On the long time behavior of equilibria in a Kuramoto Mean Field Game

joint work with A. Cesaroni

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The Kuramoto model

N coupled oscillators with phase X^i governed by

$$\dot{X}_t^i = \omega_i + \sum_{j=1}^{N} \Gamma_{ij}(X_t^i - X_t^j) \qquad\qquad i = 1, \ldots, N$$

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In the symmetric (Mean-Field) case,

$$\dot{X}_t^i = \omega + \frac{\kappa}{N} \sum_{j=1}^{N} \sin(X_t^i - X_t^j) \qquad i = 1, \dots, N$$

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In the symmetric stochastic Mean-Field case,

$$dX_t^i = \frac{\kappa}{N} \sum_{j=1}^{N} \sin(X_t^i - X_t^j)dt + dB_t^i \qquad i = 1, \dots, N$$

if $\kappa < \kappa_c$: incoherence
if $\kappa > \kappa_c$: synchronization

A Kuramoto Mean Field Game

Assume - as in [Carmona, Cormier, Soner 2022] (and [Carmona, Graves 2020]) - that each oscillator can control his own phase X_t^i

$$\mathrm{d}X_t^i = v_t^i \mathrm{d}s + \mathrm{d}B_t^i$$

to minimize the cost

$$J^i(v) = \mathbb{E} \int_0^T \frac{1}{2}|v_t^i|^2 + \frac{\kappa}{N} \sum_j \sin^2\left(\frac{X_t^i - X_t^j}{2}\right) \mathrm{d}t + u_T(X_T),$$

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Assume - as in [Carmona, Cormier, Soner 2022] (and [Carmona, Graves 2020]) - that a *typical* oscillator can control his own phase X_t:

$$dX_t = v_t dt + dB_t$$

to minimize the cost ($m_t^N = \sum_{j=1}^N \delta_{X_t^j}$)

$$J(v) = \mathbb{E} \int_0^T \frac{1}{2}|v_t|^2 + \kappa \int_{-\pi}^{\pi} \sin^2\left(\frac{X_t - y}{2}\right) m_t^N(\mathrm{d}y)\, \mathrm{d}t + u_T(X_T),$$

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In the large population limit, $m_t^N \sim m_t$.
Interesting questions regarding long-time incoherence or synchronization

- For all $\kappa > 0$, $\bar{m} = \mathrm{d}x/(2\pi)$ is a stationary equilibrium.
- For all $0 < \kappa < \kappa_c$, $\bar{m}$ is the unique stationary equilibrium and it is (locally) stable.
- For all $\kappa > \kappa_c$, there are (infinitely many) nonuniform stationary equilibria, which converge to Dirac measures as $k \to \infty$, but nothing is known on their stability.

From a PDE perspective, the problem is described by

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2}|u_x|^2 = \kappa \int_{-\pi}^{\pi} \sin^2\left(\frac{x-y}{2}\right) m(t,y)dy & =: f(m(t),x) \\ m_t - m_{xx} - (mu_x)_x = 0 \qquad \text{on } \mathbb{R}\times(0,T) \\ m(x,0) = m_0(x), \qquad u(x,T) = u_T(x) \end{cases}$$

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For even solutions, the system becomes

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2}|u_x|^2 = -\kappa \cos x \int_{-\pi}^{\pi} \cos(y) m(t,y)dy \\ m_t - m_{xx} - (mu_x)_x = 0 \end{cases}$$

Uniqueness and stability in MFG

global long-time stability $\longleftrightarrow$ uniqueness of solutions

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Known uniqueness regimes:

- Lasry-Lions monotonicity
  [Lasry-Lions '06], [Cardaliaguet-L-L-Porretta '12, '13], [Cardaliaguet-Delarue-L-L '15], ...

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- Displacement monotonicity
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  [Lions], [Bardi-Fischer '17], [Ambrose '16, ...], ...

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- [Mészáros, Graber '22], [Cecchin, Conforti '23]

Stability without monotonicity: [Bardi, Kouhkouh '23]

Lasry-Lions monotonicity

$$\int (f(x,m_1)-f(x,m_2))(m_1-m_2) \geq \alpha \int (m_1-m_2)^2 \qquad \forall m_1, m_2 \in L^2(\mathbb{R}^d)\cap\mathcal{P}(\mathbb{R}^d)$$

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Uniqueness: be $(u_1, m_1), (u_2, m_2)$ two solutions,
test HJ by $m_2 - m_1$ and FP by $u_2 - u_1$ to get $\qquad (D = \partial_x)$

$$\begin{aligned} -\frac{\mathrm{d}}{\mathrm{d}t}\int (u_1-u_2)(m_1-m_2) \geq & \int m_2\left[\frac{|Du_1|^2}{2}-\frac{|Du_2|^2}{2}-Du_2D(u_1-u_2)\right]+ \\ & \int m_1\left[\frac{|Du_2|^2}{2}-\frac{|Du_1|^2}{2}-Du_1D(u_2-u_1)\right]+ \\ & \int [f(x,m_1)-f(x,m_2)][m_1-m_2]. \end{aligned}$$

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$$\begin{aligned}
-\frac{\mathrm{d}}{\mathrm{d}t}\int (u_1 - u_2)(m_1 - m_2) \geq & \int m_2 \left[\frac{|Du_1|^2}{2} - \frac{|Du_2|^2}{2} - Du_2 D(u_1 - u_2)\right] + \\
& \int m_1 \left[\frac{|Du_2|^2}{2} - \frac{|Du_1|^2}{2} - Du_1 D(u_2 - u_1)\right] + \\
& \int [f(x,m_1) - f(x,m_2)][m_1 - m_2].
\end{aligned}$$

By uniform convexity of $|\cdot|^2$, and α-monotonicity of f,

$$\int_0^T \int (m_1 + m_2)|Du_1 - Du_2|^2 + \alpha \int (m_1 - m_2)^2 \leq \int (u_1 - u_2)(m_1 - m_2)\Big|_T^0$$

$$\alpha \int (m_1 - m_2)^2 \leq \int (u_1 - u_2)(m_1 - m_2)\Big|_T^0 = 0$$

$\Rightarrow m_1 \equiv m_2$ (that implies $u_1 \equiv u_2$).

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Stability; be $(u_1, m_1), (u_2, m_2)$ two solutions with different initial/final data. For every t_1, t_2,

$$\int_{t_1}^{t_2} \int (m_1 + m_2)|Du_1 - Du_2|^2 + \alpha(m_1 - m_2)^2 \leq \int (u_1 - u_2)(m_1 - m_2)\Big|_{t_2}^{t_1}$$

By the strong maximum principle:

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$$\overset{\text{Young}}{\leq} \sum_{i=1,2} \int (u_1 - u_2)^2 + (m_1 - m_2)^2(t_i)$$

$$\overset{\text{Poincaré}}{\lesssim} \sum_{i=1,2} \int (Du_1 - Du_2)^2 + (m_1 - m_2)^2(t_i),$$

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$$\overset{\text{Poincaré}}{\lesssim} \sum_{i=1,2} \int (Du_1 - Du_2)^2 + (m_1 - m_2)^2(t_i),$$

which reads, setting $\Phi(t) = \int |Du_1 - Du_2|^2 + (m_1 - m_2)^2\,(t)$,

$$\int_{t_1}^{t_2} \Phi(t)dt \lesssim \Phi(t_1) + \Phi(t_2) \qquad \forall t_1, t_2 \in [0, T].$$

Assuming finally that $||m_i||_\infty, ||Du_i||_\infty \leq C$, then $\Phi(0), \Phi(T)$ are bounded uniformly in T, and $\exists \delta, K > 0$ such that

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Consequences:

- "exponential" convergence of $Du_1 - Du_2$, $m_1 - m_2$ to zero.
- if (u_2, m_2) is stationary, (u_1, m_1) convergences to the stationary equilibrium (u_2, m_2)
- stationary equilibria are unique

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One can to prove that, on (t_1, t_2),

$$\int_{t_1}^{t_2}\int (m_1 - m_2)^2 dt \leq \int (m_1 - m_2)^2(t_1) + \gamma \int_{t_1}^{t_2}\int (Du_1 - Du_2)^2 dt.$$

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We plug this into

$$\int_{t_1}^{t_2}\int |Du_1 - Du_2|^2 + \alpha(m_1 - m_2)^2 \le \int (u_1 - u_2)(m_1 - m_2)\Big|_{t_2}^{t_1}$$

to get

$$\int_{t_1}^{t_2}\int \frac{1}{2}|Du_1 - Du_2|^2 + (\alpha + \frac{1}{2\gamma})(m_1 - m_2)^2$$
$$\lesssim \sum_{i=1,2}\int (Du_1 - Du_2)^2 + (m_1 - m_2)^2(t_i),$$

which is again of the form

$$\int_{t_1}^{t_2} \Phi(t)dt \lesssim \Phi(t_1) + \Phi(t_2) \qquad \forall t_1, t_2 \in [0, T].$$

provided that

$$\int (f(x, m_1) - f(x, m_2))(m_1 - m_2) > -\frac{1}{2\gamma}(m_1 - m_2)^2,$$

which is a sort of “mild” anti-monotonicity.

Back to Kuramoto

Goal: long time behaviour of

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2}|u_x|^2 = -\kappa \cos x \int_{-\pi}^{\pi} \cos(y) m(t,y) dy \\ m_t - m_{xx} - (m u_x)_x = 0 \end{cases}$$

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Intuition from the following scaling:

$$w(t,x) = u(t\kappa^{-\frac{1}{2}}, x\kappa^{-\frac{1}{4}}), \qquad \mu(t,x) = \kappa^{-\frac{1}{4}} m(t\kappa^{-\frac{1}{4}}, x\kappa^{-\frac{1}{4}})$$

solve

$$-w_t - w_{xx} + \frac{1}{2}|w_x|^2 = V_\kappa(x) - \frac{V_\kappa(x)}{\sqrt{\kappa}} \int_{-\pi\kappa^{-\frac{1}{4}}}^{\pi\kappa^{-\frac{1}{4}}} V_\kappa(y)\mu(t,y)dy \qquad \text{on } (-\pi\kappa^{\frac{1}{4}}, \pi\kappa^{\frac{1}{4}}).$$

where $V_\kappa(x) = \kappa^{\frac{1}{2}}(1 - \cos(x\kappa^{-\frac{1}{4}}))$ satisfies $\quad \frac{x^2}{6} \leq V_k(x) \leq \frac{x^2}{2}$.

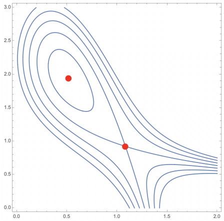
Since $V_k(x) \to x^2$, the problem is "close", for very large κ, to

$$\begin{cases} -\tilde{w}_t - \tilde{w}_{xx} + \frac{1}{2}|\tilde{w}_x|^2 = x^2 - \frac{x^2}{\sqrt{\kappa}} \int y^2 \tilde{\mu}(t,y)dy & x \in \mathbb{R}, \\ \tilde{\mu}_t - \tilde{\mu}_{xx} - (\tilde{\mu}\tilde{w}_x)_x = 0 \end{cases}$$

which is solved by

$$\tilde{w}(t,x) = \mathbf{a}(t)\frac{x^2}{2} + b(t), \qquad \mathbf{M}(t) = \int_{\mathbb{R}} y^2 \tilde{\mu}(t,y)dy.$$

It has two stationary solutions. For large κ,



Issues:

- $(-\pi\kappa^{\frac{1}{4}}, \pi\kappa^{\frac{1}{4}}) \to (-\infty, \infty)$
- Poincaré constant, bounds from below of m, stability estimates for $m_1 - m_2$ in L^2 deteriorate !

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But for (u, m) close to $(\text{quadratic}, \text{gaussian})$, instead of looking at

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we need to keep track of

$$\int m_1 |Du_1 - Du_2|^2, \qquad \int (f(m_1) - f(m_2))(m_1 - m_2) \sim \int x^2 (m_1 - m_2)^2$$

and

$$\int \frac{1}{m_1}(m_1 - m_2)^2.$$

Theorem (local stability)

For κ large, assume that (m, u), $(\bar{m}, \bar{u})$ are even solutions such that

$$m, \bar{m}(x,t) \leq c\kappa^{1/2} e^{-\kappa^{1/2} x^2} \qquad \text{for all } (x,t) \in (-\pi, \pi) \times (0, T).$$

and that $\bar{m}$ is stationary. Then, for some $K > 0$,

$$||m(t) - \bar{m}||_{L^2(e^{\kappa^{1/2} x^2})} \leq K(e^{-\omega t} + e^{-\omega(T-t)})$$

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For any m_0 in this set of profiles, there exists a solution (u,m) to

$$\begin{cases} -u_t - u_{xx} + \frac{1}{2}|u_x|^2 = -\kappa \cos x \int_{-\pi}^{\pi} \cos(y) m(t,y) dy \\ m_t - m_{xx} - (m u_x)_x = 0 \end{cases}$$

and a stationary $\bar{m}$ such that "$m(t) \to \bar{m}$ as $t \to \infty$".

Stationary solutions, up to translation, satisfy

$$\begin{cases} -u'' + \frac{1}{2}|u'|^2 + \bar{\lambda} = -\kappa \cos x \int_{-\pi}^{\pi} \cos(y) m(y) dy \\ -m'' - (mu')' = 0 \\ \int_{-\pi}^{\pi} m(y) dy = 1, m > 0 \end{cases}$$

Theorem (characterization of equilibria for large κ)

There exists $\kappa_0 > 2$ such that for all $\kappa \geq \kappa_0$ the stationary Kuramoto MFG system admits, besides the incoherent solution $m \equiv 1/(2\pi)$, a unique "self organizing" solution (u, λ, m).

The MFG is a usual fixed point $a = F_\kappa(a)$, where

$$\begin{cases} -u'' + \frac{1}{2}|u'|^2 + \bar{\lambda} = -\kappa(1 - \cos x)[1 - a] \\ -m'' - (mu')' = 0, \quad \int_{-\pi}^{\pi} m(y)dy = 1, m > 0 \\ F_\kappa(a) = \int_{-\pi}^{\pi}(1 - \cos(y))m(y)dy. \end{cases}$$

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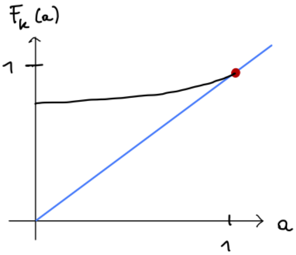
We prove our result by studying $[0,1] \ni a \mapsto F_\kappa(a)$:

κ small



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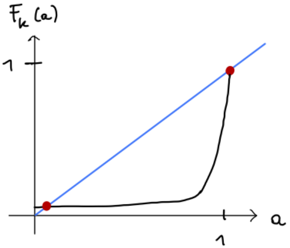

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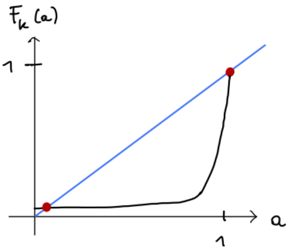
κ large

- $F'_\kappa(a) = O(\kappa^{-1/2})$ for $a \leq \underline{a} < 1$
- $F'_\kappa(a) = O(\kappa)$ for $1 - 1/\kappa < a < 1$



Since

$$-u'' + \frac{1}{2}|u'|^2 + \bar{\lambda} = \kappa(1 - \cos x)[1 - a],$$

if we let $\varphi = e^{-u} = \sqrt{m}$, then φ is a Mathieu function, i.e. it solves

$$-\varphi'' + (b_{\bar{\lambda}} + q_a \cos x)\varphi = 0, \qquad \int \varphi^2 = 1.$$

Since $F(a) = -\bar{\lambda}'/\kappa$,
convexity of F could be deduced by the sign of $\bar{\lambda}'''$...

Differentiating the HJ equation in a,

$$-v_a'' + v_a' u_a' + \lambda_a' = -\kappa V(x).$$

Moreover,

$$F_\kappa'(a) = \frac{1}{\kappa} \int_{-\pi}^{\pi} (v_a'(y))^2 m_a(y) dy$$

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Moreover,

$$F_\kappa'(a) = \frac{1}{\kappa}\int_{-\pi}^{\pi}(v_a'(y))^2 m_a(y)dy = -\int_{-\pi}^{\pi}\left(\frac{\lambda_a'}{\kappa} + V(y)\right)v_a m_a dy.$$

Differentiating the HJ equation in a,

$$-v_a'' + v_a' u_a' + \lambda_a' = -\kappa V(x).$$

Moreover,

$$F_\kappa'(a) = \frac{1}{\kappa} \int_{-\pi}^{\pi} (v_a'(y))^2 m_a(y) dy = - \int_{-\pi}^{\pi} \left(\frac{\lambda_a'}{\kappa} + V(y) \right) v_a m_a dy.$$

to control v_a, use $\pm u_a$ as a sub/supersolution.

Thanks for your attention!