Leapfrogging in Fluid dynamics

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Long Time Behavior and Singularity Formation in PDEs

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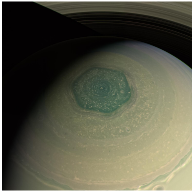
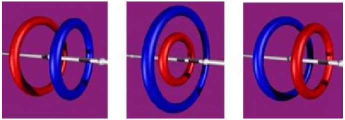
joint work with Z. Hassainia & N. Masmoudi

Coherent structures in turbulent flows

1. Rotating vortices : Saturn's hexagon,



2. Leapfrogging of two coaxial rings (Simulation below from Niemi 2005)



General problem

- Given a dynamical system (finite/infinite dimensional)

$$\dot{X}(t) = v(X(t)), \quad v : E \to F$$

  - Find the equilibria (stationary solutions) : $v(\overline{X}) = 0$
  - Analyze the phase portrait around the equilibrium state (whether periodic or quasi-periodic solutions can be captured ? !)

2d Euler equations

- Helmholtz equation (1858) :

$$\partial_t \omega + v(t,x) \cdot \nabla \omega = 0, \quad v = \nabla^{\perp} \psi = (-\partial_2 \psi, \partial_1 \psi)$$

with $\omega : [0, T] \times \mathbb{R}^2 \to \mathbb{R}$ and

$$\psi(t,x) = \frac{1}{2\pi} \int_{\mathbb{R}^2} \log(|x-y|)\omega(t,y)dy.$$

- We have a large family of stationary radial solutions :

$$\omega(t,x) = f(|x|), \; f \in L^\infty_c.$$

Find time periodic solutions ?

We distinguish two cases :

1. Rigid periodic motion (traveling waves) :

$$\omega(t, x) = \omega_0\left(e^{-i\Omega t} x\right)$$

2. Nonrigid periodic solutions : there exists $T > 0$ such that

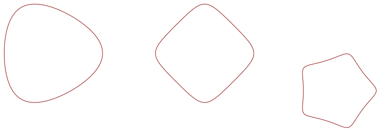
$$\omega(T, x) = \omega_0(x)$$

We may explore them around equilibria of type :

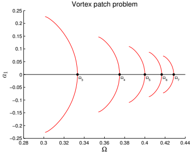
1. Vortex patches.
2. Nonuniform vortices. (It will be discussed in Claudia's talk)
3. Point vortex system.

Rigid time periodic solutions

- Vortex patches : If $\omega_0 = \mathbf{1}_{D_0}$ then for any $t \in \mathbb{R}$, $\omega(t) = \mathbf{1}_{D_t}$.
- Radial shaped patches (discs, annulus,..) are stationary solutions.
- Kirchhoff ellipses : any ellipse rotates uniformly with angular velocity $\Omega = \frac{ab}{(a+b)^2}$
- Numerical observation Deem-Zabusky 1978 : existence of $m-$fold rotating patches

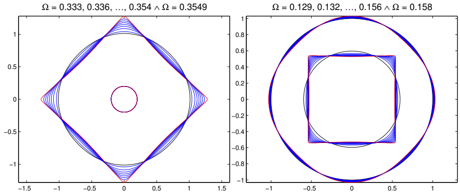


- Burbea (1982) : There exists a family of rotating patches $(V_m)_{m\geq 2}$ bifurcating from the disc at the spectrum $\Omega \in \{\frac{m-1}{2m}, m \geq 2\}$.



Doubly connected case

- de la Hoz-H.-Mateu-Verdera(2016) :Let $\mathcal{C}(b,1)$ be the annulus of small radius b, $m \geqslant 3$ and assume that $1 + b^m - \frac{1-b^2}{2}m < 0$. There are two branches of non trivial m-fold doubly connected periodic patches bifurcating from the annulus at two different angular velocities $\Omega_m^{\pm}$.



More topics :

- Boundary regularity.
- Extension to active scalar equations : gSQG, SWQG, 3D QG,..
- Geometry effects (Euler on the disc or on the sphere).
- Rigidity and flexibility of stationary solutions.
- Quai-periodic patches.
- Contributions : Berti, Cao, Castro, Córdoba,de la Hoz, Dritschel, García, Gómez-Serrano, Hassainia, H., Houamed, Ionescu, Mateu, Masmoudi, Park, Renault, Roulley, Soler, Verdera, Wheeler, L. Xue, Z. Xue, Yao,..

Point vortex model

- **Helmholtz** (1856) : If $\omega_0 = \sum_{j=1}^{N} \gamma_j \delta_{z_j}$, $z_j \in \mathbb{R}^2, \gamma_j \in \mathbb{R}^\star$ then

$$\omega(t,x) = \sum_{j=1}^{N} \gamma_j \delta_{z_j(t)},$$

  with

$$\frac{d\overline{z_j(t)}}{dt} = \frac{1}{2i\pi} \sum_{k\neq j} \frac{\gamma_j}{z_j - z_k}, \qquad j = 1, ..., N$$

- Kirchhoff (1876) : the system is Hamiltonian with

$$\gamma_j \frac{d\overline{z_j(t)}}{dt} = i\partial_{z_j} H, \quad H(z_1, ..z_N) = -\frac{1}{\pi} \sum_{1\leqslant j\neq k\leqslant N} \gamma_j \gamma_k \log|z_j - z_k|$$

- Gröbli (1877)-Poincaré (1893) : this system is integrable for $N \leqslant 3$.
- It is not integrable in general for $N \geqslant 4$.

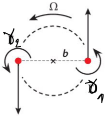
- The equations are given by

$$\frac{d\overline{z_1(t)}}{dt} = \frac{1}{2i\pi}\frac{\gamma_1}{z_1 - z_2}, \qquad \frac{d\overline{z_2(t)}}{dt} = \frac{1}{2i\pi}\frac{\gamma_2}{z_2 - z_1}$$

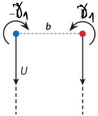
- Thus the vector $Z(t) = z_1(t) - z_2(t)$ satisfies

$$\frac{d\overline{Z(t)}}{dt} = \frac{\gamma_1 + \gamma_2}{2i\pi}\frac{1}{Z(t)}$$

- We distinguish two scenarios :
  1. Case $\gamma_1 + \gamma_2 \neq 0$. The pairs rotate uniformly about the center of mass, with $\Omega = \frac{\gamma_1+\gamma_2}{2\pi d^2}$



  2. Case $\gamma_1 + \gamma_2 = 0$. The pairs translate uniformly with $U = \frac{\gamma_1}{2\pi d}$.



Rotating configuration : link with polynomials

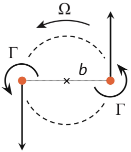
- Rotating configurations : $z_j(t) = e^{i\Omega t} z_j(0), j = 1, ..., N$
  - Taking $\gamma_j = 1$ and rescaling the time we find the system

$$\overline{z_j} = \sum_{k \neq j} \frac{1}{z_j - z_k}, j = 1, ..., N$$

  - Let $P(z) = \prod_{j=1}^{N} (z - z_j)$, then
    1. Stieltjes 1885 : Case $z_j \in \mathbb{R}$. $P'' - 2zP' + 2NP = 0$ (Hermite polynomials).
    2. Thomson (1883) : Case $z_j \in \mathbb{T}$. We find $P(z) = z^N - 1$: regular N-gon.
    3. Aref 2012 : Different nested polygons were discovered.
  - More contributions : Aref, Clarkson, Demina, Kudrayshov, P. Newton, O'neil, Tkachenko,..

Desingularization of point vortices

- Contour dynamics approach
  1. Deem-Zabusky 1978, Saffman-Szeto 1980 : Numerical evidence of a curve of concentrated rotating symmetric pairs of patches connected to the pairs of the point vortex system (simulations below from Luzzatto-Fegiz and Williamson(2010).

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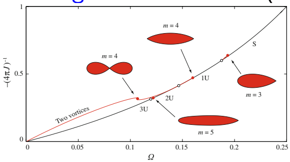
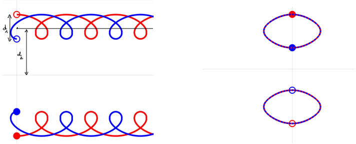


  2. Turkington 1985 : Existence of co-rotating pairs (lack of information on its topology and geometry).
  3. H.-Mateu (2017) : We gave an analytical proof using the contour dynamics equation and implicit function theorem (for Euler and gSQG)
  4. H.-Hassainia(2020) : similar result with asymmetric patches.
  5. Garcia-Haziot(2022) : Global bifurcation results.
- Variational approach-Gluing methods : Gravejat-Smets 2019, Godard-Cadillac 2020, Cao-Lai-Zhan 2020, Davilla-del Pino-Musso-Wei 2020,...

Leapfrogging of two symmetric dipoles

- Aref, Eckhardt, Pomphrey(1980-1988) :
  – The system of 4 point vortices is not integrable and Chaos may emerge.
  – The system is integrable when $\gamma_1 + \gamma_2 + \gamma_3 + \gamma_4 = 0$.
- Symmetric case (pairs of vortex dipoles). Love(1893) : If $0 < \frac{d_1}{d_2} < \sqrt{2}$ then the 4-points leapfrog (non-rigid time periodic motion in the translating frame)



Motion equation

- Take 4 vortices $(z_1,\pi),(z_2,\pi),(\overline{z_1},-\pi),(\overline{z_2},-\pi)$ and denote $z_1-z_2=\eta+\mathrm{i}\xi$. Then

$$\left\{\begin{array}{ll}\dot{\eta} & =\partial_\xi H(\eta,\xi),\\ \dot{\xi} & =-\partial_\eta H(\eta,\xi),\end{array}\right. \qquad H(\eta,\xi)=-\frac{1}{2}\log\left(\frac{1}{y_0^2-\xi^2}-\frac{1}{y_0^2+\eta^2}\right).$$

  with $y_0=\mathrm{Im}(z_1+z_2)$, which is a constant of the motion.

- The orbits are contained in the algebraic set

$$\left\{(\eta,\xi)\in\mathbb{R}^2,\quad \left(\eta^2+\frac{y_0^4}{\xi_0^2}\right)\left(\xi^2+\frac{y_0^4}{\xi_0^2}-2y_0^2\right)=y_0^4\left(\frac{y_0^2}{\xi_0^2}-1\right)^2\right\},$$

- The orbit is periodic iff $0<\frac{\xi_0}{y_0}<\frac{\sqrt{2}}{2}$. The period takes the form

$$T(\xi_0)=\frac{8\xi_0^2(1-\alpha_0)}{(1-2\alpha_0)}\left[\frac{(1-\alpha_0)^2}{\alpha_0^2}\mathtt{E}\left(\frac{\alpha_0}{1-\alpha_0}\right)-\frac{1-2\alpha_0}{\alpha_0^2}\mathtt{K}\left(\frac{\alpha_0}{1-\alpha_0}\right)\right],\quad \alpha_0=\frac{\xi_0}{y_0}$$

- The period is strictly increasing.

- We desingularize the 4 points by concentrated symmetric patches

$$\omega(t) = \tfrac{1}{\varepsilon^2}\mathbf{1}_{D^\varepsilon_{t,1}} + \tfrac{1}{\varepsilon^2}\mathbf{1}_{D^\varepsilon_{t,2}} - \tfrac{1}{\varepsilon^2}\mathbf{1}_{\overline{D^\varepsilon_{t,1}}} - \tfrac{1}{\varepsilon^2}\mathbf{1}_{\overline{D^\varepsilon_{t,2}}},$$

$$D^\varepsilon_{t,k} \triangleq \varepsilon O^\varepsilon_{t,k} + z_k(t), \quad |O^\varepsilon_{t,k}| = \pi, \quad k = 1, 2,$$

with $O^\varepsilon_{t,k}$ being simply connected domains localized around the unit disc. By a symmetry reduction we find out that a particular solution is given by

$$\forall t \in \mathbb{R}, \quad O^\varepsilon_{t,2} = O^\varepsilon_{t+\frac{T(\xi_0)}{2},1}.$$

Theorem (Hassainia-H.-Masmoudi 2023)

Let $y_0 > 0$ and $0 < a < b < \frac{y_0}{\sqrt{2}}$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there exists a Cantor type set $\mathcal{C}_\varepsilon \subset [a, b]$ with

$$\lim_{\varepsilon \to 0} |\mathcal{C}_\varepsilon| = b - a$$

and for any $\xi_0 \in \mathcal{C}_\varepsilon$, Euler equation admits a solution satisfying

$$\forall t \in \mathbb{R}, \quad O^\varepsilon_{t+T(\xi_0),1} = O^\varepsilon_{t,1}.$$

Here $T(\xi_0)$ is the period of the four points vortices.

Some Remarks

1. This the first derivation of the long time leapfrogging motion.
2. These structures are captured far away the equilibria.
3. The domain $O^{\varepsilon}_{t,1}$ is time periodic, but not rigidly rotating.
4. In $3d$ case, Davila, del Pino, Musso and Wei (2023) established a weak form of short time leapfrogging of multi rings.
5. Jerrard-Smets (2018) : Leapfrogging for 3D Gross-Pitaevskii equation (weak form).

Contour dynamics equation in the symmetric case

- Symmetry reduction : from the ansatz

$$\omega(t) = \frac{1}{\varepsilon^2}\mathbf{1}_{D^\varepsilon_{t,1}} + \frac{1}{\varepsilon^2}\mathbf{1}_{D^\varepsilon_{t,2}} - \frac{1}{\varepsilon^2}\mathbf{1}_{\overline{D^\varepsilon_{t,1}}} - \frac{1}{\varepsilon^2}\mathbf{1}_{\overline{D^\varepsilon_{t,2}}},$$

$$D^\varepsilon_{t,k} \triangleq \varepsilon O^\varepsilon_{t,k} + z_k(t), \quad k = 1,2, \quad O^\varepsilon_{t+T(\xi_0),1} = O^\varepsilon_{t,1}$$

  we reduce the 4 equations to just one equation on the boundary of $D^\varepsilon_{t,1}$.

- First $z_1(t) - z_2(t) = \sqrt{q(\omega_0 t)}e^{\mathrm{i}\Theta(\omega_0 t)}$, with ω_0 the frequency of the 4-point system.
- We parametrize the domain $O^\varepsilon_{t,1}$ as

$$\theta \in \mathbb{T} \mapsto e^{\mathrm{i}\Theta(\omega_0 t)}\sqrt{1 + 2\varepsilon r(\omega_0 t, \theta)}\, e^{\mathrm{i}\theta}$$

  with $r : (\varphi, \theta) \in \mathbb{T}^2 \mapsto r(\varphi, \theta) \in \mathbb{R}$. Then the contour dynamics equation writes

$$G(r)(\varphi, \theta) \triangleq \varepsilon^2\omega_0\partial_\varphi r - \varepsilon^2\omega_0\dot{\Theta}(\varphi)\partial_\theta r + \partial_\theta\Big[F(\varepsilon, q, r)\Big] = 0.$$

Linearization

- First, $G(0) = O(\varepsilon)$.
- By linearization at any small state r, we get

$$\partial_r G(r)[h] = \varepsilon^2 \omega_0 \partial_\varphi h + \partial_\theta \Big[\Big(\tfrac{1}{2} - \tfrac{\varepsilon}{2} r - \varepsilon^2 \mathrm{g} + \varepsilon^3 V^\varepsilon(r) \Big) h \Big] - \tfrac{1}{2} \mathcal{H}[h] - \varepsilon^2 Q_0[h] + \varepsilon^3 \mathcal{R}^\varepsilon(r)[h],$$

- with $\mathcal{H}$ the Hilbert transform in the toroidal case and Q_0 is given by

$$Q_0[h](\varphi,\theta) \triangleq \frac{1}{q(\varphi)} \partial_\theta \Big[\int_{\mathbb{T}} h(\varphi,\eta) \cos(\eta+\theta) d\eta \Big],$$

$$\mathrm{g}(\varphi,\theta) = \mathrm{Re}\Big\{ \Big(\frac{1}{q(\varphi)} + \frac{e^{\mathrm{i}2\Theta(\varphi)}}{\big(\sqrt{q(\varphi)}\sin(\Theta(\varphi)) + y_0\big)^2} - \frac{e^{\mathrm{i}2\Theta(\varphi)}}{\big(\sqrt{q(\varphi)}\cos(\Theta(\varphi)) + \mathrm{i}y_0\big)^2} \Big) e^{\mathrm{i}2\theta} \Big\}.$$

- For $\varepsilon = 0$, the operator is degenerating (in time),

$$\partial_r G(r)[h] = \tfrac{1}{2} \big(\partial_\theta - \mathcal{H} \big) h$$

The spatial modes ± 1 are trivial resonances !

Formal Nash-Moser scheme

- Newton scheme : To construct a solution to $F(r) = 0$ we use the scheme :

  r_0 is given such that $F(r_0)$ is small enough, $r_{n+1} = r_n + h_n, h_n := -F'(r_n)^{-1}F(r_n)$

  To do that, it is enough that $F : X \to Y$ is C^1 and $F'(r_0) : X \to Y$ is an isomorphism.

- In our context, $F'(r_0)$ is not an isomorphism !
- Nash-Moser scheme is a regularization of Newton scheme where we require that $F'(r_n)$ admits a right inverse (with a loss of regularity+ suitable tame estimates)

A toy model (Resonance and loss of regularity)

Consider the operator : $L_0 h = \varepsilon^2 \omega_0(\xi_0)\partial_\varphi h + \partial_\theta h$

- To solve $L_0 h = f$, we use Fourier expansion

$$h(\varphi,\theta) = \sum_{k,n\in\mathbb{Z}} h_{k,n} e^{i(k\varphi+n\theta)}, \qquad h_{k,n} = -\mathrm{i}\frac{f_{k,n}}{\varepsilon^2\omega_0(\xi_0)k+n}$$

- In the Cantor set

$$\mathcal{C}_0 = \Big\{\xi_0 \in [a,b], \forall (k,n) \neq (0,0),\ |\varepsilon^2\omega_0(\xi_0)k+n| \geqslant \tfrac{\varepsilon^{2+\delta}}{(1+|n|)^\tau}\Big\},$$

  we get

$$\|L_0^{-1} f\|_{H^s} \leqslant \varepsilon^{-2-\delta}\|f\|_{H^{s+\tau}}$$

- We know that $\xi_0 \mapsto \omega_0(\xi)$ does not degenerate,

$$\inf_{\xi_0\in[\xi_*,\xi^*]} |\omega'(\xi_0)| > 0.$$

  Hence for $\tau > 1$

$$|\mathcal{C}_0| \geqslant b - a - C\varepsilon^\delta$$

Good approximation and new scaling

- We cannot start from $r_0 = 0$ because

$$G(0) = O(\varepsilon), \quad (\partial_r G)^{-1}(0) = O(\varepsilon^{-2-\delta}), \quad (\partial_r G)^{-1}(0)G(0) = O(\varepsilon^{-1-\delta})$$

- We have to find a good approximation. Actually we obtain the following result : there exists $\overline{r_\varepsilon}$ such that

$$\overline{r_\varepsilon} = O(\varepsilon) \quad \text{and} \quad G(\overline{r_\varepsilon}) = O(\varepsilon^4)$$

- The functional that we will use is ($\mu \in (0,1)$)

$$\mathcal{F}(\rho) = \frac{1}{\varepsilon^{1+\mu}} G(\overline{r_\varepsilon} + \varepsilon^{1+\mu}\rho), \quad \mathcal{F}(0) = O(\varepsilon^{3-\mu})$$

- We show that in a suitable Cantor set

$$(\partial_\rho \mathcal{F})^{-1}(0) = O(\varepsilon^{-2-\delta}), \quad (\partial_\rho \mathcal{F})^{-1}(0)\mathcal{F}(0) = O(\varepsilon^{1-\delta-\mu})$$

Invertibility of the linearized operator and strategy

- The linear operator is given by

$$\partial_\rho \mathcal{F}(\rho)[h] = \varepsilon^2 \omega_0 \partial_\varphi h + \partial_\theta \left[\mathcal{V}^\varepsilon(\rho) h\right] - \tfrac{1}{2}\mathcal{H}[h] - \tfrac{\varepsilon^2}{q(\varphi)} Q_0[h] + \varepsilon^3 \partial_\theta \mathcal{R}_0^\varepsilon(\rho)[h],$$

where

$$\mathcal{V}^\varepsilon(\rho)(\varphi,\theta) \triangleq \tfrac{1}{2} - \varepsilon^2 \omega_0 \dot{\Theta} - \varepsilon^2 \mathrm{g}(\varphi,\theta) - \tfrac{\varepsilon^{2+\mu}}{2}\rho(\varphi,\theta) + \varepsilon^3 V^\varepsilon(\rho)(\varphi,\theta)$$

- Is it possible to invert the operator $\partial_\rho \mathcal{F}(\rho)$, for ρ and ε small enough ?

- Difficulties :

1. The operator is quasi-linear (variable coefficients at the main order).
2. Small divisor problems.
3. Trivial resonance of the spatial modes ± 1.
4. Degeneracy in ε in the time direction

- Tools :

1. KAM techniques in the spirit of the works of Berti-Montalto and Feola-Giuliani-Procesi, to conjugate the linear operator into a Fourier multiplier.
2. Monodromy matrix to handle the modes ± 1.
3. Nash Moser scheme to construct solutions to the nonlinear problem.
4. Measure of the Cantor set.

Thank you for your attention !

Reduction of the transport part

- There exists a change of coordinates transform $\mathscr{B}$ such that on the Cantor set

$$\mathcal{C}(\rho) = \bigcap_{\substack{(k,n)\in\mathbb{Z}^2 \\ |n|\geqslant 1}} \left\{ \xi_0 \in (a,b);\ \left|\varepsilon^2\omega(\xi_0)k + n\,\mathtt{c}(\varepsilon,\xi_0)\right| \geqslant \tfrac{\varepsilon^{2+\delta}}{|n|^{\tau}} \right\}$$

we have

$$\mathscr{B}^{-1}\partial_\rho\mathcal{F}(\rho)\mathscr{B} = \varepsilon^2\omega_0\partial_\varphi + \mathtt{c}(\varepsilon,\xi_0)\partial_\theta - \tfrac{1}{2}\mathcal{H} - \varepsilon^2 Q_1 + \varepsilon^{2+\mu}\mathcal{R}_1$$

with

$$Q_1[h](\varphi,\theta) \triangleq \tfrac{1}{q(\varphi)}\partial_\theta\left[\int_{\mathbb{T}} h(\varphi,\eta)\cos\left(\eta+\theta-2\left[\Theta(\varphi)-\varphi\right]\right)d\eta\right]$$

- It remains to analyze the mode 1 based on the study of the monodromy matrix.
- The invertibility is achieved by a perturbative argument.