sum_{n=1}^{\infty} a_n\sin(\mathrm{m}n\varphi)\,\middle|\,\sum_{n=1}^{\infty} b_n\sin(\mathrm{m}n\varphi)\right) = \sum_{n=1}^{\infty} a_n b_n$$

Transversality

$$\partial_c d_f\mathscr{F}\left(c_{\mathrm{m}}(\widetilde{\gamma}),0\right)[\varphi \mapsto \cos(\mathrm{m}\varphi)] = -[\varphi \mapsto \mathrm{m}\sin(\mathrm{m}\varphi)]$$

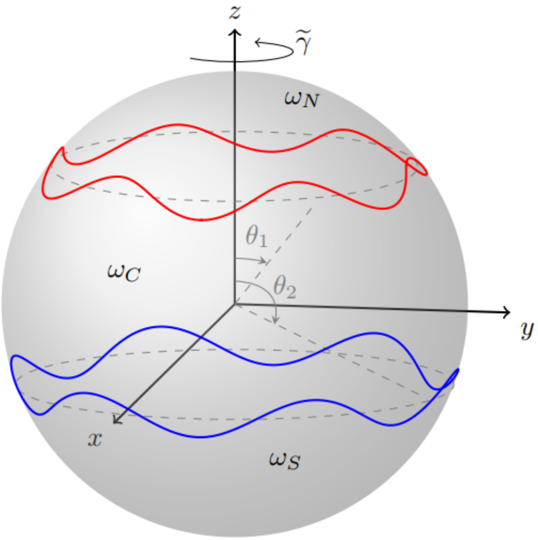


Figure – Representation of two interfaces (in red and blue) vortex cap solutions with 6-fold symmetry

Theorem (García, Hassainia, R. 23')

Let $\widetilde{\gamma}\in\mathbb{R}$ and $0<\theta_1<\theta_2<\pi$. Consider $\omega_N,\omega_C,\omega_S\in\mathbb{R}$ such that

$$\omega_N+\omega_S=(\omega_N-\omega_C)\cos(\theta_1)+(\omega_C-\omega_S)\cos(\theta_2).$$

Assume the non-degeneracy condition

$$\omega_S\cos^2\left(\tfrac{\theta_1}{2}\right)+\omega_N\sin^2\left(\tfrac{\theta_2}{2}\right)\neq 0.$$

Then, there exists $N(\theta_1,\theta_2)=N(\theta_1,\theta_2,\omega_N,\omega_S,\omega_C)\in\mathbb{N}^*$ such that for any $\mathrm{m}\in\mathbb{N}^*$ with $\mathrm{m}\geqslant N(\theta_1,\theta_2)$, there exists two branches of m-fold uniformly rotating vortex cap solutions bifurcating from

$$\overline{\Omega}_{\mathbf{FC2}}(\theta)=\omega_N\mathbf{1}_{0<\theta<\theta_1}+\omega_C\mathbf{1}_{\theta_1\leqslant\theta<\theta_2}+\omega_S\mathbf{1}_{\theta_2\leqslant\theta<\pi}$$

at the velocity

$$c_{\mathrm{m}}^{\pm}(\widetilde{\gamma},\theta_1,\theta_2)=\widetilde{\gamma}+\frac{\omega_S}{4\sin^2\left(\frac{\theta_2}{2}\right)}-\frac{\omega_N}{4\cos^2\left(\frac{\theta_1}{2}\right)}+\frac{\omega_N-\omega_S}{4\mathrm{m}}$$
$$\pm\frac{1}{4}\sqrt{\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{\mathrm{m}}\right)^2+\frac{1}{\mathrm{m}^2}(\omega_N-\omega_C)(\omega_C-\omega_S)\alpha_{\theta_1,\theta_2}^{2\mathrm{m}}}$$

where $\alpha_{\theta_1,\theta_2}=\tan\left(\frac{\theta_1}{2}\right)\cot\left(\frac{\theta_2}{2}\right)\in(0,1)$.

Alternative non-degeneracy conditions

$$\omega_S \cos^2\left(\frac{\theta_1}{2}\right) + \omega_N \sin^2\left(\frac{\theta_2}{2}\right) = 0$$

supplemented with one of the following properties

(H1+) $\omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0,$

(H2+) $\omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0$ and $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right),$

(H3+) $\omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0,$

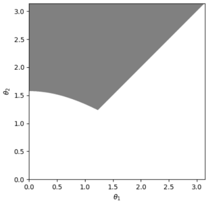
(H4+) $\omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0$ and $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$

(H1−) $\omega_C > 0, \quad \omega_N < 0, \quad \omega_S > 0,$

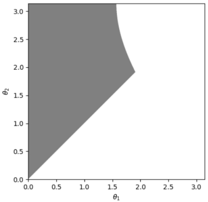
(H2−) $\omega_C > 0, \quad \omega_N > 0, \quad \omega_S < 0$ and $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right),$

(H3−) $\omega_C < 0, \quad \omega_N < 0, \quad \omega_S > 0,$

(H4−) $\omega_C < 0, \quad \omega_N > 0, \quad \omega_S < 0$ and $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right).$



(a) $2\sin^2\left(\frac{\theta_2}{2}\right) > \cos^2\left(\frac{\theta_1}{2}\right).$



(b) $2\cos^2\left(\frac{\theta_1}{2}\right) > \sin^2\left(\frac{\theta_2}{2}\right).$

Proof (1/6) : Ansatz

$$\overline{\Omega}(t,\theta,\varphi)=\omega_N\big(f_1(t,\varphi),f_2(t,\varphi)\big)\mathbf{1}_{0<\theta<\theta_1+f_1(t,\varphi)}+\omega_C\mathbf{1}_{\theta_1+f_1(t,\varphi)\leqslant\theta<\theta_2+f_2(t,\varphi)}+\omega_S\mathbf{1}_{\theta_2+f_2(t,\varphi)\leqslant\theta<\pi}$$

with

$$|f_1(t,\varphi)|,|f_2(t,\varphi)|\ll 1$$

and $\omega_N(f_1,f_2)$ chosen to satisfy the Gauss constraint. In particular

$$\omega_N(0,0)=\omega_N,\qquad d_{(f_1,f_2)}\omega_N(0,0)=0$$

Parametrizations

$$\forall k\in\{1,2\},\quad z_k(t,\varphi)=C_1\big(\theta_k+f_k(t,\varphi),\varphi\big)$$

Vortex cap system

$$\partial_t f_k(t,\varphi)=\frac{\partial_\varphi\Big(\Psi\big(t,z_k(t,\varphi)\big)\Big)}{\sin\big(\theta_k+f_k(t,\varphi)\big)}$$

Proof (2/6) : Reformulation

Looking for traveling solutions with speed $c \in \mathbb{R}$

$$f_k(t,\varphi) = f_k(\varphi - ct)$$

Insert

$$\mathscr{G}(c,f_1,f_2)=0, \qquad \mathscr{G}=(\mathscr{G}_1,\mathscr{G}_2), \qquad \mathscr{G}_k(c,f_1,f_2)(\varphi) = c\partial_\varphi f_k(\varphi) + \frac{\partial_\varphi\Big(\Psi\{f_1,f_2\}(\theta_k+f_k(\varphi),\varphi)\Big)}{\sin\left(\theta_k+f_k(\varphi)\right)}$$

where

$$\begin{aligned}\Psi\{f_1,f_2\}(\theta,\varphi) &= \Psi_{\mathbf{FC2}}(\theta) + \tfrac{\omega_N-\omega_C}{2\pi}\int_0^{2\pi}\int_{\theta_1}^{\theta_1+f_1(\varphi')}\log\left(D(\theta,\theta',\varphi,\varphi')\right)\sin(\theta')d\theta' d\varphi' \\ &+ \tfrac{\omega_C-\omega_S}{2\pi}\int_0^{2\pi}\int_{\theta_2}^{\theta_2+f_2(\varphi')}\log\left(D(\theta,\theta',\varphi,\varphi')\right)\sin(\theta')d\theta' d\varphi' \\ &+ \frac{1}{4\pi}\int_0^{2\pi}\Big(\omega_N\big(f_1(t,\varphi'),f_2(t,\varphi')\big)-\omega_N\Big)\int_0^{\theta_1+f_1(t,\varphi')}\log\Big(D(\theta,\theta',\varphi,\varphi')\Big)\sin(\theta')d\theta' d\varphi'\end{aligned}$$

Proof (3/6) : Structure of the linearized operator

Linearized operator Fredholm

$$d_{(f_1,f_2)}\mathscr{G}(c,0,0)=\begin{pmatrix}\left(c+\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}-\widetilde{\gamma}\right)\partial_\varphi+\frac{\omega_N-\omega_C}{2}\mathcal{H} & \frac{\omega_C-\omega_S}{2n}\frac{\sin(\theta_2)}{\sin(\theta_1)}\partial_\varphi\mathcal{Q}*\cdot\\ \frac{\omega_N-\omega_C}{2n}\frac{\sin(\theta_1)}{\sin(\theta_2)}\partial_\varphi\mathcal{Q}*\cdot & \left(c-\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}-\widetilde{\gamma}\right)\partial_\varphi+\frac{\omega_C-\omega_S}{2}\mathcal{H}\end{pmatrix}$$

where

$$\mathcal{Q}(\varphi)=\log\left(1-\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)\cos(\varphi)\right)$$

In Fourier

$$d_{(f_1,f_2)}\mathscr{G}(c,0,0)\left[\sum_{n=1}^{\infty}h_n^{(1)}\cos(\mathrm{m}n\varphi),\sum_{n=1}^{\infty}h_n^{(2)}\cos(\mathrm{m}n\varphi)\right]=\sum_{n=1}^{\infty}\mathrm{m}nM_{\mathrm{m}n}(c,\theta_1,\theta_2)\begin{pmatrix}h_n^{(1)}\\ h_n^{(2)}\end{pmatrix}\sin(\mathrm{m}n\varphi)$$

where

$$M_n(c,\theta_1,\theta_2)=\begin{pmatrix}-c+\frac{\omega_N-\omega_C}{2n}-\frac{\omega_N}{2\cos^2\left(\frac{\theta_1}{2}\right)}+\widetilde{\gamma} & \frac{\omega_C-\omega_S}{2n}\frac{\sin(\theta_2)}{\sin(\theta_1)}\alpha_{\theta_1,\theta_2}^{n}\\ \frac{\omega_N-\omega_C}{2n}\frac{\sin(\theta_1)}{\sin(\theta_2)}\alpha_{\theta_1,\theta_2}^{n} & -c+\frac{\omega_C-\omega_S}{2n}+\frac{\omega_S}{2\sin^2\left(\frac{\theta_2}{2}\right)}+\widetilde{\gamma}\end{pmatrix}$$

Proof (4/6) : Spectral study

$\det\left(M_n(c,\theta_1,\theta_2)\right)\in\mathbb{R}_2[c]$ with discriminant

$$\Delta_n(\theta_1,\theta_2)=\frac{1}{4}\left[\left(\frac{\omega_S}{\sin^2\left(\frac{\theta_2}{2}\right)}+\frac{\omega_N}{\cos^2\left(\frac{\theta_1}{2}\right)}-\frac{\omega_N+\omega_S-2\omega_C}{n}\right)^2+\frac{(\omega_N-\omega_C)(\omega_C-\omega_S)}{n^2}\alpha_{\theta_1,\theta_2}^{2n}\right]$$

1. Non-degeneracy condition 1
   1. 2 distinct real roots asymptotically
   2. Two different limits
   3. Asymptotic monotonicity
2. Non-degeneracy condition 2
   1. $\omega_N+\omega_S=\omega_C$ and $\omega_N\omega_S<0$ so $\Delta_n(\theta_1,\theta_2)>0$ for any $n\in\mathbb{N}^*$
   2. Accumulation at the same point of the roots
   3. Asymptotic SAME monotonicity

The last two points require a refined analysis of spectral collisions.

Proof (5/6) : Spectral collisions

Assume $\omega_C > 0$. We want to (not) solve

$$c_{\mathrm{m}}^{+}(\widetilde{\gamma},\theta_1,\theta_2) = c_{k\mathrm{m}}^{-}(\widetilde{\gamma},\theta_1,\theta_2), \qquad k \in \mathbb{N}^*$$

This is equivalent to

$$-k(\omega_N + \mathtt{r}_{\mathrm{m}}) = \omega_S + \mathtt{r}_{k\mathrm{m}}, \qquad 0 < \mathtt{r}_n = O(1/n^p)$$

1. If $\omega_N > 0$, then
$$0 = \omega_C + (k-1)\omega_N + k\mathtt{r}_{\mathrm{m}} + \mathtt{r}_{k\mathrm{m}}$$
2. If $\omega_N < 0$, then $\omega_S > 0$ and if
$$2|\omega_N| > \omega_S \qquad \text{i.e.} \qquad 2\cos^2\left(\tfrac{\theta_1}{2}\right) > \sin^2\left(\tfrac{\theta_2}{2}\right)$$
then, for m large enough
$$\forall k \geqslant 2, \quad k(|\omega_N| - \mathtt{r}_{\mathrm{m}}) \geqslant 2(|\omega_N| - \mathtt{r}_{\mathrm{m}}) > \omega_S + \mathtt{r}_{2\mathrm{m}} \geqslant \omega_S + \mathtt{r}_{k\mathrm{m}}$$

Proof (6/6) : Kernel / Range / Transversality

Kernel

$$\ker\left(d_{(f_1,f_2)}\mathscr{G}\left(c_{\mathrm{m}}^{\pm}(\widetilde{\gamma},\theta_1,\theta_2),0,0\right)\right)=\mathtt{span}(x_0)$$

where

$$x_0=\varphi\mapsto u_0\cos(\mathrm{m}\varphi),\qquad u_0\in\ker\left(M_{\mathrm{m}}\left(c_{\mathrm{m}}^{\pm}(\theta_1,\theta_2),\theta_1,\theta_2\right)\right)$$

Scalar product on $Y_{\mathrm{m}}^{\alpha}\times Y_{m}^{\alpha}$

$$\left(\left(\sum_{n=1}^{\infty}a_n\sin(\mathrm{m}n\varphi),\sum_{n=1}^{\infty}c_n\sin(\mathrm{m}n\varphi)\right)\Big|\left(\sum_{n=1}^{\infty}b_n\sin(\mathrm{m}n\varphi),\sum_{n=1}^{\infty}d_n\sin(\mathrm{m}n\varphi)\right)\right)_2=\sum_{n=1}^{\infty}a_nb_n+c_nd_n$$

Range

$$\mathrm{Im}\left(d_{(f_1,f_2)}\mathscr{G}\left(c_{\mathrm{m}}^{\pm}(\widetilde{\gamma},\theta_1,\theta_2),0,0\right)\right)=\mathtt{span}^{\perp(\cdot|\cdot)_2}(y_0)$$

where

$$y_0=\varphi\mapsto v_0\sin(\mathrm{m}\varphi),\qquad v_0\in\ker\left(M_{\mathrm{m}}^{\top}\left(c_{\mathrm{m}}^{\pm}(\theta_1,\theta_2),\theta_1,\theta_2\right)\right)$$

Transversality

$$\left(d_{(f_1,f_2)}\mathscr{G}\left(c_{\mathrm{m}}^{\pm}(\widetilde{\gamma},\theta_1,\theta_2),0,0\right)[x_0]\Big|y_0\right)=\alpha_{\mathrm{m}}\beta_{\mathrm{m}},\qquad\begin{cases}\alpha_{\mathrm{m}}=0\Leftrightarrow\Delta_{\mathrm{m}}(\theta_1,\theta_2)=0,\\ \beta_{\mathrm{m}}=0\Leftrightarrow\left(\omega_C=\omega_N\text{ or }\omega_C=\omega_S\right)\end{cases}$$

To be continued...

Thank you for your attention !