Stability Threshold of the 2D Couette flow in a homogeneous magnetic field

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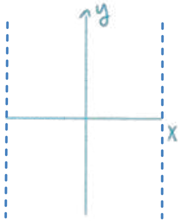
Introduction

The 2D Navier-Stokes MagnetoHydroDynamics system:

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla P = \nu \Delta u + b \cdot \nabla b \\ \partial_t b + u \cdot \nabla b - b \cdot \nabla u = \mu \Delta b \\ \operatorname{div} u = \operatorname{div} b = 0 \\ u|_{t=0} = u^{in}, \quad b|_{t=0} = b^{in} \end{cases}$$

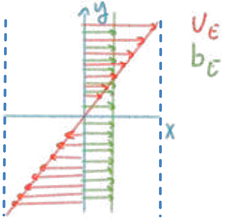
$t > 0$ $\quad (x,y) \in \mathbb{T} \times \mathbb{R}$



- $\nu = Re^{-1}$, $\mu = Re_m^{-1}$
- $Pr_m := \dfrac{\nu}{\mu} \leq 1$ and $0 < \nu \leq \mu \ll 1$
- u velocity, b magnetic field. $b \equiv 0 \to NS$.

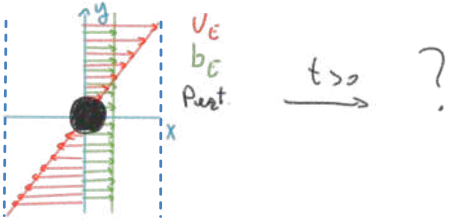
Laminar Solution

$u_E = (y, 0)\,, \quad b_E = (B, 0) \quad B \in \mathbb{R}$



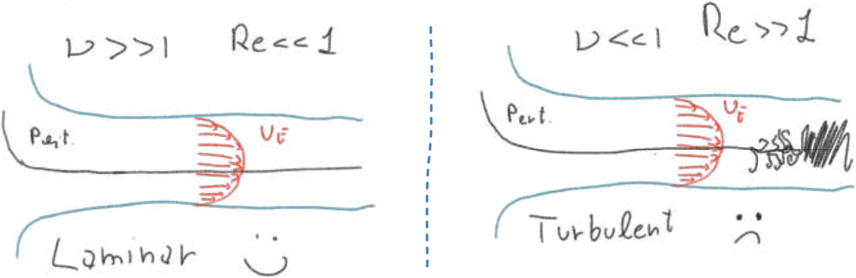
- It is a Steady State!

Q: What happens when perturbing it?



Onset of Turbulence

Reynolds 1883, NS in a channel. $v_i \sim (1-y^2, 0)$



Kelvin 1887: If the perturbation is small w.r.t $\nu \Rightarrow$ Laminar regime persists

Stability Threshold (def à la Bedrossian|Germain|Masmoudi '15)

Let $f = (v,b)$. Take $f^{in} = f_E + f_{Per}^{in}$.

What is the smallest $\gamma \geq 0$ s.t.

If $\| f_{Per}^{in} \|_{H^N} \ll \nu^{\gamma} \Rightarrow \| f_{Per}(t) \|_{H^{N'}} \ll 1 \quad N' \leq N$

and $f_{Per}(t) \xrightarrow[t \to +\infty]{} 0$ (or to a LAMINAR STATE)

- Quantitative way to measure the basin of attraction of a given equilibrium state
- Long History in many different situations

Romanov '73, Trefethen, Schmid, Henningson '90s → Numerics ...
Bedrossian|Masmoudi|Vicol '14, Coti Zelati|Elgindi|Widmayer '19 (Del Zotto '21) ...
... Chen|Ding|Lin|Zhang '23

Known Results related to our setting

- 2D NS ($b \equiv 0$) around $v_E : (y, 0)$ in $\mathbb{T} \times \mathbb{R}$

- Masmoudi \ Zhao '19 $\quad \gamma = \frac{1}{3}$ in $H^N(\mathbb{T} \times \mathbb{R})$, $N \geq 40$

  $\| v_{per}^{in} \|_{H^{40}} \ll \nu^{1/3} \rightarrow$ Laminar ok

- M \ Z '19 $\quad \gamma = \frac{1}{2}$ in $H_x^{log} L_y^2 (\mathbb{T} \times \mathbb{R})$

- Li \ M \ Z '22 $\quad \gamma = \frac{1}{2}$ SHARP in $H_x^1 L_y^2$ (allows for pert. with unstable eigenvalues)

  $\| w_{per}^{in} \|_{H_x^1 L_y^2} \sim \nu^{1/2}$ but $\| v_{per}(T) \|_{L^2} \gg 1$

Known Results related to our setting

- MHD around Couette with constant magnetic field
  - Liss '18 3D NS-MHD → $\gamma = 1$ in $H^N(\mathbb{T}\times\mathbb{R}\times\mathbb{T})$
    $u_i := (y,0,0)$, $b_i := (\beta,0,1)$ $N \sim 40$
  - Zhao/Zi '23 2D $\bar{v}$-MHD → $\gamma = 1$ in Gevrey-2^+
    $|\beta| >> 1$, $\nu = 0$, $\mu > 0$ μ^γ (see also Knobel/Zillinger '22)
  - Chen/Zi forthcoming 2D NS-MHD → $\gamma = 5/6$ in $H^N(\mathbb{T}\times\mathbb{R})$
    $|\beta| >> 1$, $\nu = \mu > 0$ $\gamma = \frac{5}{6}^+$ $N \sim 40$
  - Knobel/Zillinger '23 $\mu = (\mu^*, 0)$ → $\gamma = 3/2$ in $H^N(\mathbb{T}\times\mathbb{R})$
    $\nu > 0$

Main Result

Let $v = (y, 0) + V$, $b = (\beta, 0) + h$

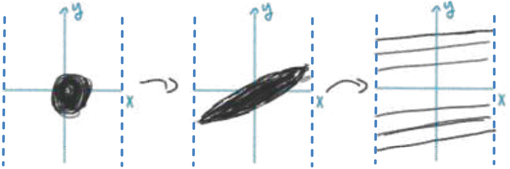
$\omega = \nabla^{\perp} \cdot v = \partial_x v^x - \partial_y v^y$ — vorticity

$J = \nabla^{\perp} \cdot h$ — current density

$\nabla^{\perp} = \begin{pmatrix} -\partial_y \\ \partial_x \end{pmatrix}$

$$\begin{cases} \partial_t \omega + y \partial_x \omega - \beta \partial_x J = \nu \Delta \omega + NL_\omega \\ \partial_t J + y \partial_x J - \beta \partial_x \omega + 2 \partial_{xy} \Delta^{-1} J = \mu \Delta J + NL_J \end{cases}$$

Transport, Oscillation, Stretching, Dissipation, Nonlinearity



$\omega^{in}, t=0$ | $t \sim 1$ | $t \gg 1$

Remark

If $\partial_x \omega = \partial_x J = 0$

$\Downarrow$

1D Heat eqt

Theorem (D. '23)

$0 < \nu \leq \mu \ll 1$, $|\beta| > \frac{1}{2}$, $N \geq 11$, $\nu \gtrsim \mu^3$

If $\|(\omega^{in}, J^{in})\|_{H^N} = \varepsilon \ll \nu^{2/3}$

Then, denoting $f_0(y) := \fint_{\mathbb{T}} f(x,y)\,dx$, $f_{\neq} = f - f_0$

- $\|(\omega_{\neq}, J_{\neq})(t)\|_{L^2} \lesssim \varepsilon (1+t) e^{-\nu^{1/3} t}$
- $\|(v^x_{\neq}, h^x_{\neq})(t)\|_{L^2} + (1+t)\|(v^y, h^y)(t)\|_{L^2} \lesssim \varepsilon e^{-\nu^{1/3} t}$
- $\|(\omega_0, J_0)(t)\|_{H^{N-1}} \lesssim \varepsilon$

Remarks

- $\gamma = 2/3$ in H^N. Why?

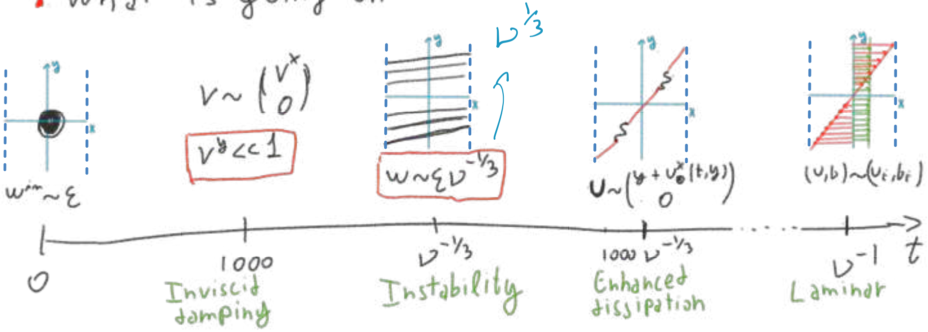
If NL perturbative $\Rightarrow \int_0^{+\infty} \|u \cdot \nabla w\| \, dt \lesssim \varepsilon^2 \int_0^{+\infty} t \, e^{-\nu^{1/3} t} \, dt$

we try to bootstrap bounds

$$\lesssim \varepsilon \left(\varepsilon \nu^{-2/3} \right) << \varepsilon$$

$\varepsilon \nu^{-1/3}$

- What is going on?



Linearized Problem

1) Follow the characteristics

- $X = x - yt$, $Y = y$ $\rightarrow$ $\Omega(t, X, Y) = w(t, x, y)$
  $J(t, X, Y) = j(t, x, y)$

- $\partial_x \longrightarrow \partial_X$ $\qquad \Delta \longrightarrow \Delta_L := \partial_{XX} + (\partial_Y - t\partial_X)^2$
  $\partial_y \longrightarrow \partial_Y - t\partial_X$

2) Fourier transform

$$\hat{f}(k, \eta) = \int_{\mathbb{R}} \int_{\mathbb{T}} e^{i(kx + \eta y)} f(x, y)\, dx\, dy$$

$$p(t, k, \eta) = k^2 + (\eta - kt)^2 = -\hat{\Delta}_L \rightsquigarrow p \sim t^2$$

$\partial_t w + y\partial_x w - \beta\partial_x J = \nu\Delta w$

$\partial_t J + y\partial_x J - \beta\partial_x w + 2\partial_{xy}\Delta^{-1} J = \mu\Delta J$

$P = k^2 + (\eta - kt)^2$

$\frac{\partial_t P}{P} = \frac{-2k(\eta - kt)}{k^2 + (\eta - kt)^2} = \frac{-2\widehat{\partial_x(\partial_y - t\partial_x)}}{\Delta_L}$

The system for $\hat{\Omega}, \hat{J}$ is

$$\partial_t \hat{\Omega} = i\beta k \hat{J} - \nu p(t) \hat{\Omega}$$

$$\partial_t \hat{J} = i\beta k \hat{\Omega} + \frac{\partial_t p(t)}{p(t)} \hat{J} - \mu p(t) \hat{\Omega}$$

since $\widehat{-2\partial_{xy}\Delta^{-1}} \to \frac{\partial_t P}{P}$

Equivalently

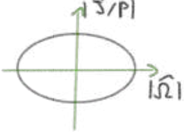
$$\frac{d}{dt}\begin{pmatrix} \hat{\Omega} \\ \hat{J}/p(t) \end{pmatrix} = \begin{bmatrix} -\nu p(t) & i\beta k\, p(t)\, t^2 \\ \frac{i\beta k}{p(t)} \frac{1}{t^2} & -\mu p(t) \end{bmatrix} \begin{pmatrix} \hat{\Omega} \\ \hat{J}/p(t) \end{pmatrix}$$

2×2 Nonautonomous ODE

Remark: If $\nu = \mu = 0$ and $p(t) \equiv$ constant

$J \sim t^2$ if $\beta = 0$

$\Omega \sim 1$



Symmetric Variables

Q: How to effectively study this system?

1) Study $\partial_{tt}\hat{J}$ and use asymptotics of special functions
$\rightsquigarrow$ Not flexible enough

2) Use Elsässer variables $E^{\pm} = \hat{\Omega} \pm \hat{J} \rightsquigarrow$ $\nu = \mu$ ✓ (Liss, Chen/Zi)
$\nu \neq \mu$?

3) "Symmetrize" $\rightsquigarrow$ Good unknowns for which oscillatory behaviour is precisely captured
$\sim$ Renormalize to absorb growing stuff

Antonelli/D./Marcati '20 2D compressible NS (Linear)

Bianchini/Coti Zelati/D. '20 2D Stratified fluids (Linear)

Bedrossian/Bianchini/Coti Zelati/D '21 2D Euler-Boussinesq (NonLinear)

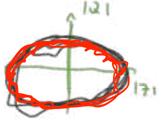
Symmetrized System

- $(z, Q) := \sqrt{\frac{K^2}{P(t)}}\,(\hat{\Omega}, \hat{J}) \quad \rightsquigarrow \sqrt{\partial_x x(\Delta)}^{-1}(\omega, J)$

$$\frac{d}{dt}\begin{pmatrix} z \\ Q \end{pmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{\partial_t P}{P}\frac{1}{\epsilon} & i\beta K \\ i\beta K & \frac{1}{2}\frac{\partial_t P}{P}\frac{1}{\epsilon} \end{bmatrix}\begin{pmatrix} z \\ Q \end{pmatrix} - \nu P(t)\begin{pmatrix} z \\ \frac{\mu}{\nu} Q \end{pmatrix}$$

- Trace of ⤴ is zero and antidiagonal terms ok ☺

$\frac{\partial_t P}{P} \sim \frac{1}{t} \xrightarrow[t \to \infty]{} \begin{bmatrix} 0 & i\beta K \\ i\beta K & 0 \end{bmatrix}$ oscillatory ☺



- If $|z|, |Q| \sim 1$ $\quad \nu = \mu = 0$

$\Rightarrow |\hat{\Omega}| = \sqrt{\frac{P}{K^2}}\,|z| \sim t$ $\quad$ ($\sqrt{\frac{P}{K^2}} \rightsquigarrow t$)

$|z| + |Q| \sim 1 \; (e^{-\nu^{1/3} t})$

Energy Functional

$-\frac{1}{2}\frac{\partial_t P}{P}\left(|z|^2-|Q|^2\right)$

Weighted energy estimates:

- Let $m(t,k,\eta)$ be a bounded Fourier multiplier ($|m|\sim 1$)

$$\frac{d}{dt}\begin{pmatrix} m^{-1}z \\ m^{-1}Q\end{pmatrix} = -\frac{\partial_t m}{m}\begin{pmatrix} z \\ Q\end{pmatrix} + \begin{bmatrix} -\frac{1}{2}\partial_t P/P & i\beta k \\ i\beta k & \frac{1}{2}\frac{\partial_t P}{P}\end{bmatrix}\begin{pmatrix} z \\ Q\end{pmatrix} - \nu p(t)\begin{pmatrix} z \\ \pm Q\end{pmatrix}$$

Damping if $\frac{\partial_t m}{m} > 0$

Lemma $\exists\, m$ s.t.

$$E_{sym}(t) = \frac{1}{2}\left(\|m^{-1}z\|^2 + \|m^{-1}Q\|^2 - \left(Re\left\langle \frac{\partial_t P}{i\beta k P} m^{-1}z, m^{-1}Q\right\rangle\right)\right)$$

$\frac{1}{t}$

$$E_{sym}(t) \approx e^{-\nu^{1/3}t}$$

Remark: $\frac{|\partial_t P|}{P} = \frac{2|k||\eta-kt|}{k^2+(\eta-kt)^2} \leq 1 \Rightarrow E_{sym}$ coercive if $|B| > 1/2$:)

- If $\nu=\mu=0$, $P\equiv$ constant $\Rightarrow E_{sym}$ LYAPUNOV FUNCTIONAL.

$$\mu \geq \nu^3$$

$$\frac{1}{2}\frac{d}{dt}\left(\|m^{-1}z\|^2 + \|m^{-1}Q\|^2\right) = -\frac{1}{2}\frac{\partial_t P}{P}\left(\|m^{-1}z\|^2 - \|m^{-1}Q\|^2\right)$$

$$\frac{1}{\epsilon} \quad \langle i\beta k m^{-1}z, m^{-1}Q\rangle + \langle i\beta n^{-1}Q, m^{-1}z\rangle$$

$$- \sim \|\sqrt{p}\,z\|^2 \cdot - \sim \|\sqrt{p}\,Q\|$$

$$\frac{d}{dt} z = \left(i\beta k\, Q - \frac{1}{2}\frac{\partial_t P}{P} z\right) \Big| \; Q - \frac{1}{i\beta k}\frac{1}{2}\frac{\partial_t P}{P}.$$

$$\frac{d}{dt} Q = \left(i\beta u\, z + \frac{1}{2}\frac{\partial_t P}{P} Q\right) \Big/ \; z$$

$$\frac{1}{2}\frac{d}{dt}\left\langle \frac{\partial_t P}{i\beta n P} m^{-1}z, m^{-1}Q\right\rangle = \frac{1}{2}\frac{\partial_t P}{P}\left(\|m^{-1}z\|^2 - \|m^{-1}Q\|^2\right)$$

$$u \gtrsim \nu^3$$

$$+ \left\langle \frac{\partial_t P}{i\beta n P}\, DP \cdot z, Q\right\rangle$$

$$\frac{1}{\sqrt{P}} \quad \sim \sqrt{P}$$

The weight for enhanced dissipation

Toy problem

$$\partial_t f = -\nu p(t) f = -\nu\left(k^2 + (\eta - kt)^2\right) f$$

$$\Rightarrow f(t) = e^{-\nu\int_0^t (\sim)\,d\tau} f^{in} \lessapprox e^{-\nu t^3} f^{in} \sim e^{-\nu^{1/3} t} f^{in}$$

Explicit solution not flexible!

- Choice of a weight m s.t. $\frac{\partial_t m}{m} > 0$, $|m| \sim 1$

$$\partial_t(m^{-1} f) = -\nu\left(k^2 + (\eta - kt)^2 + \frac{\partial_t m}{m}\right)(m^{-1} f)$$

We want $\boxed{\partial_t(m^{-1} f) \leq -\nu^{1/3}(m^{-1} f) \Rightarrow f(t) \lessapprox e^{-\nu^{1/3} t}}$

i) If $|\eta - \kappa t| > \nu^{-1/3} \Rightarrow -\nu(\eta-\kappa t)^2 \leq \nu^{1/3}$

$$\Rightarrow \partial_t(m^{-1}f) = -\nu\left(\kappa^2+(\eta-\kappa t)^2 + \frac{\partial_t m}{m}\right)(m^{-1}f)$$

$$\leq -\nu^{1/3}(m^{-1}f) \quad \checkmark \quad \left(\text{If } \frac{\partial_t m}{m} > 0\right)$$

ii) If $|\eta - \kappa t| \leq \nu^{1/3}$? Choose m as Bedrossian/Germain/Masmoudi '15

$$\boxed{\frac{\partial_t m}{m} = \frac{\nu^{1/3}}{1+\nu^{2/3}(\eta/\kappa - t)^2} > 0} \rightsquigarrow \begin{array}{l} m = \exp\left(\arctan\left(\nu^{1/3}(\eta/\kappa - t)\right)\right) \\ |m| \sim 1 \end{array}$$

$\Rightarrow \partial_t(m^{-1}f) \leq -\nu^{1/3}(m^{-1}f)$ ALWAYS! ☺

$$\Rightarrow \|f\| \sim \|m^{-1}f\| \leq e^{-\nu^{1/3}t} \quad \checkmark$$

Nonlinear Issues

Try to Bootstrap the behavior observed for Linear Pb.

$$\partial_t z = L z + \sqrt{\frac{K^2}{P}}\, NL_w$$

- Good Energy at the Linearized level
  $\rightsquigarrow$ use it $\rightsquigarrow$ $E_{sym} \lesssim \varepsilon^2$ ... Some bad NL term :(

- In NL_w $\quad f_{\neq} = f - \oint f\, dx$

$$u \cdot \nabla w = u_{\neq} \nabla w_{\neq} + u_0^x \partial_x w_{\neq}$$ $\rightsquigarrow$ OK, $w_{\neq} \sim t e^{-\nu^{1/3} t}$ $\Rightarrow \sim \varepsilon^2 t e^{-\nu^{1/3} t}$ ✓

$$+ u_{\neq}^2 \partial_y w_0$$ $\rightarrow$ BAD

$u^2 \sim \frac{1}{t}$ $\qquad$ $u_{\neq}^2 \sim \frac{1}{t}$ NOT INTEGRABLE

Error at highest order of Regularity !!!

$z \sim \partial_x \| \nabla \|^{-1} \Omega$ $\qquad$ $\partial_y \Omega \sim \partial_{yy} z$ not even dissipation is enough :(

$\sqrt{\frac{n^2}{P}} \sim \tau^{-1/2}$

To control $v_x^2 \partial_y w_0$, we need

$$E(t) = \frac{1}{2}\left(\|m^{-1}\Omega\|^2 + \|m^{-1} J\|^2\right)$$

$\varepsilon^2 t^2$

ORIGINAL VARIABLES

"Are you crazy?"

- Ω, J no good energy structure due to stretching term. But

$$\frac{1}{2}\frac{d}{dt}\left(\|\Omega\|_{L^2}^2 + \|J\|_{L^2}^2\right) = -\langle 2\partial_x(\partial_y - t\partial_x)\Delta_L^{-1} J, J\rangle$$

$$= \langle \frac{\partial_t p}{p} \hat{J}, \hat{J}\rangle$$

$$= \langle \frac{\partial_t p}{|k|\sqrt{p}} \sqrt{\frac{k^2}{p}} \hat{J}, \hat{J}\rangle \leq 2\|Q\|_{L^2}\|J\|_{L^2}$$

Q

$\frac{\partial_t p}{|k|\sqrt{p}} \leq 2$

$\Rightarrow$ If $\|Q\| \lesssim \varepsilon \Rightarrow \|\Omega\| + \|J\| \lesssim \varepsilon t$ → This is what we expect :)

$$\frac{d}{dt} E \leq E\sqrt{E} \rightsquigarrow \sqrt{E} \lesssim \varepsilon t$$