Unveiling the Role of Diffusion-Driven Flow in Micro-Fluidic Transport: A Slow Manifold-Based Approach to Reduced Modeling

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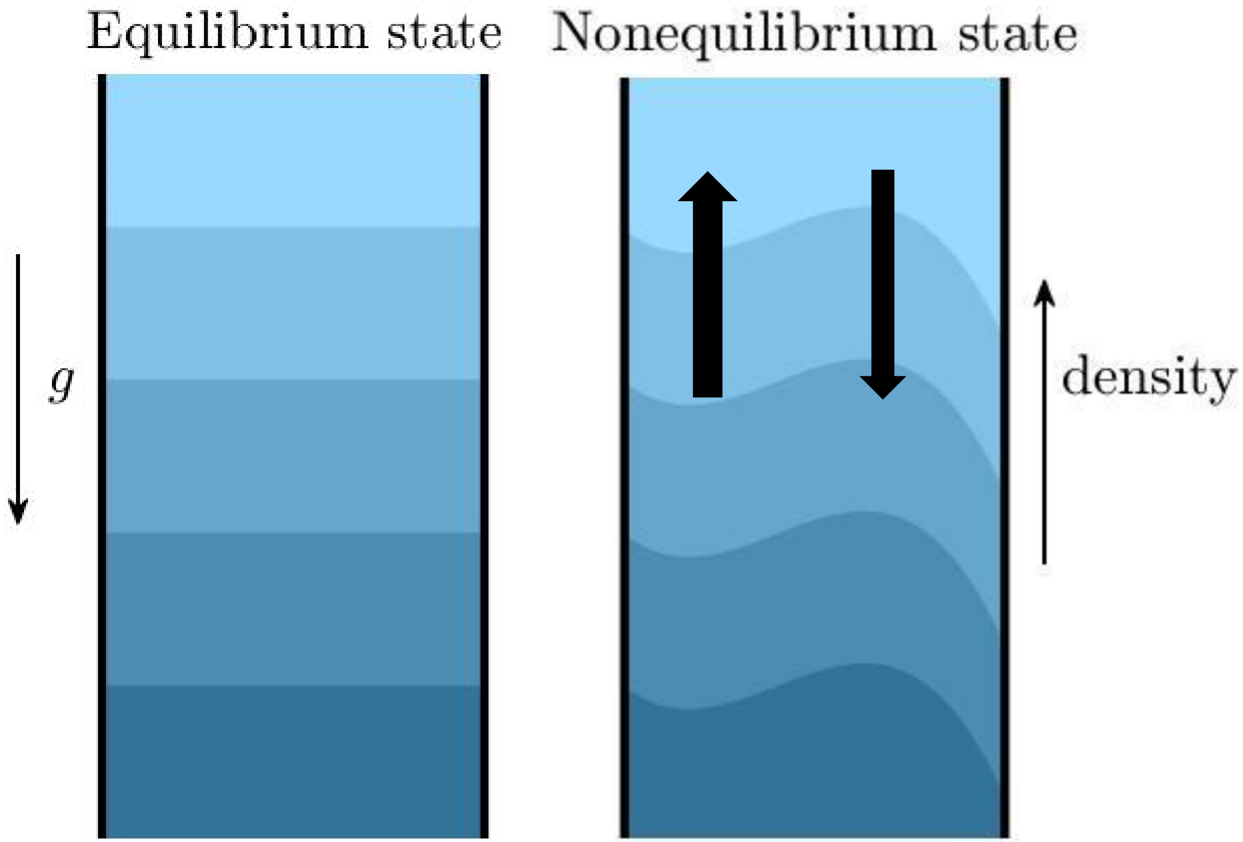
Outline

- What is diffusion driven flow?
- Applications of diffusion driven flow
- Model the dispersion induced diffusion driven flow
- Lubrication approximation for thin film fluid
- Example of some models that involve low dimensional slow manifold
- A Slow Manifold-Based Approach to Reduced Modeling

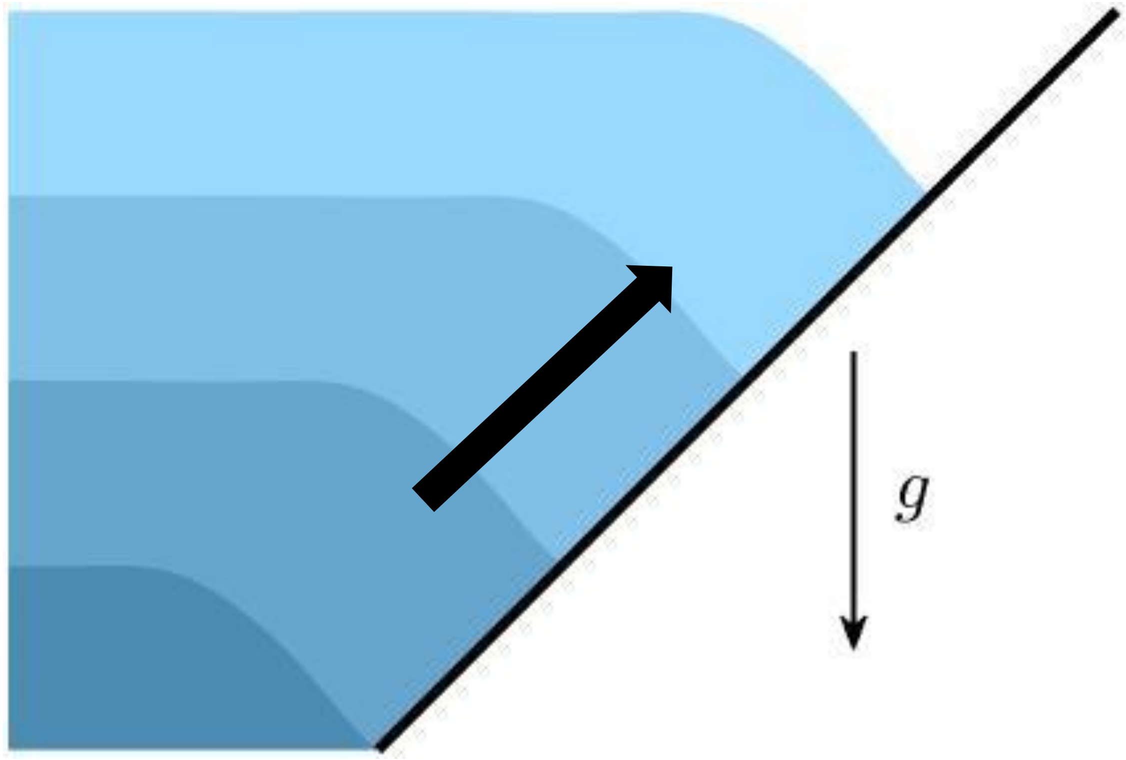
When will a density-stratified fluid reach an equilibrium state?



- hydrostatic equilibrium: The isoline of density field (isopycnal) should be perpendicular to the direction of gravity. Otherwise, the buoyance force will generate a flow.

Diffusion driven flow

For a diffusing solute, the impermeable (i.e. no-flux) boundary condition requires that the isopycnals must always be perpendicular to an impermeable boundary to ensure that there is no diffusive flux normal to the boundary.



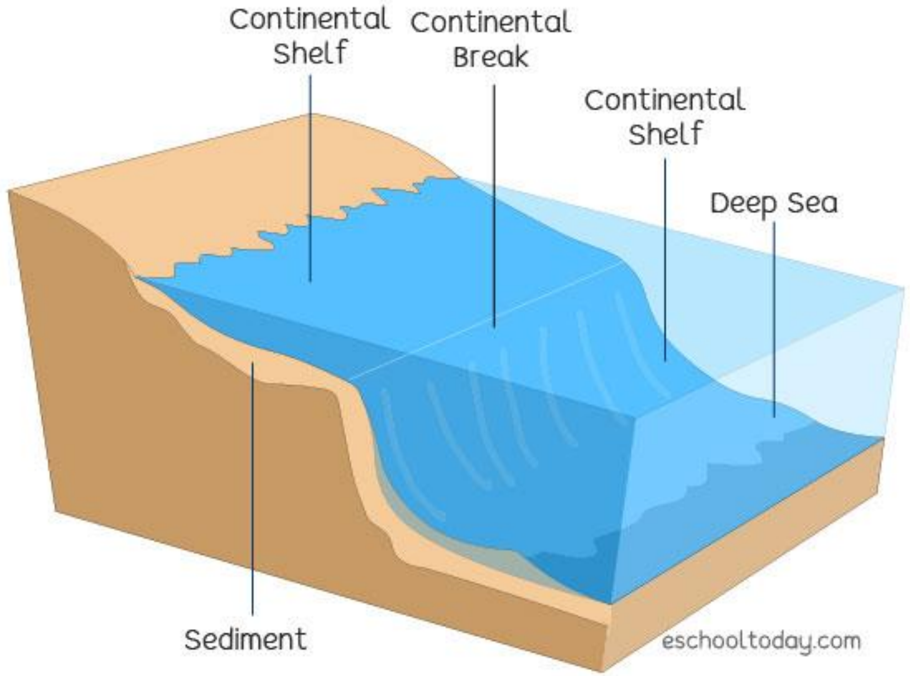
Three conditions: 1 density stratified fluid, 2 diffusing scalar, 3 boundary is not parallel to the gravitational direction.

Ocean

Three conditions: 1 density stratified fluid, 2 diffusing scalar, 3 boundary is not parallel to the gravitational direction.

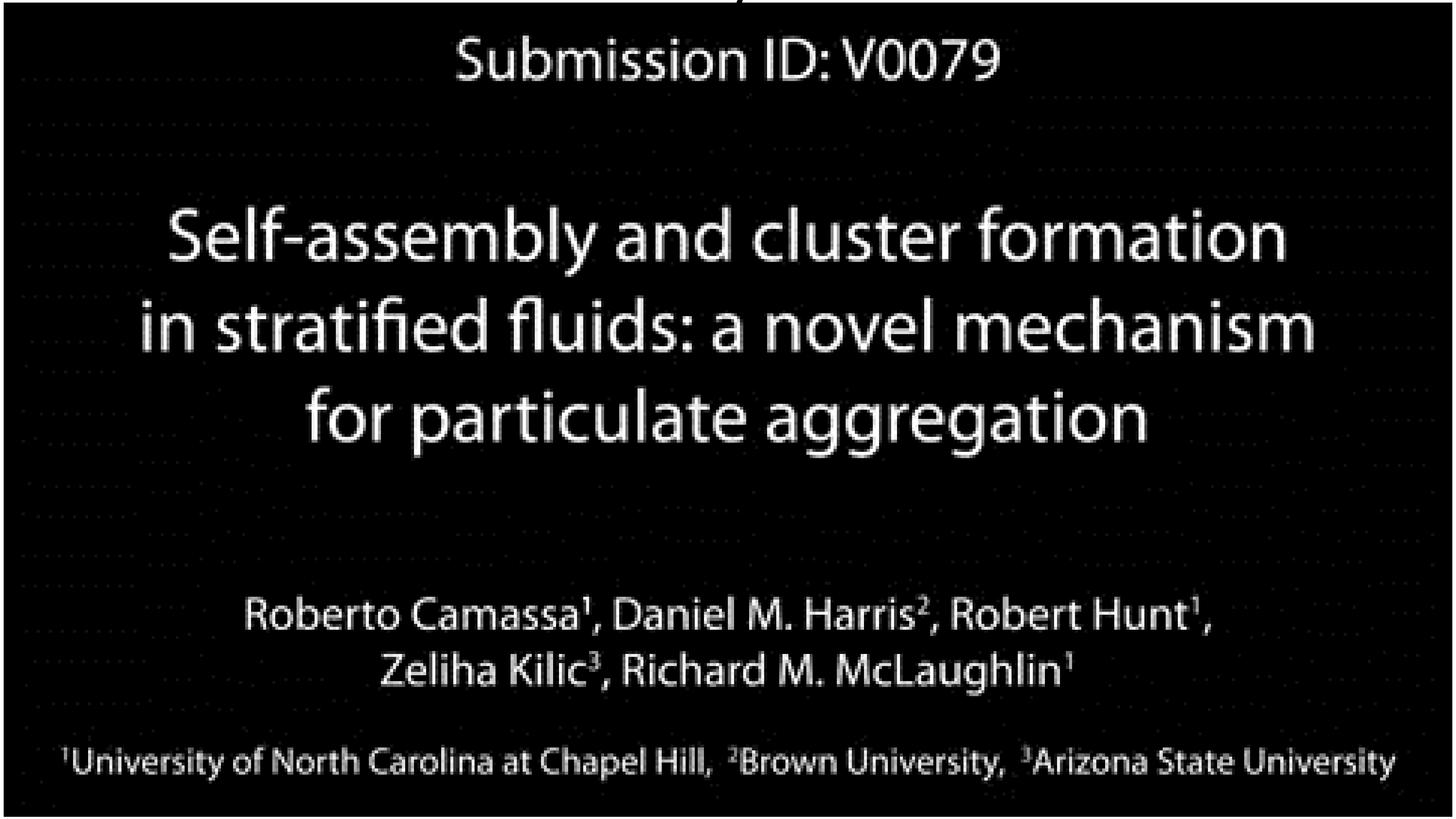
In ocean: 1 Stratified by salt 2 salt diffuse 3 continental shelf is not parallel to the gravitational direction.

This upwelling diffusion driven flow plays a crucial role in facilitating the vertical exchange of oceanic properties



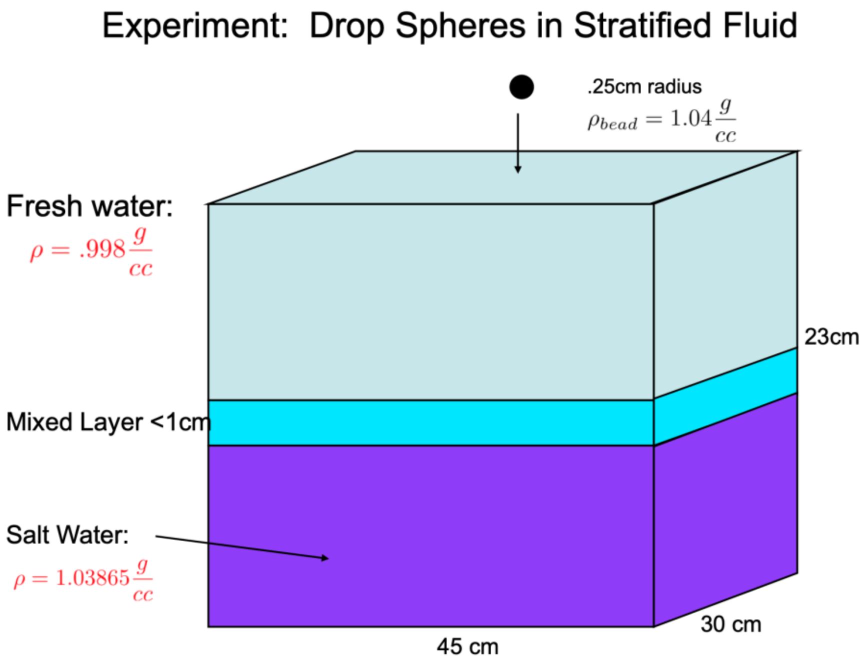
1. OM Phillips, 1970,On flows induced by diffusion in a stably stratified fluid
2. Wunsch, Carl, 1970, On oceanic boundary mixing.
3. RW Dell, LJ Pratt ,2015,Diffusive boundary layers over varying topography

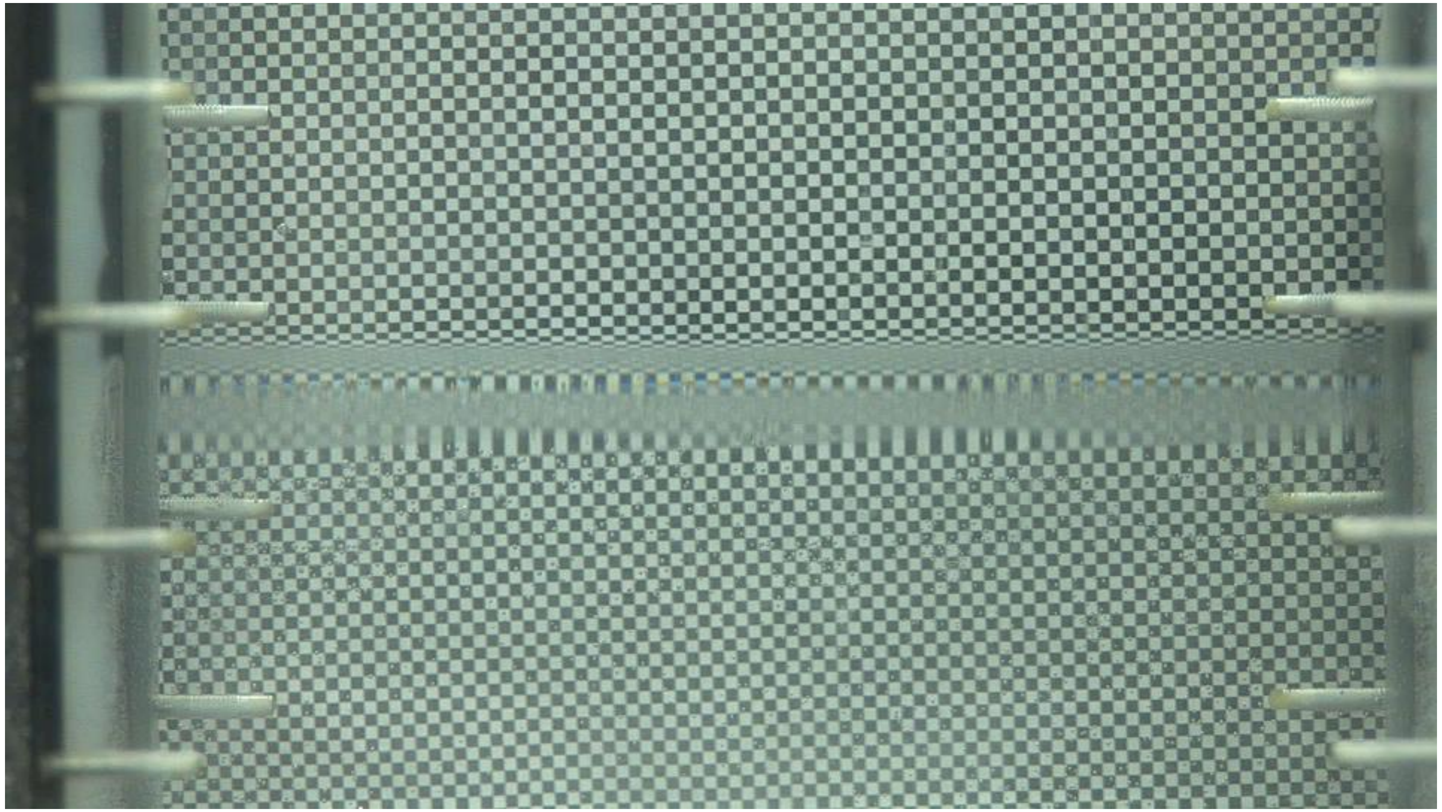
Particle self-assembly



Roberto Camassa, Daniel M. Harris, Robert Hunt, Zeliha Kilic, Richard M. McLaughlin, A first-principle mechanism for particulate aggregation and self-assembly in stratified fluids, Nature Communications,2019

Discovery of Particle self-assembly phenomenon

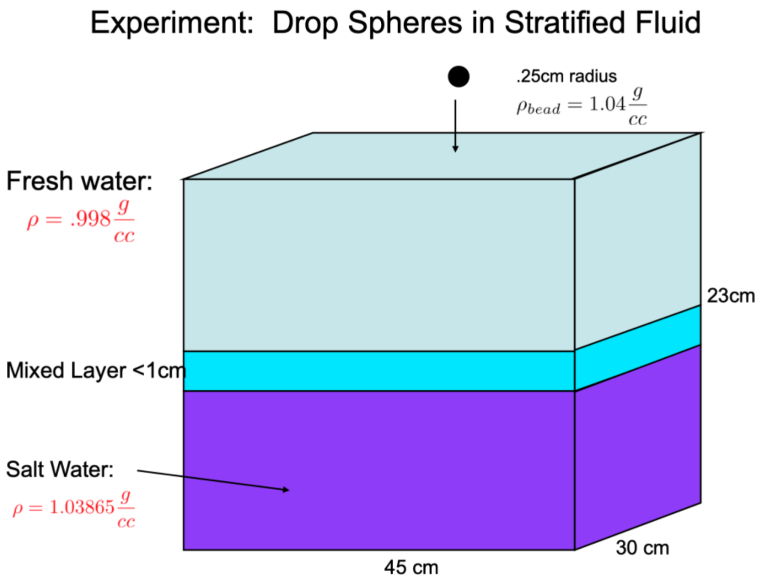


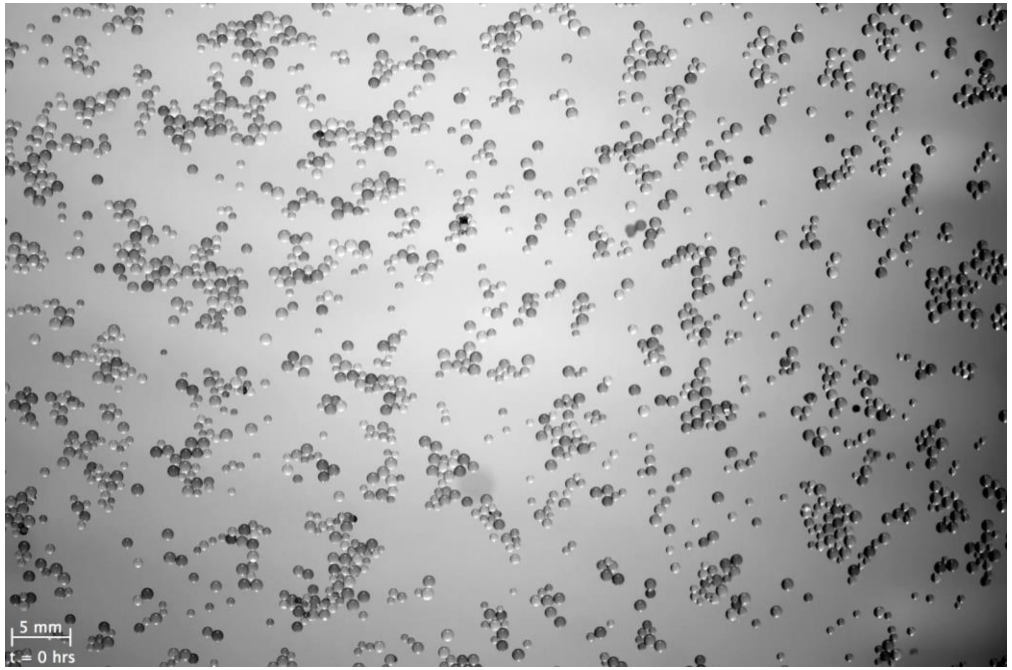


Roberto Camassa, Lingyun Ding, Richard M McLaughlin, Robert Overman, Richard Parker, and Ashwin Vaidya. Critical density triplets for the arrestment of a sphere falling in a sharply stratified fluid.

if $\rho_2 < \rho_b < \rho_b^*$, $\rho_b^* = 1.03\rho_2 - 0.0295\rho_1$, sphere can levitate. ρ_1 is the top fluid density ρ_2 is the bottom fluid density ρ_b is the sphere density

Discovery of Particle self-assembly phenomenon





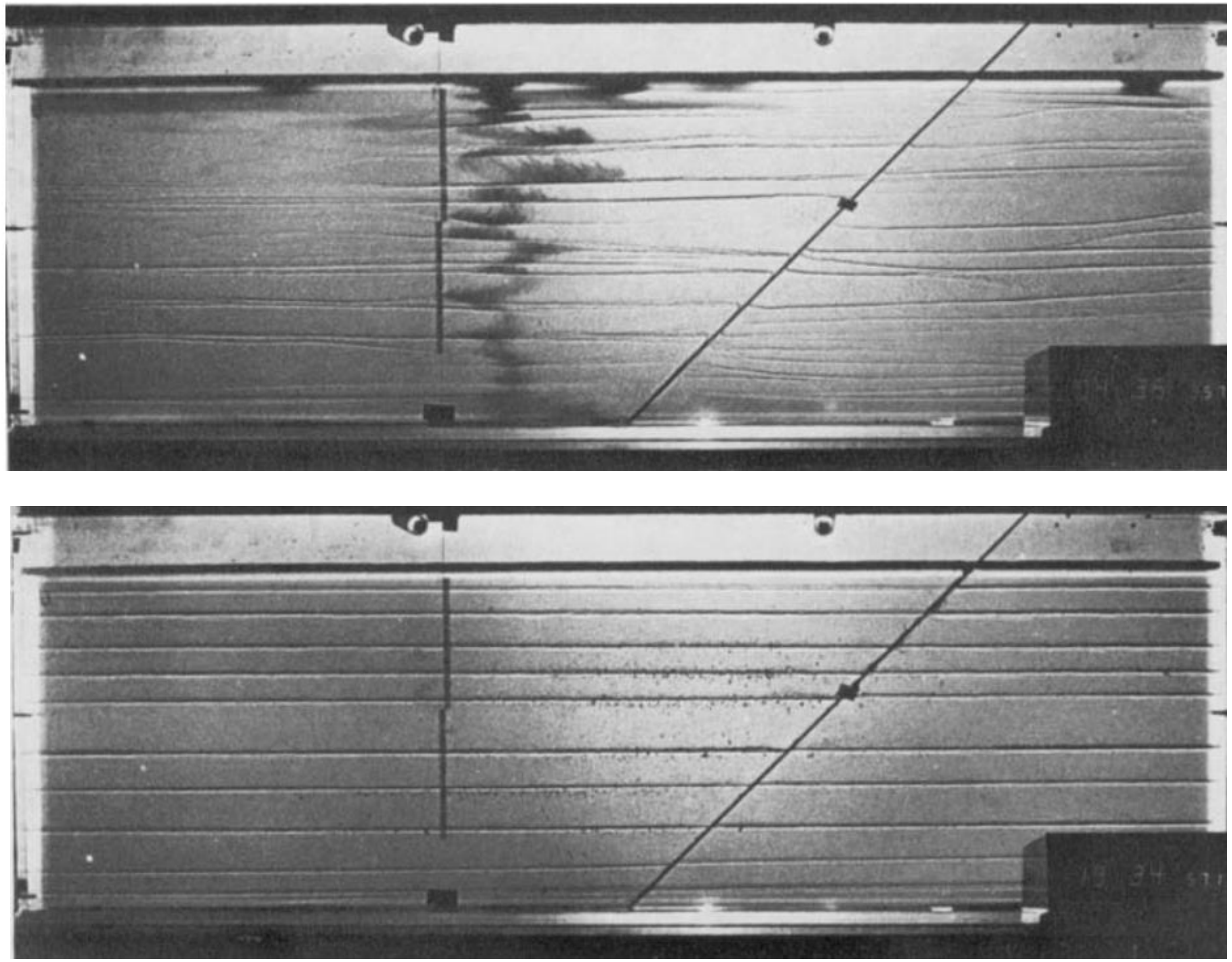
Due to an error in calculating the density of the particles, the experiment failed. My supervisor professor Richard M McLaughlin decided to take a break and returned the next day to clean the experimental apparatus. When he came to the lab the next day, he was surprised to find that all the particles had aggregated together.

Having a break is important.

The Diffusion Fish

Michael R. Allshouse, Michael F. Barad & Thomas Peacock, Nature Physics, 2010, Propulsion generated by diffusion-driven flow

Layer formation in Double-diffusive systems

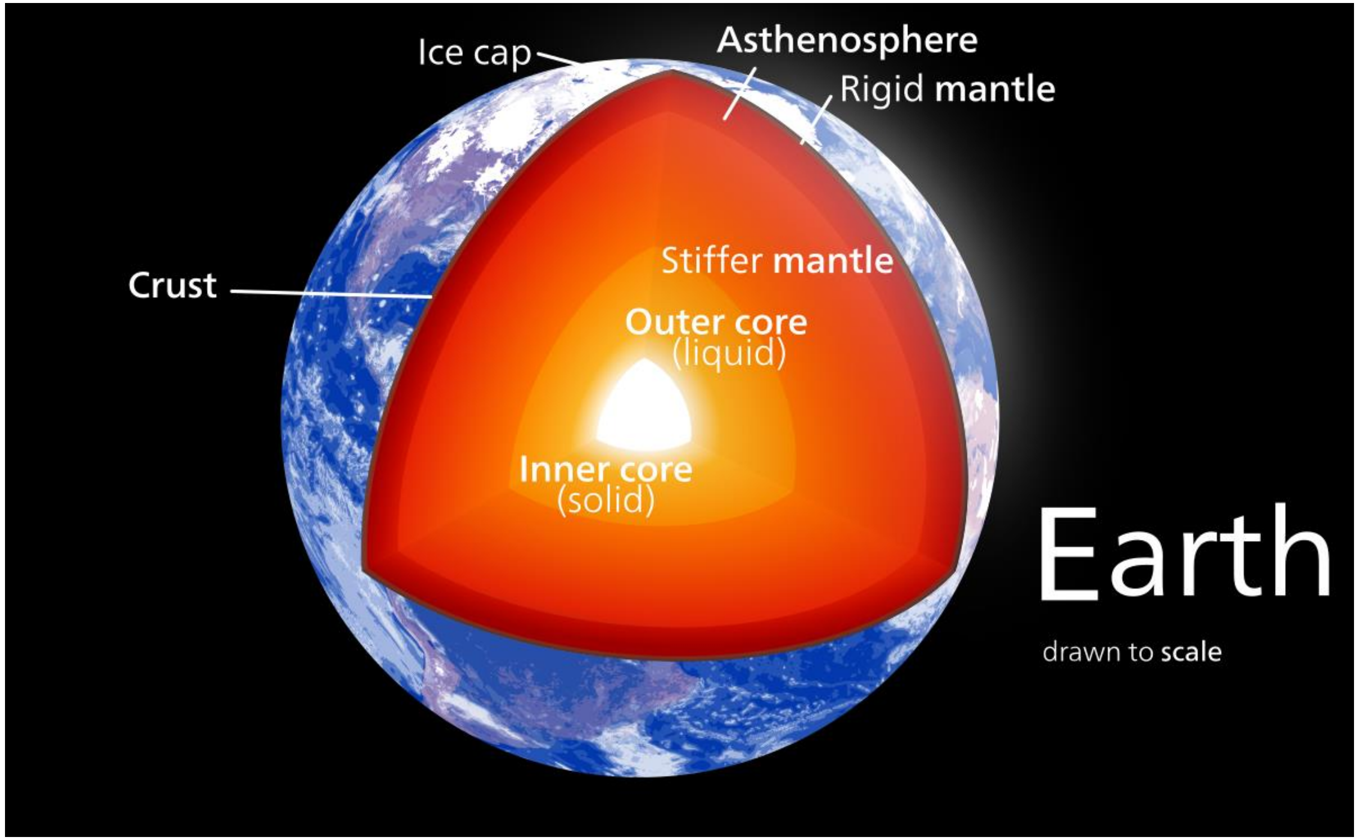


Temperature stratification and salt concentration stratification

PF Linden, JE Weber, 1977, JFM, The formation of layers in a double-diffusive system with a sloping boundary

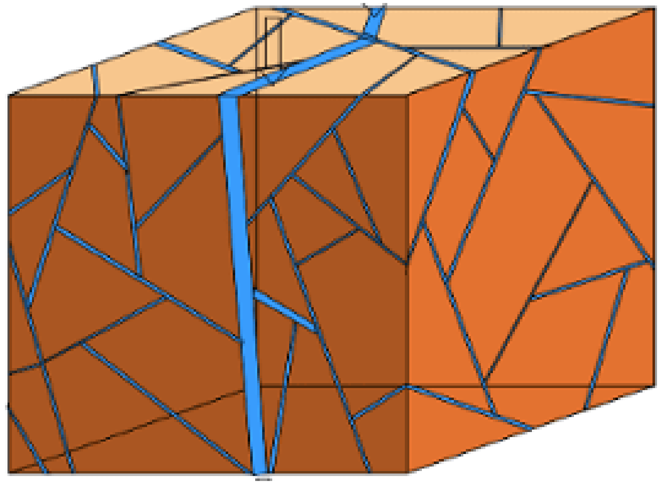
Earth’s geothermal gradient induces convective flows in the fluid contained in long narrow rock fractures, which leads to significant solute dispersion on geological timescales

AW Woods, SJ Linz, 1992, Natural convection and dispersion in a tilted fracture



The **stratifying scalar** causes non-uniform density distributions in the fluid and drives diffusion-driven flow. In contrast, the **passive scalar** represents the concentration of a different solute that doesn’t contribute to density variation but is instead passively advected by the fluid flow.

Temperature field is the stratifying scalar, the solute concentration field is passive scalar.



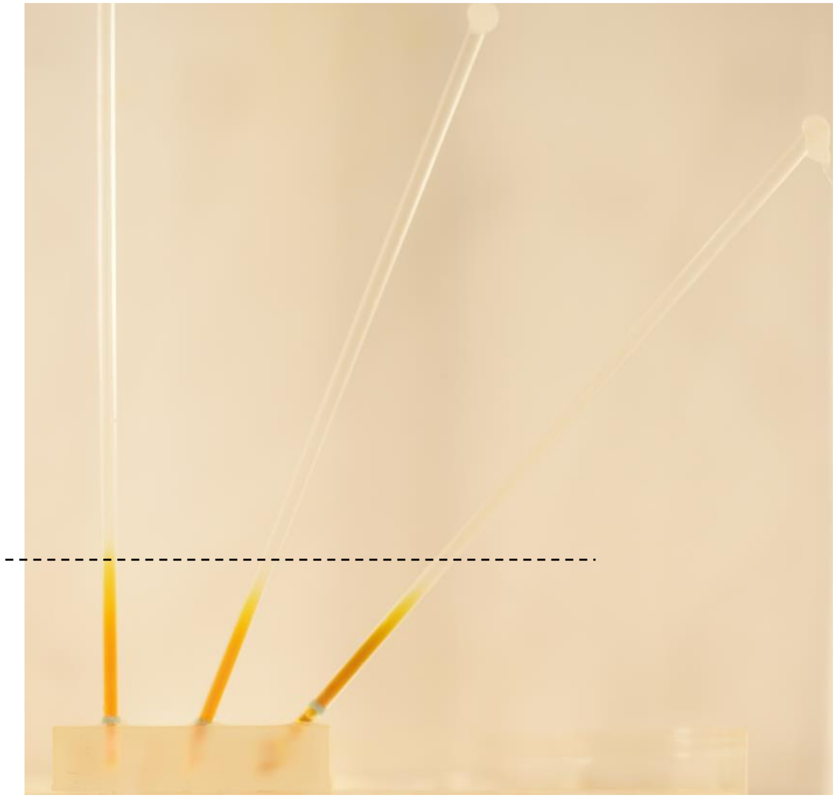
Model the dispersion induced diffusion driven flow

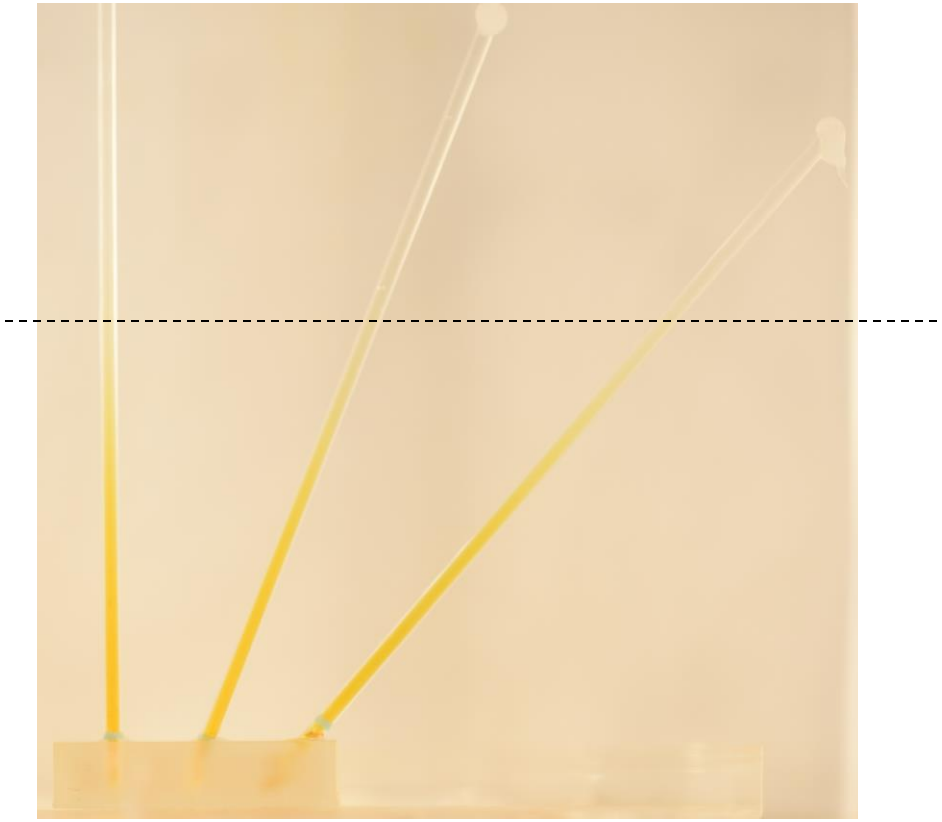
Gap:

1. Most existing theories primarily focus on linearly stratified fluids, and there is a scarcity of theoretical investigations into diffusion-driven flow in nonlinearly stratified fluids. $\rho = a - bz$
2. Many studies focus on how diffusion-driven flow enhances the dispersion of a passive scalar and the corresponding analysis for the stratifying scalar are rare.

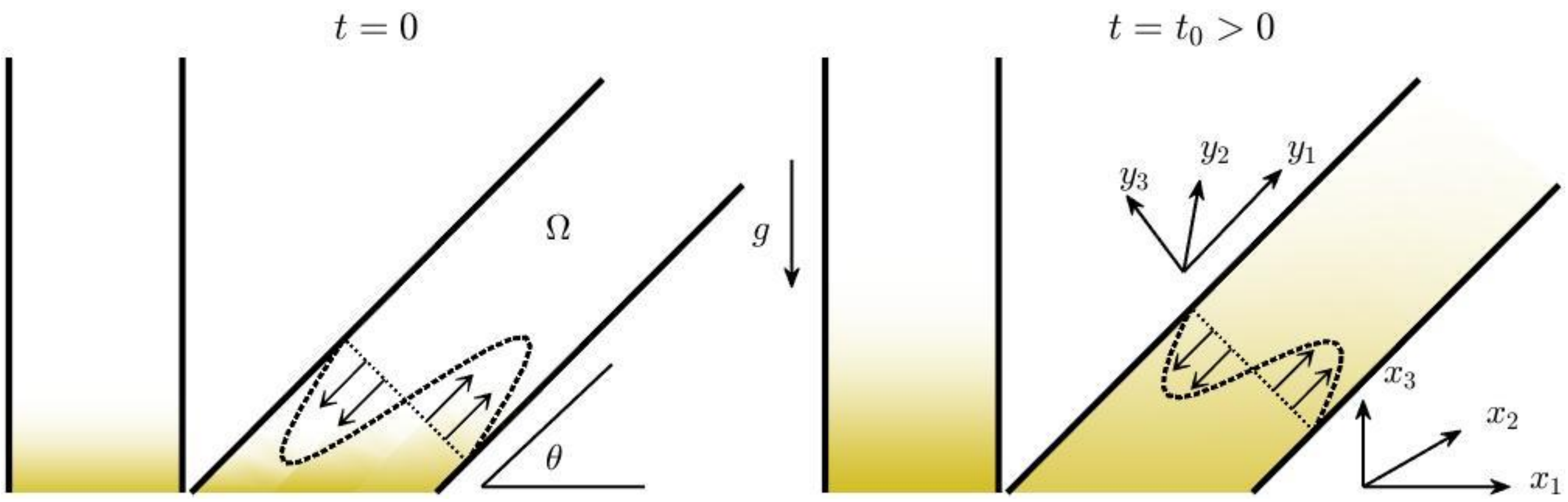
Model the non-linear stratified fluids, non passive scalar.

Enhanced dispersion





Keep the capillary tube vertical to reduce diffusion



$$\rho\left(\partial_t v_1 + v_1 \partial_{y_1} v_1 + v_3 \partial_{y_3} v_1\right) = \mu \Delta v_1 - \partial_{y_1} p - \sin\theta g \rho,$$

$$\rho\left(\partial_t v_3 + v_1 \partial_{y_1} v_3 + v_3 \partial_{y_3} v_3\right) = \mu \Delta v_3 - \partial_{y_3} p - \cos\theta g \rho,$$

$$\partial_t \rho + \mathbf{v} \cdot \nabla \rho = \kappa \Delta \rho, \quad \nabla \cdot \mathbf{v} = 0, \quad \mathbf{v}|_{\mathbf{x}\in\partial\Omega} = \mathbf{0}, \ \partial_{\mathbf{n}} \rho|_{\mathbf{x}\in\partial\Omega} = 0.$$

Lubrication approximation for thin film fluid

$$L_i y_i' = y_i, \quad V_i v_i' = v_i, \quad \frac{\rho_0 L_1 \nu U}{L_3^2} p' = p, \quad \frac{L_1^2}{\kappa} t' = t, \quad \rho_0 \rho' = \rho,$$

$$\epsilon = \frac{V_3}{V_1} = \frac{L_3}{L_1} \ll 1, \quad \mathrm{Re} = \frac{V_1 L_3}{\nu}, \quad \mathrm{Pe} = \frac{V_1 L_3}{\kappa}, \quad \mathrm{Sc} = \frac{\mu}{\rho_0 \kappa}, \quad \mathrm{Ri} = \frac{L_3 g}{V_1^2}.$$

Dropping primes and rearranging the equation results

$$\rho\left(\frac{\epsilon^2}{\mathrm{Sc}} \partial_t v_1 + \epsilon \mathrm{Re}\left(v_1 \partial_{y_1} v_1 + v_3 \partial_{y_3} v_1\right)\right) = \left(\epsilon^2 \partial_{y_1}^2 v_1 + \partial_{y_3}^2 v_1\right) - \partial_{y_1} p - \mathrm{Re}\mathrm{Ri}\rho \sin\theta,$$

$$\rho\left(\frac{\epsilon^4}{\mathrm{Sc}} \partial_t v_3 + \epsilon^3 \mathrm{Re}\left(v_1 \partial_{y_1} v_3 + v_3 \partial_{y_3} v_3\right)\right) = \epsilon^2\left(\epsilon^2 \partial_{y_1}^2 v_3 + \partial_{y_3}^2 v_3\right) - \partial_{y_3} p - \epsilon \mathrm{Re}\mathrm{Ri}\rho \cos\theta,$$

$$\epsilon^2 \partial_t \rho + \epsilon \mathrm{Pe}\left(v_1 \partial_{y_1} \rho + v_3 \partial_{y_3} \rho\right) = \epsilon^2 \partial_{y_1}^2 \rho + \partial_{y_3}^2 \rho,$$

$$\partial_{y_1} v_1 + \partial_{y_3} v_3 = 0.$$

We consider the formal power series expansions for the velocity, density and pressure

$$v_i = \sum_{k=0}^{\infty} v_{i,k} \epsilon^k, \quad \rho = \sum_{k=0}^{\infty} \rho_k \epsilon^k, \quad p = \sum_{k=0}^{\infty} p_k \epsilon^k$$

$$L_i y_i' = y_i, \quad V_i v_i' = v_i, \quad \frac{\rho_0 L_1 \nu U}{L_3^2} p' = p, \quad \frac{L_1^2}{\kappa} t' = t, \quad \rho_0 \rho' = \rho,$$

$$\epsilon = \frac{V_3}{V_1} = \frac{L_3}{L_1} \ll 1, \quad \mathrm{Re} = \frac{V_1 L_3}{\nu}, \quad \mathrm{Pe} = \frac{V_1 L_3}{\kappa}, \quad \mathrm{Sc} = \frac{\mu}{\rho_0 \kappa}, \quad \mathrm{Ri} = \frac{L_3 g}{V_1^2}.$$

$$\rho = \rho_0 + \epsilon^2 \rho_2, \quad \partial_t \rho_0 = \partial_{y_1}^2 \rho_0,$$

$$\rho_2 = (\partial_{y_1} \rho_0)^2 \, \frac{\mathrm{PeReRi}\left(1 - 2y_3^3(10 + 3y_3(2y_3 - 5))\right)\cos(\theta)}{1440},$$

$$v_1 = -\epsilon \partial_{y_1} \rho_0 \mathrm{ReRi} \cos\theta \frac{y_3(y_3 - 1)(2y_3 - 1)}{12},$$

$$v_3 = -\epsilon \partial_{y_1}^2 \rho_0 \mathrm{ReRi} \cos\theta \frac{y_3^2(y_3 - 1)^2}{24}.$$

The fact that $\int_0^1 \rho_2 dy_3 = 0$ implies that as ϵ tends to zero, the diffusion-driven flow only distorts the isopycnal of the density without amplifying the dispersion of the stratifying scalar in the longitudinal direction of the channel.

Potential solution:

1. Choose different scaling relation between different physical quantities

Find an alternative method such that:

1. obtain an approximation that accurately describe density dynamics across a wider parameter range
2. Don't need to deal with the complicated scaling analysis.

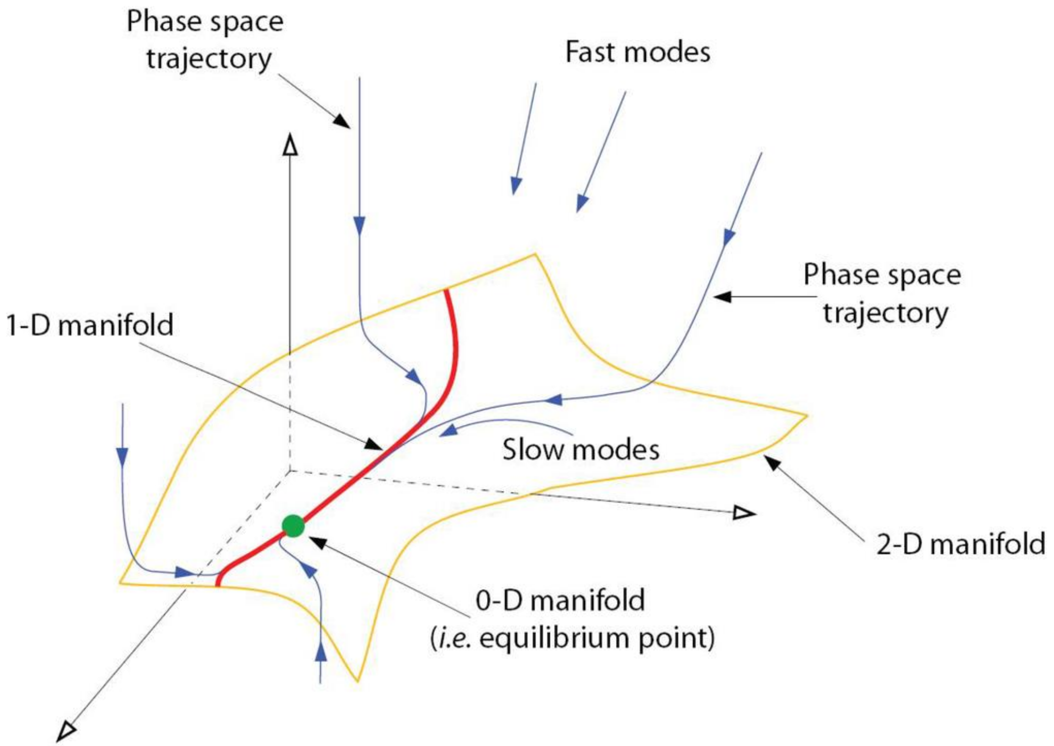
$$\rho = \rho_0 + \epsilon^2 \rho_2, \quad \partial_t \rho_0 = \partial_{y_1}^2 \rho_0,$$

$$\rho_2 = (\partial_{y_1} \rho_0)^2 \, \frac{\mathrm{PeReRi}\left(1 - 2y_3^3(10 + 3y_3(2y_3 - 5))\right)\cos(\theta)}{1440},$$

$$v_1 = -\epsilon \partial_{y_1} \rho_0 \mathrm{ReRi} \cos\theta \frac{y_3(y_3 - 1)(2y_3 - 1)}{12},$$

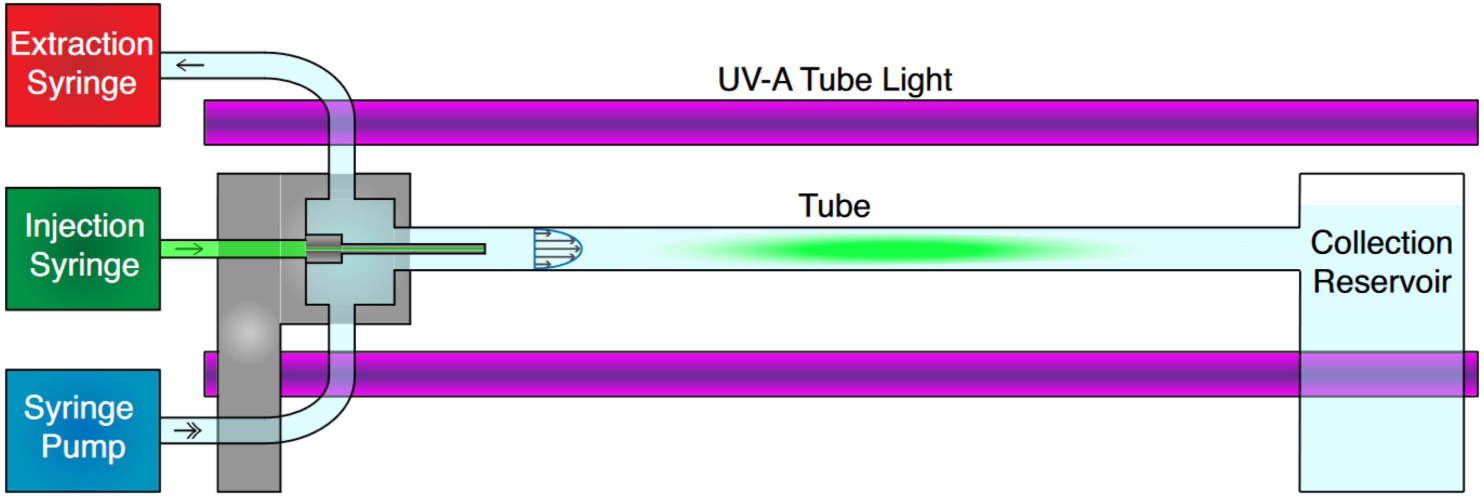
$$v_3 = -\epsilon \partial_{y_1}^2 \rho_0 \mathrm{ReRi} \cos\theta \frac{y_3^2(y_3 - 1)^2}{24}.$$

All quantities are function of $\partial_{y_1}\rho_0$, ρ_0 is a slow manifold of the system



Take a look at some example involves slow manifolds.

Shear dispersion



$$\partial_t T + \mathrm{Pe}u(y)\partial_x T = \Delta T, \quad T(x,y,0) = T_I(x,y), \quad \partial_{\mathbf{n}} T|_{\partial\Omega} = 0$$

As t tends to infinity, we have the approximation

$$T = \bar{T} + \theta(y)\partial_x \bar{T}, \quad \Delta\theta = -\mathrm{Pe}u, \quad \partial_{\mathbf{n}}\theta|_{\partial\Omega} = 0$$

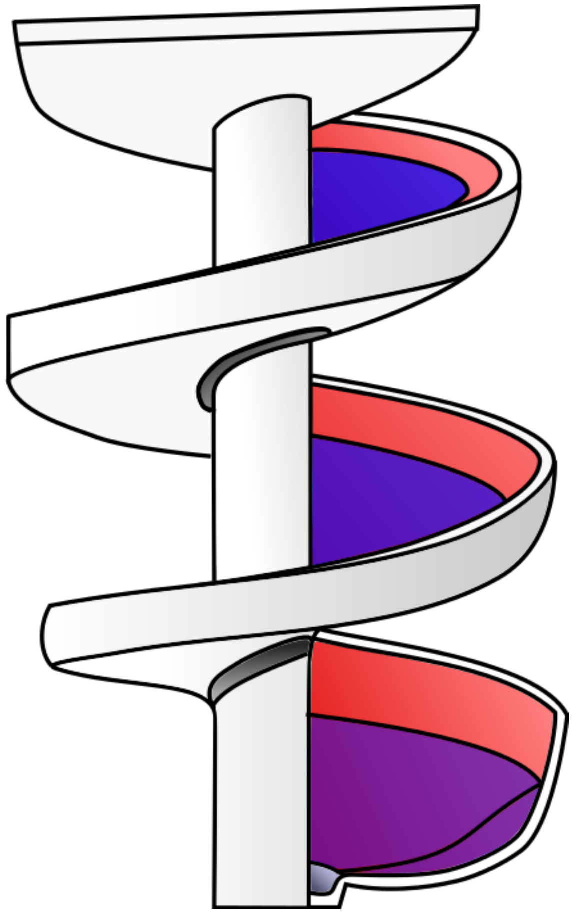
$$\partial_t \bar{T} + \mathrm{Pe}\bar{u}\partial_x \bar{T} = \kappa_{\mathrm{eff}}\partial_x^2 \bar{T}, \quad \kappa_{\mathrm{eff}} = 1 + \overline{u\theta}$$

GI Taylor,1953,Dispersion of soluble matter in solvent flowing slowly through a tube

$\bar{T}$ is the cross-sectional average concentration, which is the slow manifold in this problem.

Spiral separator

From https://www.youtube.com/watch?v=fvA0nkKCVf4



1. separate solid components in a slurry
2. Widely used in mining and food industries
3. The depth of the fluid layer is small compared to the channel width.

L Ding, S Burnett, A Bertozzi, Separation of bidensity suspensions in gravity-driven thin-film flow in helical channels

Lubrication approximation yields

$$u_r = -\frac{\mathrm{Re}\mathrm{Ri}z\left(-15zh+6h^2+8z^2\right)\left(\left(1+\frac{\alpha^2}{R^2r^2}\right)r^3\partial_r\rho+4\frac{\alpha^2}{R^2}\rho\right)}{48\left(1+\frac{\alpha^2}{R^2r^2}\right)^3r^3\mu}$$

$$-\frac{\alpha^2\mathrm{Re}^3\mathrm{Ri}^2z\rho^3\left(-42z^4h+70z^3h^2-72zh^4+32h^5+7z^5\right)}{840R^2\left(1+\frac{\alpha^2}{R^2r^2}\right)^5r^3\mu^3}$$

h is the fluid depth, which is the slow manifold in this problem. Notice that velocity is a function of the slow manifold.

Slow Manifold-Based Approach

$$T = \sum_{k=0}^{\infty} \epsilon^k T_k \Rightarrow T = \bar{T} + a_1 \partial_x \bar{T} + a_2 \partial_x^2 \bar{T}$$

Use the expansion of derivative directly.

Shear dispersion
1. W. Gill and R. Sankarasubramanian, 1970, Exact analysis of unsteady convective diffusion
2. G. Mercer and A. Roberts, 1990. A centre manifold description of contaminant dispersion in channels with varying flow properties
3. Lingyun Ding, Richard M. McLaughlin, 2022, Determinism and invariant measures for diffusing passive scalars advected by unsteady random shear flows

Thin film fluid
1. A. Roberts, 1993, The invariant manifold of beam deformations
2. A. Roberts, 1996, Low-dimensional models of thin film fluid dynamics,

Centre Manifold Theory
1. J. Carr, 2012,Applications of Centre Manifold Theory
2. B. Aulbach and T. Wanner, 1996,Integral manifolds for Carathéodory type differential equations in Banach spaces

How about considering the following expansion?

$$v_1 = v_{1,0}(\mathbf{y},t) + v_{1,1}(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + v_{1,2}(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots,$$

$$v_3 = v_{3,0}(\mathbf{y},t) + v_{3,1}(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + v_{3,2}(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots,$$

$$\rho = \bar{\rho}(\mathbf{y},t) + \rho_1(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + \rho_2(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots,$$

$$p = p_0 + p_1(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + p_2(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots, \qquad \mathbf{y} = (y_1, y_3).$$

Where $\bar{f}(y_1,t) = \int_0^1 f(y_1,y_3,t)dy_3$ denotes the cross-sectional average

The resulting leading order approximation is the same as the result obtained by the **l**ubrication approximation. This method also can not predict the enhancement dispersion induced by the diffusion driven flow.

$$
\begin{aligned}
v_1 &= v_{1,0}(\mathbf{y},t) + v_{1,1}\,(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + v_{1,2}\,(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
v_3 &= v_{3,0}(\mathbf{y},t) + v_{3,1}\,(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + v_{3,2}\,(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
\rho &= \bar{\rho}(\mathbf{y},t) + \rho_1\,(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + \rho_2\,(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
p &= p_0 + p_1\,(\mathbf{y},t)\,\partial_{y_1}\bar{\rho} + p_2\,(\mathbf{y},t)\,\partial^2_{y_1}\bar{\rho} + \ldots\,, \qquad \mathbf{y} = (y_1,y_3).
\end{aligned}
$$

Let’s introduce nonlinearity!

$$
\begin{aligned}
v_1 &= v_{1,0}(\mathbf{y},t) + v_{1,1}\,(\mathbf{y},t,\partial_{y_1}\bar{\rho})\,\partial_{y_1}\bar{\rho} + v_{1,2}\left(\mathbf{y},t,\partial_{y_1}\bar{\rho},\partial^2_{y_1}\bar{\rho}\right)\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
v_3 &= v_{3,0}(\mathbf{y},t) + v_{3,1}\,(\mathbf{y},t,\partial_{y_1}\bar{\rho})\,\partial_{y_1}\bar{\rho} + v_{3,2}\left(\mathbf{y},t,\partial_{y_1}\bar{\rho},\partial^2_{y_1}\bar{\rho}\right)\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
\rho &= \bar{\rho}(\mathbf{y},t) + \rho_1\,(\mathbf{y},t,\partial_{y_1}\bar{\rho})\,\partial_{y_1}\bar{\rho} + \rho_2\left(\mathbf{y},t,\partial_{y_1}\bar{\rho},\partial^2_{y_1}\bar{\rho}\right)\partial^2_{y_1}\bar{\rho} + \ldots\,,\\
p &= p_0(\mathbf{y},t) + p_1\,(\mathbf{y},t,\partial_{y_1}\bar{\rho})\,\partial_{y_1}\bar{\rho} + p_2\left(\mathbf{y},t,\partial_{y_1}\bar{\rho},\partial^2_{y_1}\bar{\rho}\right)\partial^2_{y_1}\bar{\rho} + \ldots\,.
\end{aligned}
$$

This expansion yields precise approximations for velocity, density, and pressure fields.

Relation between derivatives

To provide an intuitive justification, let's assume that flow effects are negligible, and diffusion is the dominant process in the system.

$$\partial_t \bar{\rho} = \partial^2_{y_1} \bar{\rho},$$

$$\bar{\rho} = \frac{1}{2}\mathrm{erfc}\left(\frac{y_1}{2\sqrt{t}}\right), \quad \mathrm{erfc}(z) = \frac{2}{\sqrt{\pi}}\int_z^\infty e^{-t^2}\,dt.$$

$$\partial_{y_1}\bar{\rho} = -\frac{e^{-\frac{y_1^2}{4t}}}{2\sqrt{\pi}\sqrt{t}}, \quad \partial^2_{y_1}\bar{\rho} = -\frac{y_1 e^{-\frac{y_1^2}{4t}}}{4\sqrt{\pi}t^{3/2}}, \quad \partial^n_{y_1}\bar{\rho} = -\frac{1}{\sqrt{\pi}}\left(-\frac{1}{2\sqrt{t}}\right)^n H_{n-1}\left(\frac{y_1}{2\sqrt{t}}\right) e^{-\left(\frac{y_1}{2\sqrt{t}}\right)^2}.$$

$$\partial^n_{y_1}\bar{\rho} = \mathcal{O}(t^{-n+\frac{1}{2}}), \quad \bar{\rho} \gg \partial_{y_1}\bar{\rho} \gg \partial^2_{y_1}\bar{\rho} \gg \dots \quad t \to \infty.$$

Where H_n is the Hermite polynomial of degree n.

In presence of the fluid flow, the derivative may have different decay rate. However, this relation still holds.

$$\bar{\rho} \gg \partial_{y_1}\bar{\rho} \gg \partial^2_{y_1}\bar{\rho} \gg \dots \quad t \to \infty$$

We consider the derivative as a small parameter within the framework of standard asymptotic calculations. Collecting the terms that is comparable to $\partial_{y_1}\bar{\rho}$ yields the following equation

$$0 = \partial_{y_3}^2 v_{1,1} - \mathrm{ReRi}\left(\sin\theta\rho_1 - \cos\theta(y_3 - \frac{1}{2})\right), \quad v_{1,1}\big|_{y_3=0,1} = 0,$$

$$\mathrm{Pe}v_{1,1}\partial_{y_1}\bar{\rho} = \partial_{y_3}^2\rho_1, \quad \partial_{y_3}\rho_1\big|_{y_3=0,1} = 0.$$

After Differencing the above equation twice with respect to y_3, we can decouple the density and velocity

$$0 = \partial_{y_3}^4 v_{1,1} - \mathrm{RePeRi}\sin\theta\partial_{y_1}\bar{\rho}v_{1,1}, \quad v_{1,1}\big|_{y_3=0,1} = 0, \quad \partial_{y_3}^3 v_{1,1}\big|_{y_3=0,1} = -\mathrm{ReRi}\cos\theta,$$

$$\partial_{y_3}^4\rho_1 = \mathrm{RePeRi}\partial_{y_1}\bar{\rho}\left(\sin\theta\rho_1 - \cos\theta(y_3 - \frac{1}{2})\right), \quad \partial_{y_3}\rho_1\big|_{y_3=0,1} = 0, \quad \partial_{y_2}^2\rho_1\big|_{y_3=0,1} = 0.$$

The solutions are

$$v_{1,1} = \frac{2\gamma\cot(\theta)}{\mathrm{Pe}\partial_{y_1}\bar{\rho}}\frac{\sin(\gamma y_3)\sinh(\gamma(1-y_3)) - \sin(\gamma(1-y_3))\sinh(\gamma y_3)}{\sin(\gamma)+\sinh(\gamma)},$$

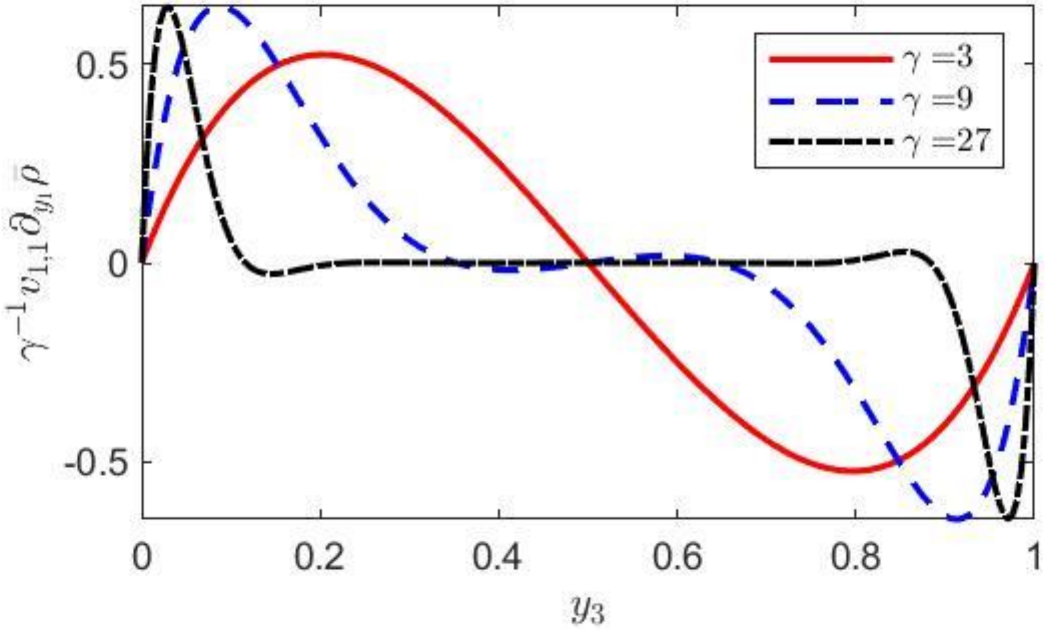
$$\rho_1 = \cot\theta\left(y_3 - \frac{1}{2} - \frac{\cos(\gamma(1-y_3))\cosh(\gamma y_3) - \cos(\gamma y_3)\cosh(\gamma(1-y_3))}{\gamma(\sin(\gamma)+\sinh(\gamma))}\right),$$

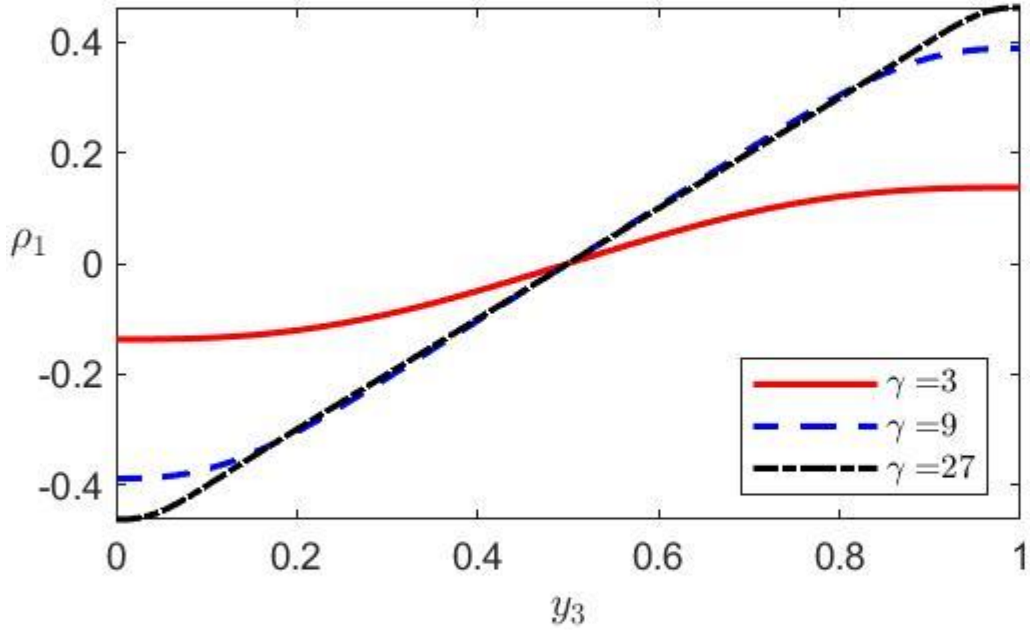
$$\gamma = \frac{1}{\sqrt{2}}\left(-\mathrm{RePeRi}\sin\theta\partial_{y_1}\bar{\rho}\right)^{\frac{1}{4}}.$$

$$v_{1,1} = \frac{2\gamma\cot(\theta)}{\mathrm{Pe}\partial_{y_1}\bar{\rho}} \frac{\sin(\gamma y_3)\sinh(\gamma(1-y_3)) - \sin(\gamma(1-y_3))\sinh(\gamma y_3)}{\sin(\gamma)+\sinh(\gamma)},$$

$$\rho_1 = \cot\theta\left(y_3 - \frac{1}{2} - \frac{\cos(\gamma(1-y_3))\cosh(\gamma y_3) - \cos(\gamma y_3)\cosh(\gamma(1-y_3))}{\gamma(\sin(\gamma)+\sinh(\gamma))}\right),$$

$$\gamma = \frac{1}{\sqrt{2}}\left(-\mathrm{RePeRi}\sin\theta\partial_{y_1}\bar{\rho}\right)^{\frac{1}{4}}.$$





γ^{-1} serves as an effective indicator of the boundary layer's thickness,
γ can be considered the characteristic velocity of the system.

Converge to Lubrication approximation in the limit

As $\partial_{y_1}\bar{\rho} \to 0$, we have

$$
\begin{aligned}
v_{1,1} = & -\frac{\mathrm{Re}\mathrm{Ri}y_3(y_3-1)(2y_3-1)\cos(\theta)}{12} \\
& -\frac{\mathrm{Pe}\mathrm{Re}^2\mathrm{Ri}^2 y\left(2(y_3(2y_3-7)+7)y_3^4-7y_3+3\right)\sin(2\theta)\partial_{y_1}\bar{\rho}}{40320}+\mathcal{O}\left((\partial_{y_1}\bar{\rho})^2\right), \\
\rho_1 = & \frac{\mathrm{Pe}\mathrm{Re}\mathrm{Ri}\left(1-2y_3^3(3y_3(2y_3-5)+10)\right)\cos(\theta)\partial_{y_1}\bar{\rho}}{1440}+\mathcal{O}\left((\partial_{y_1}\bar{\rho})^2\right).
\end{aligned}
$$

Effective equation

$$\partial_t \rho + \mathrm{Pe}\left(v_1 \partial_{y_1} \rho + v_3 \partial_{y_3} \rho\right) = \Delta \rho,$$

Taking the cross-sectional average on both side of the equation and utilizing the compressibility condition and non-flux boundary condition yields

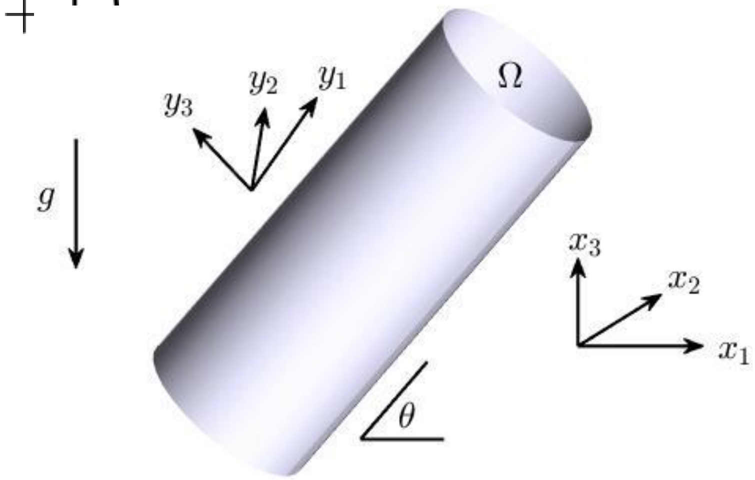
$$\partial_t \bar{\rho} + \mathrm{Pe}\partial_{y_1} \overline{v_1 \rho} = \partial_{y_1}^2 \bar{\rho}$$

Substituting the expansion of the velocity field and density field to the above equation yields

$$\partial_t \bar{\rho} = \partial_{y_1}\left(\kappa_{\mathrm{eff}} \partial_{y_1} \bar{\rho}\right),$$

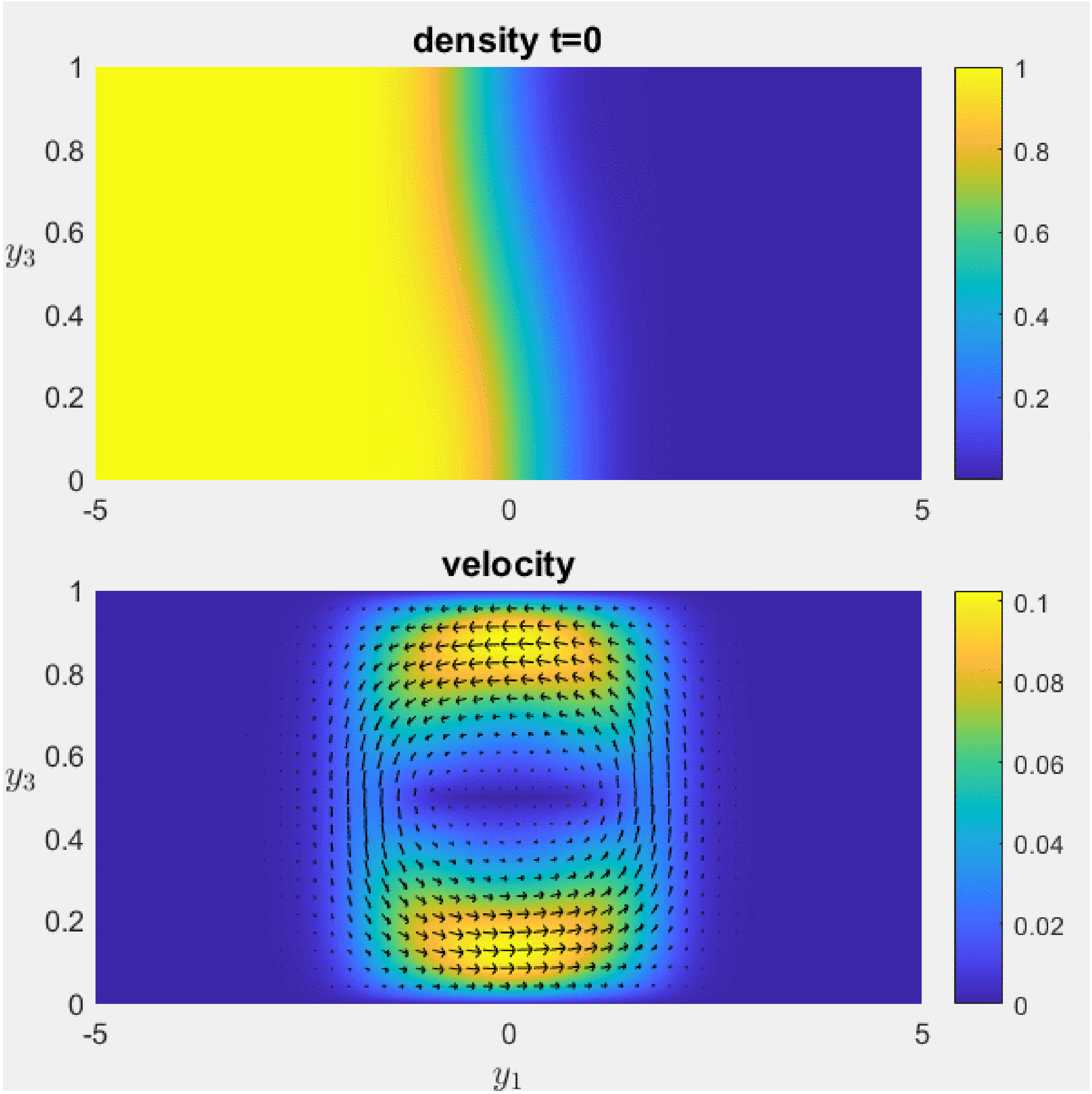
$$\kappa_{\mathrm{eff}} = 1 + \cot^2(\theta)\left(\frac{\sin(\gamma)\sinh(\gamma)}{(\sin(\gamma)+\sinh(\gamma))^2} + \frac{5(\cos(\gamma)-\cosh(\gamma))}{2\gamma(\sin(\gamma)+\sinh(\gamma))} + 1\right)$$

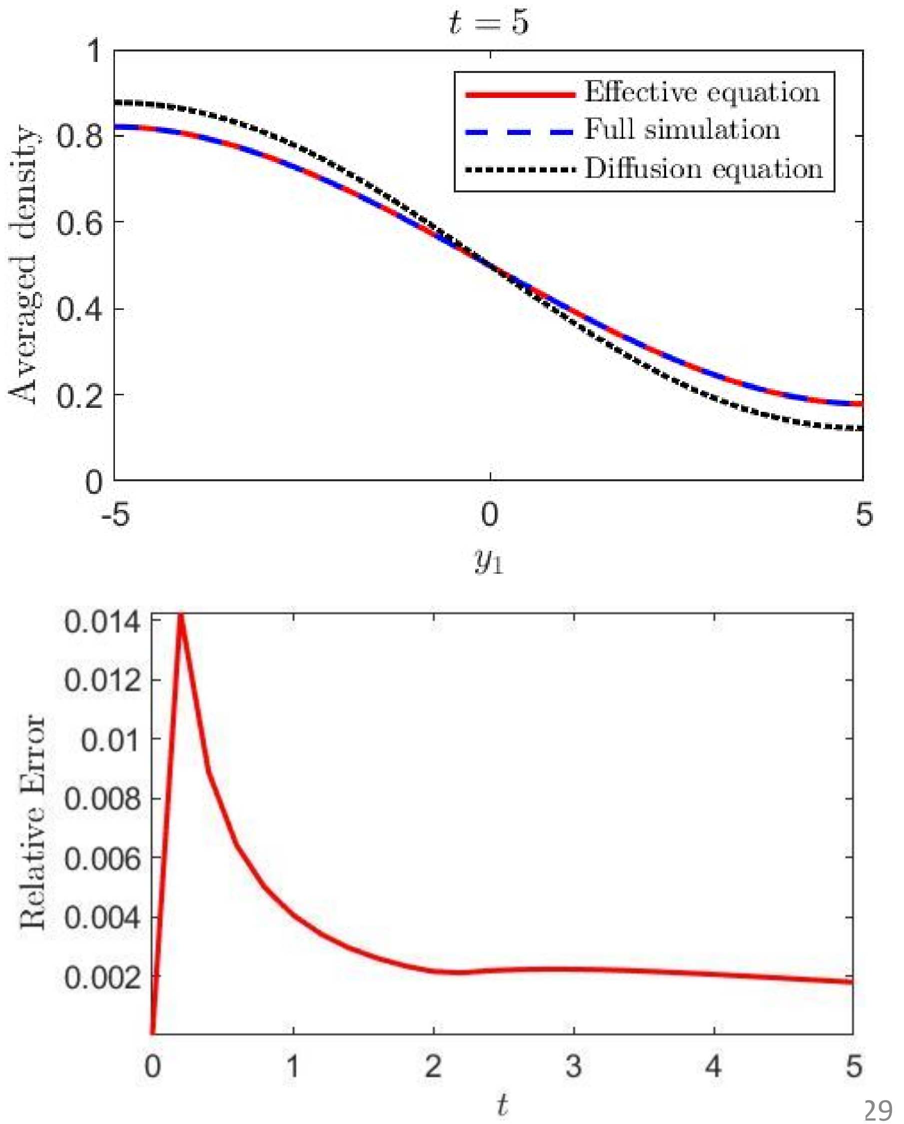
Dimension reduction: $(y_1, y_3, t) \rightarrow (y_1, t)$, (2+1)->(1+1)
For three dimension diffusion driven flow:$(y_1, y_2, y_3, t) \rightarrow (y_1, t)$, (3+1)-> (1+1)



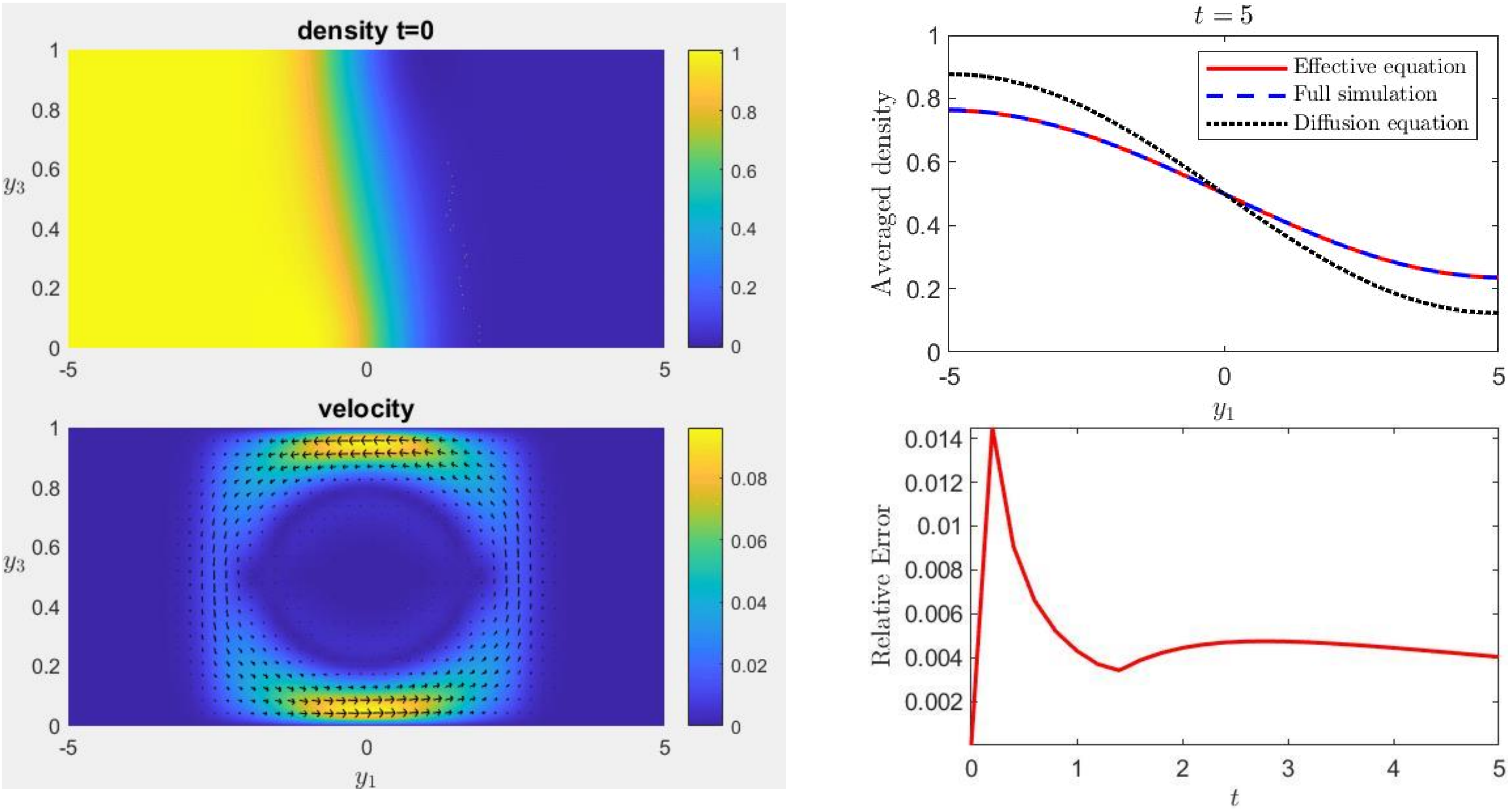
Numerical simulation

$$\mathrm{Re} = 1, \quad \mathrm{Ri} = 400, \quad \mathrm{Pe} = 40, \quad \mathrm{Sc} = 1, \quad \theta = \frac{\pi}{4}$$





$$\mathrm{Re} = 0.1, \quad \mathrm{Ri} = 50000, \quad \mathrm{Pe} = 100, \quad \mathrm{Sc} = 1, \quad \theta = \frac{\pi}{4}$$

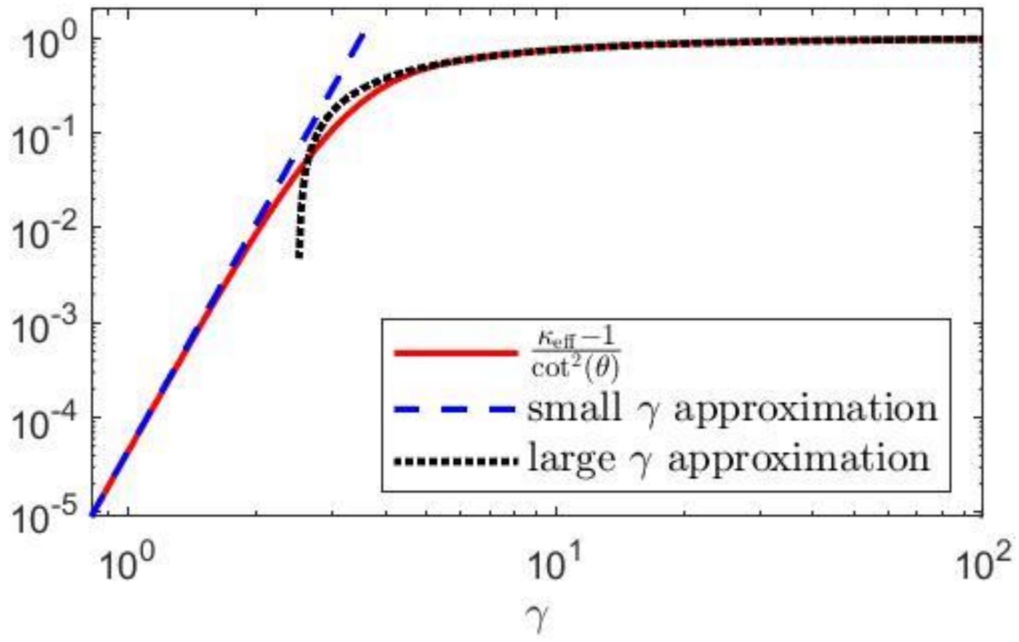


$$\partial_t \bar{\rho} = \partial_{y_1} \left(\kappa_{\text{eff}} \partial_{y_1} \bar{\rho} \right),$$

$$\kappa_{\text{eff}} = 1 + \cot^2(\theta) \left(\frac{\sin(\gamma)\sinh(\gamma)}{(\sin(\gamma)+\sinh(\gamma))^2} + \frac{5(\cos(\gamma)-\cosh(\gamma))}{2\gamma(\sin(\gamma)+\sinh(\gamma))} + 1 \right)$$

$$\kappa_{\text{eff}} = 1 + \cot^2(\theta) \left(\left(1 - \frac{5}{2\gamma} \right) + \frac{(2\gamma+5)\sin(\gamma) + 5\cos(\gamma)}{\gamma e^{\gamma}} + \mathcal{O}\left(\gamma^2 e^{-\gamma}\right) \right), \ \gamma \to \infty,$$

$$\kappa_{\text{eff}} = 1 + \cot^2(\theta) \left(\frac{\gamma^8}{22680} - \frac{2879\gamma^{12}}{4086482400} + \mathcal{O}\left(\gamma^{16}\right) \right), \quad \gamma \to 0.$$



The lower bound and upper bound of the effective diffusivity

$$1 \leqslant \kappa_{\text{eff}}(\gamma) < 1 + \cot^2(\theta)$$

Conclusion

1. We introduce a novel asymptotic expansion inspired by center manifold theory, which yields precise approximations for velocity, density, and pressure fields.
2. We derive a nonlinear effective equation for the density field, serving as a reduced model of the original system.
3. We find the lower bound and upper bound of the effective diffusivity induced by the diffusion-driven flow.

Future work:

1. Apply the method to other system involves the slow manifold.
2. More rigorous justification of the expansion, stability analysis.

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