Norm inflation for solutions of semi-linear one-dimensional Klein-Gordon equations

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1. Motivation

Consider a solution $(t,x) \to u(t,x)$ of the semi-linear Klein-Gordon equation

(KG)
$$(\partial_t^2 - \Delta + 1)u = \underbrace{N(u, \partial_t u, \nabla_x u)}_{\text{vanishing at order } \geq 2 \text{ at } 0}$$
$$u\big|_{t=0} = \epsilon u_0$$
$$\partial_t u\big|_{t=0} = \epsilon u_1.$$

Assume $u_0, u_1 \in C_0^\infty(\mathbb{R}^d)$.

- If $d \geq 2$, for $\epsilon \ll 1$, solutions are global.
- If $d = 1$ there are examples of initial data and nonlinearities for which solutions blow-up in finite time (Yordanov, Keel-Tao).

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- If $d = 1$ there are examples of initial data and nonlinearities for which solutions blow-up in finite time (Yordanov, Keel-Tao).

Question: If $d = 1$, can one find an *optimal* condition (C) on the nonlinearity such that, for $\epsilon \ll 1$,

$(C) \Leftrightarrow$ global existence for (KG) holds and solutions satisfy the dispersive estimate $\|u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}})$?

Assume $d = 1$ and $N(u, \partial_t u, \partial_x u) = \underbrace{P(u; \partial_t u, \partial_x u)}_{\text{hom. of order 3}}$.

Null condition: Decompose

$$P(T; Z_1, Z_2) = \sum_{k=0}^{3} P_k(T; \underbrace{Z_1, \quad Z_2}_{\text{hom. of order } k}).$$

For $y \in]-1, 1[$, set $\omega_0(y) = \frac{1}{\sqrt{1-y^2}}$, $\omega_1(y) = \frac{-y}{\sqrt{1-y^2}}$ and define

$$p_k(\omega_0(y), \omega_1(y)) = P_k(1; \omega_0(y), \omega_1(y)).$$

Definition

One says that P satisfies the null condition if and only if

$$\phi(y) \stackrel{\text{def}}{=} (p_1 + 3p_3)(\omega_0(y), \omega_1(y)) \equiv 0.$$

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Theorem (D. 2001)

If the null condition holds, then for $\epsilon \ll 1$, solutions of (KG) with nonlinearity $N = P(u; \partial_t u, \partial_x u)$ are global and

$$\|u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}}), t \to +\infty.$$

Remarks: • The result holds as well for quasi-linear equations, with quadratic and cubic nonlinearities.

• One does not need to assume the data smooth and compactly supported: the result extends to initial data $(u_0, u_1) \in H^{s+1} \times H^s$ $(s \gg 1)$ with in addition $xu_0 \in H^1$, $xu_1 \in L^2$ (Stingo).

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2. Statement of the main result

Notation: For $f_0, g_0 \in \mathcal{S}(\mathbb{R})$ and $y \in]-1, 1[$, define

$$\Gamma(y) = \frac{1}{8\pi}(1-y^2)^{-1}\left|\hat{f}_0(\omega_1(y)) - i\sqrt{1-y^2}\hat{g}_0(\omega_1(y))\right|^2.$$

Define T_* by $\frac{1}{T_*} = \sup_{y\in]-1,1[}(\Gamma(y)\phi(y))$.

- If the null condition holds, $T_* = +\infty$.
- If $T_* \in]0, +\infty[$, for any $A < T_*$, the solution of (KG) with initial conditions $u|_{t=0} = \epsilon f_0$, $\partial_t u|_{t=0} = \epsilon g_0$, exists over the interval $[0, e^{\frac{A}{\epsilon^2}}]$ and satisfies up to that time

$$\|u(t,\cdot)\|_{L^\infty} = O(\epsilon t^{-\frac{1}{2}}).$$

Goal: Show that this no longer holds for a time $T(\epsilon) \sim e^{\frac{T_*}{\epsilon^2}}$ for convenient f_0, g_0 when the null condition does not hold.

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Assumption: The maximum $\frac{1}{T_*}$ of $y \to \Gamma(y)\phi(y)$ is reached at a unique point $y_0 \in]-1,1[$ and for some $\kappa_0 \in \mathbb{N}^*$, one has

$$\partial_y^k(\Gamma(y)\phi(y))|_{y=y_0} = 0 \text{ if } k \leq 2\kappa_0 - 1$$
$$\partial_y^{2\kappa_0}(\Gamma(y)\phi(y))|_{y=y_0} < 0.$$

Fix $\delta' > 0$, $\gamma \geq 2(\delta' + 2)$ and for $\epsilon > 0$ small, set

$$\epsilon' = \epsilon^{-\frac{2+\gamma+2\delta'}{1+2\delta'}} \exp\left[-\frac{T_*}{\epsilon^2(1+2\delta')}\right] \ll 1.$$

Denote by $T(\epsilon)$ some large time close to $e^{\frac{T_*}{\epsilon^2}}$ of the form

$$T(\epsilon) = e^{\frac{T_*}{\epsilon^2}}[1 - \epsilon' + O(\epsilon'^2)] < e^{\frac{T_*}{\epsilon^2}}.$$

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Theorem

Fix small $\theta > 0, c > 0$. For large enough $s_0 \in \mathbb{N}$, there are $\epsilon_0 > 0, C > 0$ such that for any $\epsilon \in]0, \epsilon_0[$, there are functions $x \to f(x, \epsilon)$, $x \to g(x, \epsilon)$ satisfying

$$\|f(\cdot,\epsilon)\|_{H^{s_0+1}} + \|g(\cdot,\epsilon)\|_{H^{s_0}} + \|xf(\cdot,\epsilon)\|_{H^1} + \|xg(\cdot,\epsilon)\|_{L^2} \leq C\epsilon^{1-\theta}$$

so that the unique solution of $(\partial_t^2 - \partial_x^2 + 1)u = P(u, \partial_t u, \partial_x u)$ *with initial data*

$$u|_{t=0} = \epsilon(f_0(x) + f(x,\epsilon)),\ \partial_t u|_{t=0} = \epsilon(g_0(x) + g(x,\epsilon))$$

is defined on $[0, T(\epsilon)]$ *and satisfies*

$$\|u(T(\epsilon),\cdot)\|_{L^\infty} + \|\partial_t u(T(\epsilon),\cdot)\|_{L^\infty} = \frac{\epsilon}{\sqrt{T(\epsilon)}} I(\epsilon)$$
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with $I(\epsilon) \geq cT(\epsilon)^{\frac{1}{2}-c}, J(\epsilon) \geq cT(\epsilon)^{\frac{1}{2}-\frac{1}{4\kappa_0}-c}$.

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Remark: The theorem provides inflation of norms (in comparison with estimates holding true when the null condition is satisfied) and *not* blowing up. This property is stable under perturbations of the nonlinearity at the difference with blowing-up.

Toy example:
The ODE $\dot{y} = \frac{y^3}{2}$ with initial condition $y(0) = \epsilon > 0$ has the solution $y(t) = \frac{\epsilon}{\sqrt{1 - t\epsilon^2}}$ that blows-up at time $\frac{1}{\epsilon^2}$.
The solution of the perturbed equation $\dot{y} = \frac{1}{2}y^3(1 - y^2)$ with initial condition $y(0) = \epsilon$ is *global*. Though, at a time $t_\epsilon = \epsilon^{-2}(1 - c(\epsilon))$ for some convenient $c(\epsilon) \sim \epsilon^\delta$ ($\delta > 0$ small), one has $|y(t_\epsilon)| \sim \epsilon^\delta \gg \epsilon$, i.e. inflation of the size of the solution.

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References: Cazenave-Han-Martel, Cazenave-Martel-Zhao have constructed solutions to

$$(*) \qquad (D_t - \frac{1}{2}D_x^2)v = \alpha|v|^2 v, \quad \alpha \in \mathbb{C} - \mathbb{R}$$

that blow-up at time $t = 1$, and are global in the past $t < 1$.

We use these ideas, except that the preceding invariance does not hold for Klein-Gordon: this equation is essentially of the form

$$(D_t - \sqrt{1 + D_x^2})v = \underbrace{M^{(1)}(v, v, v)}_{\text{non char.}} + \underbrace{M^{(2)}(v, v, \bar{v})}_{\text{char.}} + \underbrace{M^{(3)}(v, \bar{v}, \bar{v})}_{\text{non char.}} + \underbrace{M^{(4)}(\bar{v}, \bar{v}, \bar{v})}_{\text{non char.}}.$$

To reduce to an equation with a structure similar to $(*)$, one needs to eliminate the non-characteristic terms by normal forms. This generates quintic perturbations in RHS, and thus one may expect only conclusions that are stable under such perturbations (cf. Toy Model above).

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3. Strategy of proof: Approximate solution

Define

$$u_{\rm app}(t,x) = {\rm Re}\Big[\sum_{\substack{\ell=1\\ \ell\ {\rm odd}}}^{N} \epsilon^{2-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} a_{\ell,1}(s,y,\epsilon)$$

$$+ \sum_{\substack{\ell=3\\ \ell\ {\rm odd}}}^{N} \sum_{\substack{3\le q\le N\\ q\ {\rm odd}}} \epsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{iqt\varphi(y)} a_{\ell,q}(s,y,\epsilon)\Big]\Big|_{\substack{y=x/t\\ s=\epsilon^2 \log t}}$$

where $\varphi(y) = \sqrt{1-y^2}$ and $a_{\ell,q}$ satisfies for $0 < \delta \ll 1$, any N

$$(*) \quad |a_{\ell,q}(s,y,\epsilon)| \le C(T_* - s + |y-y_0|^{2\kappa_0})^{-\frac{\ell}{2}-\delta(\ell-1)}(1-|y|)^N.$$

Remark: If $t \to e^{\frac{T_*}{\epsilon^2}}$ (i.e. $s \to T_*$), each term in the expansion blows-up. Because of that, one takes $t \le T(\epsilon) < e^{\frac{T_*}{\epsilon^2}}$. Then $u_{\rm app}(t,x)$ is a sum of "small terms".

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3. Strategy of proof: Approximate solution

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$$u_{\mathrm{app}}(t,x) = \mathrm{Re}\Big[\sum_{\substack{\ell=1\\ \ell \text{ odd}}}^{N} \epsilon^{2-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} a_{\ell,1}(s,y,\epsilon) + \sum_{\substack{\ell=3\\ \ell \text{ odd}}}^{N} \sum_{\substack{3\le q\le N\\ q \text{ odd}}} \epsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{iqt\varphi(y)} a_{\ell,q}(s,y,\epsilon)\Big]\Big|_{\substack{y=x/t\\ s=\epsilon^2 \log t}}$$

where $\varphi(y) = \sqrt{1-y^2}$ and $a_{\ell,q}$ satisfies for $0<\delta \ll 1$, any N

$$(*) \quad |a_{\ell,q}(s,y,\epsilon)| \le C(T_* - s + |y-y_0|^{2\kappa_0})^{-\frac{\ell}{2}-\delta(\ell-1)}(1-|y|)^N.$$

Remark: If $t \to e^{\frac{T_*}{\epsilon^2}}$ (i.e. $s \to T_*$), each term in the expansion blows-up. Because of that, one takes $t \le T(\epsilon) < e^{\frac{T_*}{\epsilon^2}}$. Then $u_{\mathrm{app}}(t,x)$ is a sum of "small terms".

The coefficient $a_{1,1}$ solves an ODE

$$\omega_0(y)\partial_s a_{1,1}(s,y) = \frac{1}{2}(\phi(y) + i\psi(y))|a_{1,1}(s,y)|^2 a_{1,1}(s,y)$$

with initial data $a_{1,1}(0,y)$ given explicitly in terms of the data. This implies the above $O((T_* - s + |y-y_0|^{2\kappa_0})^{-\frac{1}{2}}(1-|y|)^N)$ upper bound and the fact that $a_{1,1}(s,y_0)$ blows-up as $s \to T_*$.
The remainder

$$r_{\mathrm{app}} = (\partial_t^2 - \partial_x^2 + 1)u_{\mathrm{app}} - P(u_{\mathrm{app}}, \partial_t u_{\mathrm{app}}, \partial_x u_{\mathrm{app}})$$

may be written as

$$\begin{aligned} r_{\mathrm{app}} = 2\mathrm{Re}\Big[&\sum_{\substack{\ell=N+4 \\ \ell \text{ odd}}}^{3N} \epsilon^{6-\ell} t^{-\frac{\ell}{2}} e^{it\varphi(y)} d_{\ell,1}(s,y,\epsilon) \\ &+ \sum_{\substack{\ell=N+2 \\ \ell \text{ odd}}}^{3N} \sum_{\substack{3\le q\le N \\ q \text{ odd}}} \epsilon^{2q-\ell} t^{-\frac{\ell}{2}} e^{iqt\varphi(y)} d_{\ell,q}(s,y,\epsilon)\Big]\Big|_{\substack{y=x/t \\ s=\epsilon^2 \log t}} \end{aligned}$$

with $d_{\ell,q} = O((T_* - s + |y-y_0|^{2\kappa_0})^{-\frac{\ell}{2}-\delta(\ell-3)})$.

4. Construction of exact solution

Rewrite the second order equation as a first order system in terms of complex valued unknown $u_+, u_- = -\bar{u}_+$

$$(D_t - \sqrt{1 + D_x^2})u_+ = \text{ cubic terms in } u_+, u_-.$$

Set $v_\pm = u_\pm - u_\pm^{\mathrm{app}}$, where $u_\pm^{\mathrm{app}}$ is the complex unknown corresponding to the approximate solution u_{app} already constructed. One gets

$$\begin{aligned}(D_t - \sqrt{1 + D_x^2})v_+ = &\text{ trilinear terms in } (v_\pm, v_\pm, v_\pm)\\ &+ \text{trilinear terms in } (v_\pm, v_\pm, u_\pm^{\mathrm{app}})\\ &+ \text{trilinear terms in } (v_\pm, u_\pm^{\mathrm{app}}, u_\pm^{\mathrm{app}}).\end{aligned}$$

One wants to solve backwards this equation from $t = T(\epsilon)$ with zero final condition to $t = 0$.

In the right hand side, one has characteristic and noncharacteristic terms.

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Non-characteristic terms may be eliminated by normal form: one defines from $v_\pm$ a new unknown $w_\pm$ so that w_+ solves an equation

$$\begin{aligned}(D_t - \sqrt{1+D_x^2})w_+ = &\text{ trilinear terms in } (w_{i_1}, w_{i_2}, w_{i_3})\\ &+ \text{ trilinear terms in } (w_{i_1}, w_{i_2}, u_{i_3}^{\mathrm{app}})\\ &+ \text{ trilinear terms in } (w_{i_1}, u_{i_2}^{\mathrm{app}}, u_{i_3}^{\mathrm{app}}) + \mathcal{R}\end{aligned}$$

where $i_j = \pm$ and in all expressions $i_1 + i_2 + i_3 = 1$ (i.e. nonlinearities are characteristic) and $\mathcal{R}$ is a remainder. Essentially, one may consider as a model

$$(**) \qquad (D_t - \sqrt{1+D_x^2})w_+ = |w_+|^2 w_+ + |u_+^{\mathrm{app}}|^2 w_+ + \mathcal{R}.$$

The remainder, which comes from the error in the construction of the approximate solutions, satisfies bounds

$$\|\mathcal{R}(t,\cdot)\|_{H^{s_0}} \leq C t^{-2} \epsilon^{N_0} (T_* - \epsilon^2 \log t)^{N_0}$$
$$\|L_+\mathcal{R}(t,\cdot)\|_{L^2} \leq C t^{-1} \epsilon^{N_0} (T_* - \epsilon^2 \log t)^{N_0}$$

with $N_0 \gg 1$ and $L_+ = x + t\frac{D_x}{\langle D_x \rangle}$.

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Proposition (Bootstrap argument)

Let s_0, ρ_0 be large enough integers, $\theta > 0$ small and N_0, N_1 be integers satisfying $N_0 \gg N_1 \gg 1$. One may choose successively constants $A_0, A_1, B, \epsilon_0 > 0$ such that for $\epsilon \in]0, \epsilon_0]$, if the solution to $(**)$ with final condition $w_+(T(\epsilon), \cdot) = 0$ satisfies on some interval $[T, T(\epsilon)]$

$$\begin{aligned} \|w_+(t,\cdot)\|_{H^{s_0}} &\leq A_0 \epsilon^{2-\theta} (T_* - \epsilon^2 \log t)^{N_0} \\ \|L_+ w_+(t,\cdot)\|_{L^2} &\leq A_1 \epsilon^{2-\theta} (T_* - \epsilon^2 \log t)^{N_1} \\ \|w_+(t,\cdot)\|_{W^{\rho_0,\infty}} &\leq \frac{B}{\sqrt{t}} \epsilon^{2-\theta}, \end{aligned}$$

then actually the same estimates hold true on the same time interval with (A_0, A_1, B) replaced by $(A_0/2, A_1/2, B/2)$.

Proof of second estimate: Set $F_+ = L_+(|w_+|^2 w_+ + |u_+^{\mathrm{app}}|^2 w_+)$ so that one has $(D_t - \sqrt{1+D_x^2})L_+ w_+ = F_+ + L_+\mathcal{R}$. By energy inequality, for $t < T(\epsilon)$

$$\|L_+ w_+(t,\cdot)\|_{L^2} \leq C_0 \int_t^{T(\epsilon)} [\underbrace{\|u_+^{\mathrm{app}}(\tau,\cdot)\|_{L^\infty}^2}_{O(\epsilon^2\tau^{-1}(T_* - \epsilon^2\log\tau)^{-1})} + \|w_+(t,\cdot)\|_{L^\infty}^2]\|L_+ w_+(\tau,\cdot)\|_{L^2}\, d\tau + \int_t^{T(\epsilon)} \|L_+\mathcal{R}(t,\cdot)\|_{L^2}.$$

By Gronwall

$$\|L_+ w_+(t,\cdot)\|_{L^2} \leq K \int_t^{T(\epsilon)} \left(\frac{T_* - \epsilon^2\log t}{T_* - \epsilon^2\log\tau}\right)^{C_0} \underbrace{\|L_+\mathcal{R}(\tau,\cdot)\|_{L^2}}_{O(\tau^{-1}\epsilon^{N_0}(T_* - \epsilon^2\log\tau)^{N_0})} d\tau.$$

Using the a priori assumption on $\mathcal{R}$, one gets

$$\|L_+ w_+(t,\cdot)\|_{L^2} \leq \frac{A_1}{2}\epsilon^{2-\theta}(T_* - \epsilon^2\log t)^{N_1}.$$

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