Free Boundary Regularity and Support Propagation in Mean Field Games and Optimal Transport

Sebastian Munoz

Department of Mathematics
University of Chicago

Joint work with P. Cardaliaguet and A. Porretta
Research Center on Stability, Instability, and Turbulence
New York University Abu Dhabi

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General Setting

- We are interested in the following system of PDE:

$$\begin{cases} -u_t + \frac{1}{2}|Du|^2 = f(m(x,t)) & \text{in } \Omega \times (0,T), \\ m_t - \operatorname{div}(mDu) = 0 & \text{in } \Omega \times (0,T), \\ m(x,0) = m_0(x) & \text{in } \Omega, \end{cases}$$

- m_0 is a probability density, $\Omega \subset \mathbb{R}^d$ or $\Omega = \mathbb{T}^d$, and $f' > 0$.
- The condition at $t = T$ can be either $u(x,T) = g(m(x,T))$ (final cost problem), where $g'(m) > 0$, or $m(x,T) = m_T(x)$ (planning problem).

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- This PDE system arises in mean field games, optimal transport, among others.
- From modeling perspective, (u, m) is the Nash equilibrium of a differential game with infinitely many players, where

$$m(x,t) \equiv \text{density of players at time } t, \text{ position } x.$$

$$u(x,t) \equiv \text{optimal cost for generic player at time } t$$

Specifically, $u(x,t)$ equals

$$\inf_{\gamma \in W^{1,\infty}([t,T];\Omega),\, \gamma(t)=x} \int_t^T \left(\frac{1}{2}|\gamma'(s)|^2 + f(m(s,\gamma(s))) \right) ds + u(T,\gamma(T)). \quad (1)$$

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- Until recently, only weak solutions to (2) with limited regularity were known to exist (e.g. see Cardaliaguet and Graber (2014), Cardaliaguet and Porretta (2020)).
- However, there are strong heuristic reasons to expect the solution of (2) to be smooth, at least in the set $\{m > 0\}$.

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$$\begin{cases} -u_t + \frac{1}{2}|Du|^2 = f(m(x,t)), \\ m_t - \operatorname{div}(mDu) = 0, \end{cases}$$

Idea of P.-L. Lions: write $m = f^{-1}(-u_t + \frac{1}{2}|Du|^2)$, and eliminate m from system, obtaining second-order quasilinear equation in space-time:

$$-\operatorname{Tr}(A(D_{xt}u)D^2_{xt}u) = 0,$$

where, $A \geq 0$, and $A > 0 \Leftrightarrow mf'(m) > 0$.
So we expect smooth solutions on $\{m > 0\}$.

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Guiding questions

From the preceding analysis, two natural questions arise:

1. If the initial density m_0 is smooth and strictly positive, does the system have a smooth solution (u, m) such that $m > 0$ everywhere?
2. If the initial density m_0 is not everywhere positive, is the solution (u, m) still smooth in the set $\{m > 0\}$? Moreover, what is the shape and regularity of the free boundary $\partial\{m > 0\}$?

For Question 1, there are a number of results, which we discuss first. Our new results are on Question 2, about which nothing was known so far.

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Known regularity results

Theorem (S. Munoz (2020))

If $m_0 > 0$ and $\lim_{m \to 0^+} f(m) = -\infty$, then the solution (u, m) is globally smooth, for arbitrary dimensions $d > 0$.

Theorem (N. Mimikos and S. Munoz (2022))

Assume dimension $d = 1$.

- *If $m_0 > 0$, then the solution (u, m) is globally smooth.*
- *Assume that $m_0^{-1} \in L^p$ for an adequately large $0 < p < \infty$. Then m becomes instantly positive for times $0 < t < T$, and (u, m) is smooth on $(0, T)$.*

These results tell us nothing unless $m_0 > 0$ almost everywhere!

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Summary

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In summary, we know so far that the answer is yes if

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- To fix ideas, assume $u(\cdot, T) \equiv 0$.
- m_0 is a bump-like function. For some $\beta > 0$, $C_0 > 0$,

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Model cases

There are two distinct model cases, depending on f.

(i) $f(m) = \log(m)$. Leads to infinite speed of propagation of the support. So $m > 0$ for positive times. There is no free boundary.

(ii) $f(m) = m^\theta$, for $\theta > 0$. Leads to finite speed of propagation. Besides regularity of (u, m), one must study the regularity of the free boundary $\partial\{m > 0\}$.

We will focus here on (ii).

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Self-similar Solutions

When $f(m) = m^\theta$, the prototype for our results is the following explicit solution:

$$m(x,t) = t^{-\alpha}\left(R - \alpha(1-\alpha)\frac{x^2}{2t^{2\alpha}}\right)_+^{1/\theta}, \quad \alpha = \frac{2}{2+\theta}. \qquad (3)$$

- $m(x,0) \equiv$ Dirac mass at $x = 0$.
- Reminiscent of Barenblatt solution for the porous medium equation.
- Free boundary is a pair of convex curves spreading outward.
- $m(t) \in C^{\frac{1}{\theta}}(\mathbb{R})$ for $t > 0$, $f(m) = m^\theta \in W^{1,\infty}(\mathbb{R})$, and $f(m)_{xx} = -C(t) < 0$ inside the support.

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$$m(x,t) = t^{-\alpha}\left(R - \alpha(1-\alpha)\frac{x^2}{2t^{2\alpha}}\right)_+^{1/\theta}, \quad \alpha = \frac{2}{2+\theta}. \qquad (3)$$

- $m(x,0) \equiv$ Dirac mass at $x = 0$.
- Reminiscent of Barenblatt solution for the porous medium equation.
- Free boundary is a pair of convex curves spreading outward.
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Characterization of the Free Boundary

Theorem

Assume that $f(m_0) \in W^{1,\infty}(\mathbb{R}) \cap C^{1,\alpha}((a,b))$, *and* $f(m_0)$ *is semi–convex. Then* (u,m) *is smooth in* $\{m > 0\}$, *and the free boundary is a pair of Lipschitz curves. That is,*

$$\{m > 0\} = \{(x,t) \in \mathbb{R} \times [0,T] : \gamma_L(t) < x < \gamma_R(t)\},$$

where $\gamma_L, \gamma_R \in W^{1,\infty}((0,T))$.

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Regularity of the Free Boundary

- In the previous result, the free boundary is only Lipschitz.
- To get more regularity, we add a non-degeneracy assumption satisfied by the self-similar solutions:

$$f(m_0)_{xx} \leq 0 \text{ near } \partial[a, b].$$

- This actually implies that $f(m_0)_x(a) > 0$ and $f(m_0)_x(b) < 0$: the non-degeneracy condition for the pressure in the porous medium equation.

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In addition to the previous assumptions, suppose that $f(m_0)_{xx} \leq 0$ *for* x *near* $\partial[a,b]$. *Then* m *is Hölder continuous up to the free boundary,* $\gamma_L, \gamma_R \in W^{2,\infty}(0,T)$,

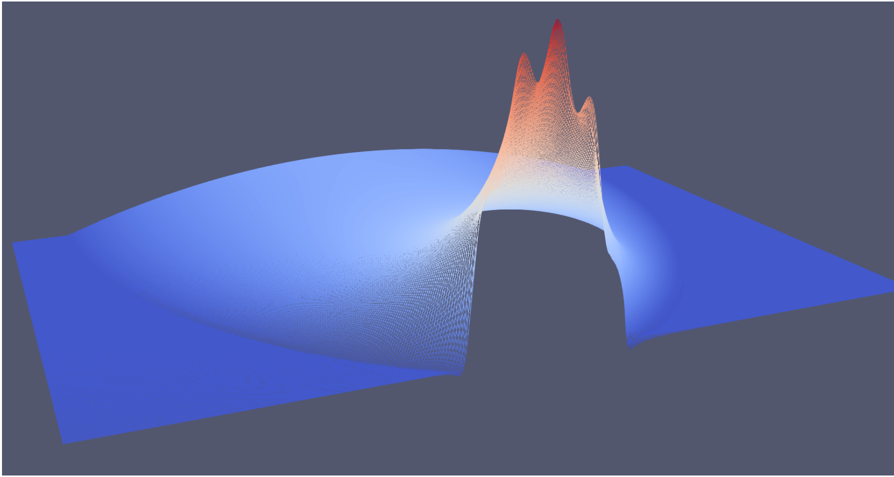
$$\ddot{\gamma}_L > 0, \ \dot{\gamma}_L < 0,$$

and

$$\ddot{\gamma}_R < 0, \ \dot{\gamma}_R > 0.$$

In other words, $\{m > 0\}$ *is convex, and is expanding outward with finite speed and a* $C^{1,1}$ *interface.*

Illustration of the Result



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Flow of Optimal Trajectories

- Key idea to study the free boundary: analyze the problem in Lagrangian coordinates.
- The function $\gamma(x,t) : [a,b] \times [0,T] \to \mathbb{R}$ defined by

$$\begin{cases} \gamma'(\cdot) = -u_x(\gamma(\cdot), \cdot) \\ \gamma(0) = x \end{cases}$$

  describes the optimal trajectories for a player with initial position $x \in [a,b]$.
- We expect free boundary to consist of the two trajectories starting at the endpoints of $\mathrm{spt}(m_0)$. That is, the curves $\gamma_L(t) = \gamma(a,t)$ and $\gamma_R(t) = \gamma(b,t)$.

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- The transformation

$$(x,t) \mapsto (\gamma(x,t),t)$$

is a change of coordinates which transforms the set $\{m>0\} \subset \mathbb{R} \times [0,T]$ into the rectangle $(a,b)\times(0,T)$.

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The Equation satisfied by the Flow of Trajectories

The flow of optimal trajectories $\gamma(x,t) : [a,b] \times [0,T] \to \mathbb{R}$ satisfies an elliptic equation:

$$\gamma_{tt} + \frac{\theta f(m_0)}{(\gamma_x)^{2+\theta}} \gamma_{xx} = \frac{f(m_0)_x}{(\gamma_x)^{1+\theta}}. \quad (4)$$

This equation is a fundamental tool. For instance, one can use (4) to prove

$$\|\gamma\|_{W^{1,\infty}(\mathbb{R}\times(0,T))} \le C.$$

This implies, in particular, that $|\gamma_t| \le C$, so $\gamma_L = \gamma(a,t)$, $\gamma_R = \gamma(b,t)$ are Lipschitz curves.

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- We saw that the bound on $|\gamma_t|$ is immediately useful.
- We now explain how the bound

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- Thus, if $\gamma_x \leq C$,

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Regularity results

- To obtain deeper results, such as $C^{1,1}$ regularity of the free boundary, and the Hölder continuity of m up to the interface, we study m in Lagrangian coordinates, namely the function $m(\gamma(x,t),t)$.
- One can show that $v(x,t) = f(m(\gamma(x,t),t))$ satisfies the divergence form equation

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