# Γ–limit for zigzag walls


Hans Knüpfer

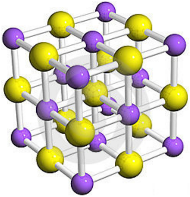
University of Heidelberg

joint work with W. Shi (Aachen)

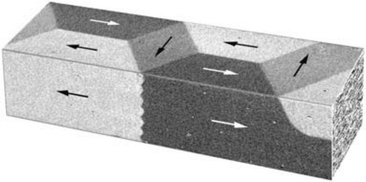

Abu Dhabi, Feb 2024

NYUAD

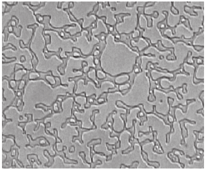
# Pattern forming systems



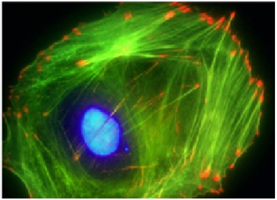
NaCl crystal



Magnetic domains in iron bar



Polymer phase separation in poor solvent



Actin cytosceleton

Understand these patterns by minimization principle?

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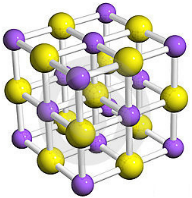

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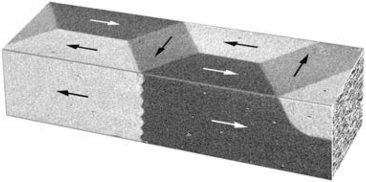

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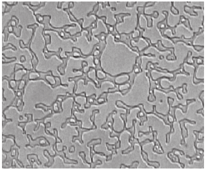

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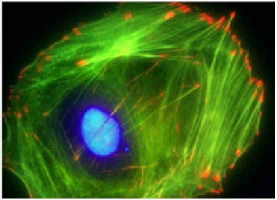

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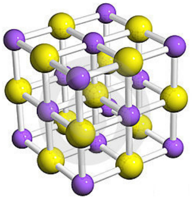

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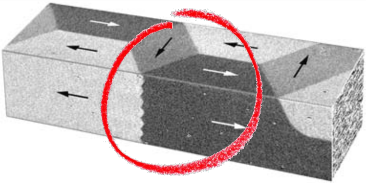

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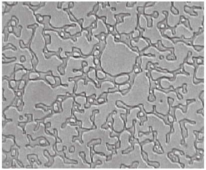

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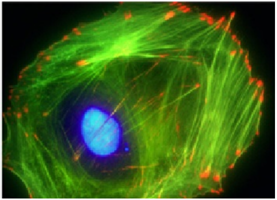

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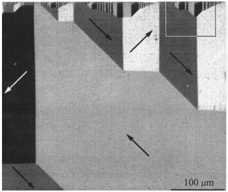
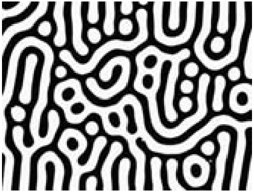
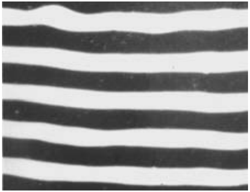
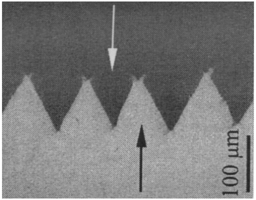
# Formation of magnetic domains in magnetic materials








Hubert & Schäfer, Magnetic Domains

Magnetization direction given unit vector $|M| = 1$.

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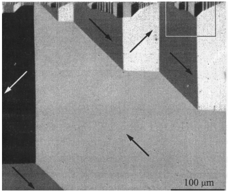


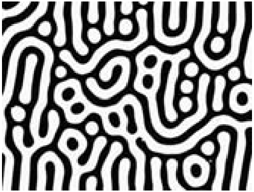
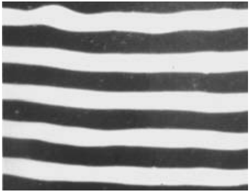
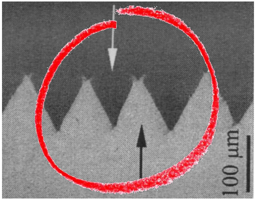


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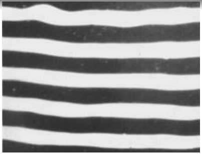
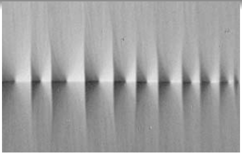
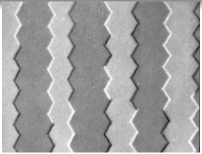
# Complex transition layers in ferrromagnetic films



Néel wall



Cross tie wall



Zigzag wall

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- Néel wall Garcia-Cervera '04, Melcher '03, '04, DeSimone, HK, Otto '06, Ignat, Otto '08
- Cross–tie wall Alouges, Riviere, Serfaty '02, '03
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- Asymmetric Bloch/Néel wall is asymptotically 1d. Döring, Ignat, Otto '14

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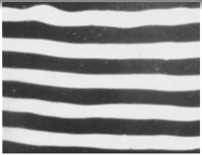

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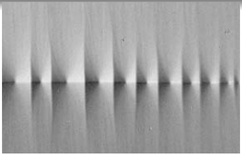

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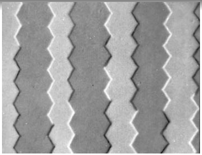

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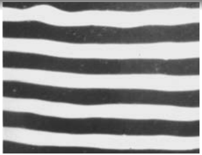

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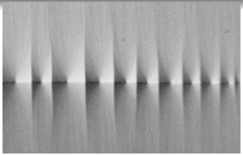

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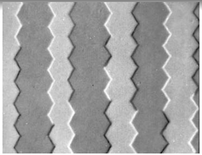

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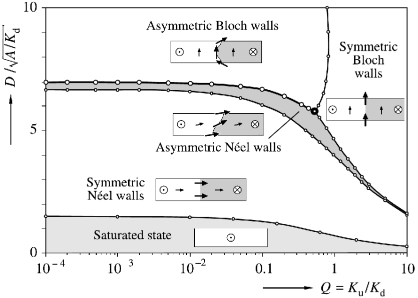

# Continuum micromagnetic model

The micromagnetic energy of a ferromagnetic sample $\Omega \subset \mathbb{R}^3$ with magnetization $M : \mathbb{R}^3 \to \mathbb{R}^3$, $|M| = \chi_\Omega$ is given by (Landau, Lifshitz '35)

$$\mathcal{E}[M] \;=\; d^2 \int_\Omega |\nabla M|^2 \, dx \;+\; Q \int_\Omega (M_1^2 + M_2^2) \, dx \;+ \int_{\mathbb{R}^3} |\nabla \varphi|^2 \, dx$$

*Stray field energy* and *stray field* $\nabla\varphi$ determined by

$$-\Delta\varphi \;=\; -\operatorname{div} M$$

In analogy to electrostatics $\operatorname{div} M$ are called *magnetic charges*

- Exchange energy prefers uniform magnetization
- Anisotropy energy prefers orientation $M \approx \pm e_3$.
- Stray field energy prefers charge free configurations

DeSimone, Müller, Kohn, Otto, Melcher, Serfaty, Riviere, Ignat, Muratov, ...

Nonconvex, nonlocal, vectorial, multiscale problem

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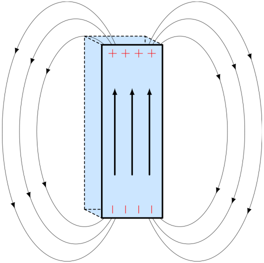


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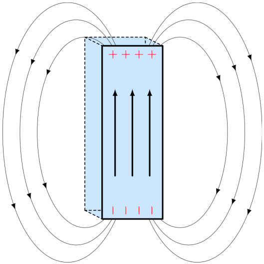


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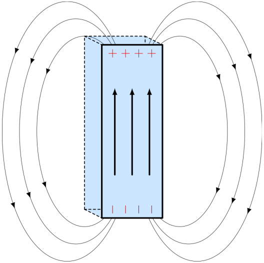


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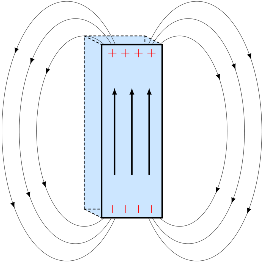

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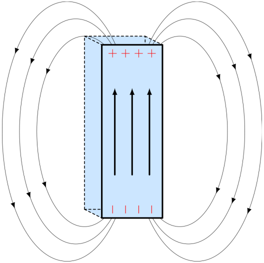


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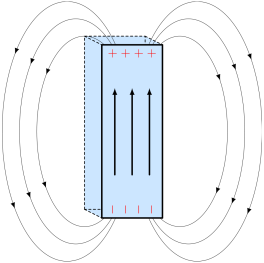


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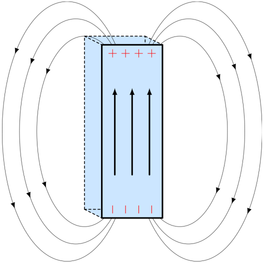


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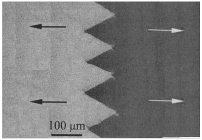
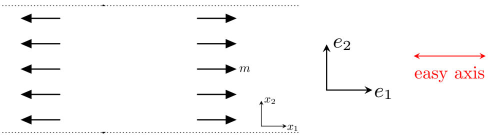
# Geometry




**Geometry.**

- Consider thin–film of form $\Omega = \underbrace{\mathbb{R} \times \mathbb{T}_\ell}_{=: Q_\ell} \times [0, t]$.
- thin film regime $t \ll d$
- $\ell$ periodicity in $x_2$ with torus $\mathbb{T}_\ell := \mathbb{R}/[0, \ell)$.
- Enforce charged transition layer by boundary conditions

$$m = \pm e_1 \qquad \text{for } \pm x_1 > 1.$$

$$\int_\Omega \underbrace{\operatorname{div} m}_{mag.charge} = 0$$

uncharged
domain wall

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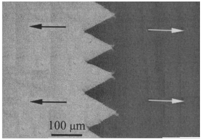


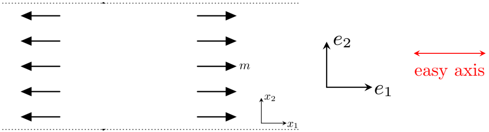


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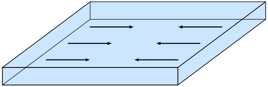

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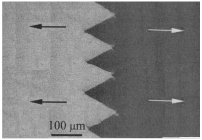


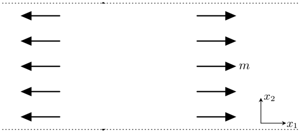


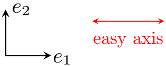


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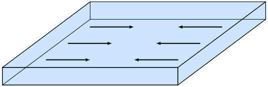

- thin film regime $t \ll d$
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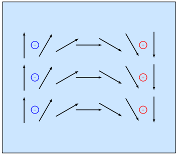

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uncharged
domain wall

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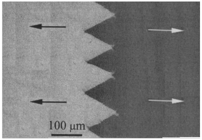


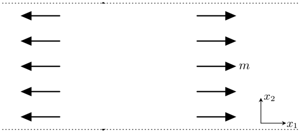


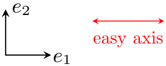


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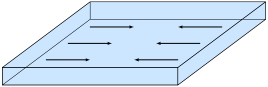

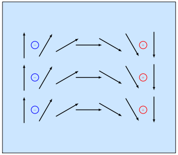

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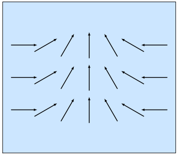

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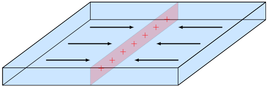
# Reduced two–dimensional energy

**Thin film geometry and approximation.**



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$$m_3 \;=\; 0, \quad m \;\neq\; m(x_3).$$

- Stray field approximation

$$\int_{\mathbb{R}^3} |\nabla \varphi|^2 = \int_{\mathbb{R}^3} \left||\nabla|^{-1} \operatorname{div} m\right|^2 \approx \int_{\mathbb{R}^2} \left||\nabla|^{-\frac{1}{2}} \operatorname{div} m\right|^2$$

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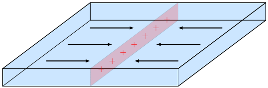

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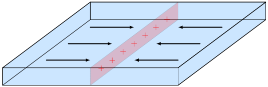

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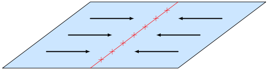

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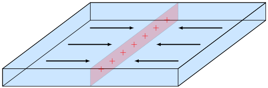

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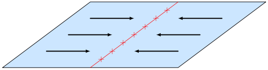

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# Classical phase transformation theory of Ginzburg–Landau

For prescribed volume $\int_\Omega u = \lambda$ consider

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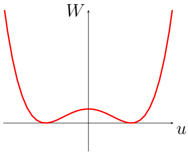


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**$\Gamma$–limit (DeGiorgi):** $X$ metric space, $\mathcal{E}_k : X \to \mathbb{R}$. Then $\mathcal{E}_k \xrightarrow{\Gamma} \mathcal{E}_0$ if

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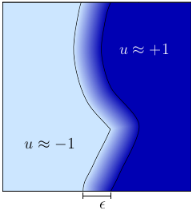


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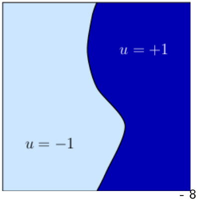

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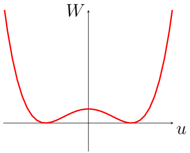


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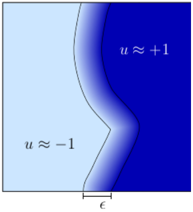


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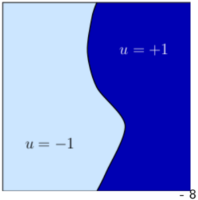

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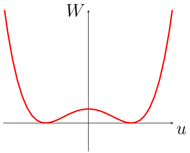


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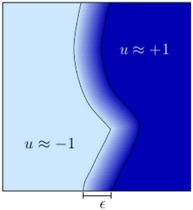


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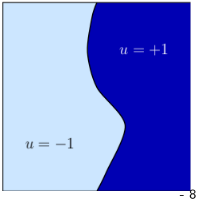

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# A look at the limit model

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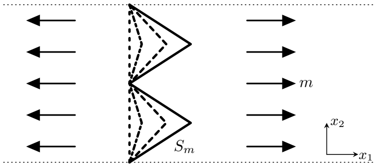


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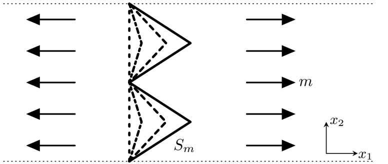
$$|n \cdot e_1| \ \geq \ \min\{1, \lambda^{-\frac{1}{2}}\}.$$




The minimal energy for $m \in \mathcal{A}_0$ is

$$\min_{m \in \mathcal{A}_0} E_0[m] \ = \ 2\ell \begin{cases} (1+\lambda) & \text{if} \lambda \le 1, \\ 2\sqrt{\lambda} & \text{if } \lambda > 1. \end{cases}$$

Anisotropic penalization of jump singularities in the limit, allows for zigzags