Mass transfer and global solutions in a micro-scale model of superfluidity

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- Certain fluids are fundamentally inviscid (superfluids). All particles must move in unison – ground state of BEC (QM phase transition)
- Helium-4 (bosonic) and Helium-3 (fermionic). Remarkable properties: critical velocity, film flow, fountain pressure

Phase diagram of He-4

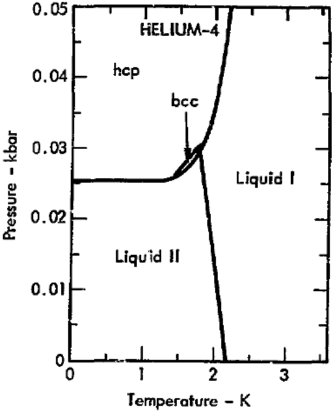

Figure: Young, David A. (1975) *Phase diagrams of the elements.* United States. https://doi.org/10.2172/4010212 - Technical report at LLNL

Conversion between the fluids

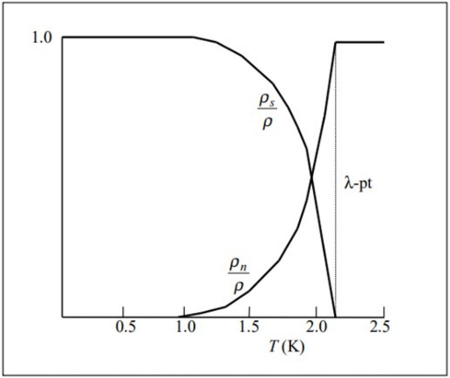

Figure: Vinen, W.F. The physics of superfluid Helium. *Technical report, CERN* (2004)

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- $T > 0$: only a part of the normal liquid will have condensed into the superfluid
- Need to characterize the retarding interaction between phases
- Important to understand these dynamics since Helium is used as a cryogen (LHC, quantum computing, dark matter searches), gravitational waves(?), a quantum solvent in spectroscopic analysis, and in gyroscopes

Pitaevskii model [Pit59]

$$\partial_t\psi + \lambda B\psi = -\frac{1}{2i}\Delta\psi + \frac{\mu}{i}|\psi|^p\psi$$

$$B = \frac{1}{2}(-i\nabla - u)^2 + \mu|\psi|^p = -\frac{1}{2}\Delta + \frac{1}{2}|u|^2 + iu\cdot\nabla + \mu|\psi|^p\psi$$

$$\partial_t\rho + \nabla\cdot(\rho u) = 2\lambda\,\mathrm{Re}(\bar{\psi}B\psi)$$

$$\partial_t(\rho u) + \nabla\cdot(\rho u\otimes u) + \nabla q - \nu\Delta u + \alpha\rho u = -2\lambda\,\mathrm{Im}(\nabla\bar{\psi}B\psi)$$

$$\nabla\cdot u = 0$$

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- $\alpha, \mu, \nu, \lambda > 0$; $p \geq 1$.
- Equip with periodic boundary conditions, on $\mathbb{T}^n$ for $n = 2, 3$.

Features of this model

1. Nonlinearly coupled NLS-NSE
2. Coupling (B) is like relative kinetic energy ($+$ self-energy)
3. NLS turns parabolic (Ginzburg-Landau)
4. Bidirectional mass and momentum transfer
5. Continuity equation has sign-indefinite source term
6. Assumed divergence-free velocity (and general power-law self-interactions)
7. Linear drag term added (to induce coercive velocity estimates)

(Weak) Solutions

Definition 2.1 ((Weak) Solutions)

For $T > 0$, and initial conditions $\psi_0 \in H^2(\mathbb{T}^n)$, $u_0 \in H^1_d(\mathbb{T}^n)$ and $\rho_0 \in L^\infty(\mathbb{T}^n)$, a triplet (ψ, u, ρ) is a ***weak solution*** *to the Pitaevskii model if:*

$$\psi \in L^\infty([0,T]; H^2(\mathbb{T}^n)) \cap L^2(0,T; H^3(\mathbb{T}^n))$$
$$u \in L^\infty([0,T]; H^1_d(\mathbb{T}^n)) \cap L^2(0,T; H^2_d(\mathbb{T}^n))$$
$$\rho \in L^\infty([0,T] \times \mathbb{T}^n)$$

and they satisfy the governing equations in the sense of distributions for all test functions $\{\varphi, \Phi, \sigma\}$ satisfying

1. *complex scalar field $\varphi \in H^1(0,T; L^2(\mathbb{T}^n)) \cap L^2(0,T; H^1(\mathbb{T}^n))$*
2. *real, divergence-free vector field $\Phi \in H^1(0,T; L^2_d(\mathbb{T}^n)) \cap L^2(0,T; H^1_d(\mathbb{T}^n))$*
3. *real scalar field $\sigma \in H^1(0,T; L^2(\mathbb{T}^n)) \cap L^2(0,T; H^1(\mathbb{T}^n))$*

Related work

- **NLS + non-local potential** to model dipolar quantum gases (Carles-Markowich-Sparber 2008; Sohinger 2011)
- NLS recast as **Korteweg-type QHD with "quantum pressure"** (Hattori-Li, Jüngel, Wang-Guo)
- **Weak solns to QHD**-type models (Antonelli-Marcati: frac. step method; Jüngel: test functions vanish at vacuum)

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- **Inhomo. incomp.**: local-weak-3D (Kazhikov 1974); including vacuum (Kim 1987); continuity at $t = 0$ and global-weak-3D (Simon 1999)
- Local-strong-3D, global-strong-2D: density bounded below (Ladyzhenskaya 1978)
- Local-strong-3D: vacuum + compatible data (Choe-Kim 2003); density bounded below + axisym. (Boldrini et al 2003)

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Differences from/similarities to existing work

- Local existence of solutions established (without drag) in [JT22a, JT22b]
- The first investigations of a **bidirectionally-coupled** NLS-NSE model. (Antonelli-Marcati 2015: NSE affects NLS but not vice versa; no source term for ρ)
- Most closely related to [Kim87], but density bounded below and has source
- (u_0, ρ_0): same regularity as [Kim87] (less than the strong solutions in Choe-Kim 2003)

Theorem 3.1 (Global existence in 2D (weak nonlinearities) [JJK23b])

Fix $p \in [1,4)$, and let $\psi_0 \in H^{\frac{5}{2}}(\mathbb{T}^2)$, $u_0 \in H^1_d(\mathbb{T}^2)$, $\rho_0 \in [m_i, M_i]$ a.e. in $\mathbb{T}^2$. Then, $\exists$ a global weak solution (ψ, u, ρ) such that the density is bounded between $m_f \in (0, m_i)$ and $M_f := M_i + m_i - m_f$, if the initial data satisfy

$$\|\psi_0\|_{H_x^{\frac{5}{2}}} + \|u_0\|_{H_x^1} + \|\psi_0\|_{L_x^{p+2}} \leq \varepsilon_0(\lambda, \mu, \nu, m_i, M_i, m_f, \alpha, p).$$

The solution has the regularity

$$\begin{aligned} \psi &\in C([0,\infty); H^{\frac{5}{2}}(\mathbb{T}^2)) \cap L^2(0,\infty; H^{\frac{7}{2}}(\mathbb{T}^2)), \\ u &\in C([0,\infty); H^1_d(\mathbb{T}^2)) \cap L^2(0,\infty; H^2_d(\mathbb{T}^2)) \\ \rho &\in L^\infty([0,\infty) \times \mathbb{T}^2) \cap C([0,\infty); L^r(\mathbb{T}^2)), \end{aligned}$$

for $1 \leq r < \infty$. The solution also satisfies the energy balance (as an equality).

Energy balance equality

$$\frac{1}{2}\|\sqrt{\rho(t)}u(t)\|_{L_x^2}^2 + \frac{1}{2}\|\nabla\psi(t)\|_{L_x^2}^2 + \frac{2\mu}{p+2}\|\psi(t)\|_{L_x^{p+2}}^{p+2}$$
$$+ \nu\|\nabla u\|_{L_{[0,t]}^2 L_x^2}^2 + \alpha\|\sqrt{\rho}u\|_{L_{[0,t]}^2 L_x^2}^2 + 2\lambda\|B\psi\|_{L_{[0,t]}^2 L_x^2}^2$$
$$= \frac{1}{2}\|\sqrt{\rho_0}u_0\|_{L_x^2}^2 + \frac{1}{2}\|\nabla\psi_0\|_{L_x^2}^2 + \frac{2\mu}{p+2}\|\psi_0\|_{L_x^{p+2}}^{p+2} \quad a.e.\ t \in [0,\infty).$$

Theorem 3.2 (Almost-global existence in 2D (strong nonlinearities) [JJK23b])

In the case of $p = 4$, the solution to the Pitaevskii model has the same regularity properties as in Theorem 3.1, except that their existence is guaranteed on $[0, T]$ such that $T \sim \exp\left(\varepsilon^{-\frac{1}{2}}\right)$, where ε is the size of the (sufficiently small) initial data.
For $p > 4$, the existence time scales polynomially with the size of the data, as $T \sim \varepsilon^{-\frac{p}{p-4}}$. In both cases, these solutions also satisfy the energy equality on $[0, T]$.

Strategy

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$$\rho(t, X_\alpha(t)) = \rho_0(\alpha) + 2\lambda \operatorname{Re} \int_0^t \bar{\psi} B \psi(\tau, X_\alpha(\tau))\, d\tau$$

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- $\|B\psi\|_{L^2_{[0,T]} L_x^\infty} \lesssim \|B\psi\|_{L^2_{[0,T]} H_x^{1+}}$ — need $\psi \in L^2 H^{3+}$

Sketch of a priori estimates

- Mass:
  - $\frac{d}{dt}\|\psi\|_{L^2}^2 + 2\lambda \operatorname{Re}\int_x \overline{\psi}B\psi = 0$
  - Use coercivity of second term: $\operatorname{Re}\int_x \overline{\psi}B\psi \geq \mu\|\psi\|_{L^{p+2}}^{p+2}$
  - $$S(t) = \|\psi(t)\|_{L_x^2}^2 \lesssim \frac{S_0}{\left(1 + S_0^{\frac{p}{2}}\, t\right)^{2/p}} \quad , \quad t \in [0, T],$$

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- Higher-order energy:
  - Test NSE by $-\Delta u$, and also by $\partial_t u$; test NLS by $-\Delta^2\psi$
  - Resulting estimates yield $u \in L^\infty H^1 \cap L^2 H^2 \cap H^1 L^2$ and $\psi \in L^\infty \dot{H}^2 \cap L^2 \dot{H}^3 \cap H^1 \dot{H}^1$
  - Importantly, we obtain $\int_t^{2t} \text{Dissipation} \lesssim Z_0 e^{-\frac{t}{C}} + \frac{S_0^{\frac{p}{2}+1}}{\left(1+S_0^{\frac{p}{2}}\, t\right)^{2/p}}$,

Sketch of a priori estimates

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  - For some $s \in (1,2)$,

$$\partial_t(-\Delta)^s\psi + \lambda(-\Delta)^s(B\psi) = -\frac{1}{2i}\Delta(-\Delta)^s\psi + \frac{\mu}{i}(-\Delta)^s(|\psi|^2\psi)$$

  - We use $s = \frac{5}{4}$ to take advantage of the interpolation of $u \in L^\infty H^1 \cap L^2 H^2 \subset L^4 H^{\frac{3}{2}}$
  - Resulting estimates yield $\psi \in L^\infty \dot{H}^{\frac{5}{2}} \cap L^2 \dot{H}^{\frac{7}{2}}$
  - Again, a dyadic time-decaying dissipation is obtained:

$$\int_t^{2t} \|\psi\|^2_{\dot{H}^{\frac{7}{2}}} \lesssim (W_0 + Z_0)e^{-\frac{t}{2C}} + \frac{S_0^{\frac{p}{2}+1}}{\left(1 + S_0^{\frac{p}{2}}t\right)^{1+\frac{2}{p}}}$$

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- To enforce density condition, split time integrals into short-time $t \in [0,1]$ and long-time $t \in [2^k, 2^{k+1}]$ estimates — former are uniformly bounded, latter lead to convergent/divergent geometric series

Approximate system

$$\partial_t \rho^N + u^N \cdot \nabla \rho^N = 2\lambda \operatorname{Re}(\overline{\psi^N} B^N \psi^N)$$

$$\begin{aligned} P^N(\rho^N \partial_t u^N + \rho^N u^N \cdot \nabla u^N) + \nu A u^N = & -2\lambda P^N \left(\operatorname{Im} \left(\nabla \overline{\psi^N} B^N \psi^N \right) \right. \\ & \left. + u^N \operatorname{Re} \left(\overline{\psi^N} B^N \psi^N \right) \right) \\ & - \alpha P^N(\rho^N u^N) \end{aligned}$$

$$\partial_t \psi^N + \frac{1}{2i} \Delta \psi^N = -Q^N \left(\lambda B^N \psi^N + i\mu |\psi^N|^p \psi^N \right)$$

where $B^N = -\frac{1}{2}\Delta + \frac{1}{2}|u^N|^2 + iu^N \cdot \nabla + \mu|\psi^N|^p$.

Galerkin, initial conditions, existence

- Galerkin velocity: $u^N(t,x) = \sum_{k=1}^N c_k^N(t) a_k(x)$ — $c_k^N(t) \in \mathbb{R}$, a_k are eigenfunctions of the Stokes operator
- Galerkin wavefunction: $\psi^N(t,x) = \sum_{k=1}^N d_k^N(t) b_k(x)$ — $d_k^N(t) \in \mathbb{C}$, b_k are eigenfunctions of the Laplacian operator

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- Initial conditions
  - $u_0^N = P^N u_0, \psi_0^N = Q^N \psi_0$ — both these approximations converge in the relevant Sobolev spaces
  - ρ_0^N is a sequence of smooth data that converges to ρ_0 in L^2

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- Existence
  - ODE theory, contraction mapping arguments for u and ψ
  - Establish $\rho^N \in C^0 C^0$ for each N using characteristics
  - Weak limits and compactness

Renormalization

- Renormalization (diPerna-Lions theory)
  - Given a weak solution $\rho \in L^\infty L^\infty$, we mollify it to ρ_h
  - Using the mollified continuity equation, we show that $\eta(\rho_h) \xrightharpoonup{\mathcal{D}} \eta(\rho)$ solves the continuity equation, for $\eta \in C^1$
  - Use $\eta(x) = x^{2n}$ so that $\rho_h \xrightarrow{C^0 L^{2n}} \rho$
  - From continuity equation, we have $\|\partial_t \rho^N\|_{L^\infty W^{-1,r}} < \infty$, uniform in N, so that $\rho^N \xrightarrow{C_w L^r} \rho \in C^0 L^{2n}$
  - Remains to show that $\|\rho^N(t^N)\|_{L^r} \to \|\rho(t)\|_{L^r}$ — prove for $r = 2$ using uniform integrability of source term; then result follows from uniqueness of limits and Lebesgue interpolation

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  - Remains to show that $\|\rho^N(t^N)\|_{L^r} \to \|\rho(t)\|_{L^r}$ — prove for $r = 2$ using uniform integrability of source term; then result follows from uniqueness of limits and Lebesgue interpolation
- Note: The range for the renormalization index can be explained by $H^1 \subset L^r$ for $1 \leq r < \infty$ in 2D. The equivalent embedding in 3D requires $1 \leq r \leq 6$.

Motivation

- Main difference is the unfavorable Sobolev embeddings
- We will need $\psi \in L^2 H^{\frac{7}{2}+}$
- The interpolation of $u \in L^4 H^{\frac{3}{2}}$ is not sufficient
- Taking more derivatives of ψ alone is not possible; we will have to go higher in u, and thus ρ as well
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- Taking more derivatives of ψ alone is not possible; we will have to go higher in u, and thus ρ as well
- We overcome this challenge using parabolic maximal regularity for NLS
- Threshold moved from $p = 4$ to $p = \infty$
- Approach works in 2D also!

Theorem 5.1 (Global existence in 3D (weak nonlinearities) [JJK23a])

Fix $p \in [1, \infty)$, $\delta \in (0, \frac{1}{3})$ such that $\delta < \frac{1}{p-1}$. Let $\psi_0 \in H^2(\mathbb{T}^3)$, $u_0 \in H^1_d(\mathbb{T}^2)$, and $\rho_0 \in [m_i, M_i]$ a.e. in $\mathbb{T}^3$. Then, $\exists$ a global weak solution (ψ, u, ρ) such that the density is bounded between $m_f \in (0, m_i)$ and $M_f := M_i + m_i - m_f$, if

$$\|\psi_0\|^2_{H^2_x} + \|u_0\|^2_{H^1_x} + \|\psi_0\|^{p+2}_{L^{p+2}_x} \leq \varepsilon_0(\lambda, \mu, \nu, m_i, M_i, m_f, \alpha, p).$$

The solution has the regularity

$$\psi \in C([0,\infty); H^2(\mathbb{T}^3)) \cap L^2(0,\infty; H^3(\mathbb{T}^3)) \cap L^{1+\delta}(0,\infty; \dot{H}^{\frac{7}{2}+\delta_1}(\mathbb{T}^3))$$
$$u \in C([0,\infty); H^1_d(\mathbb{T}^3)) \cap L^2(0,\infty; H^2(\mathbb{T}^3))$$
$$\rho \in L^\infty([0,\infty) \times \mathbb{T}^3) \cap C([0,\infty); L^s(\mathbb{T}^3)),$$

for a sufficiently small $\delta_1 > 0$, and $1 \leq s \leq 6$. The solution also satisfies the energy equality.

Uniform K, ζ ellipticity I

Definition 5.2 (Uniform (K, ζ)-ellipticity [PS01])

For a complex-valued Banach space X, consider the differential operator $A(t,x)$ with domain $D(A(t,x)) \subset X$, and given by

$$A(t,x) = \sum_{|\alpha|=2m} a_\alpha(t,x)\partial^\alpha.$$

The coefficients a_α are bounded and uniformly continuous from $[0,T] \times \mathbb{R}^n$ to $\mathbb{C}^{N\times N}$. The principal symbol associated with this operator is

$$\tilde{A}(t,x,\xi) = (-1)^m \sum_{|\alpha|=2m} a_\alpha(t,x)\xi^\alpha.$$

Uniform K, ζ ellipticity II

Definition 5.2 (Uniform (K, ζ)-ellipticity [PS01])

*The operator $A(t,x)$ is said to be **uniformly** (K,ζ)-**elliptic** if there exist $K \geq 1$ and $\zeta \in \left[0, \frac{\pi}{2}\right)$ such that*

- $\sum_{|\alpha|=2m} \|a_\alpha\|_{L^\infty_{t,x}} \leq K$,
- $\left|\tilde{A}(t,x,\xi)^{-1}\right| \leq K$, *and*
- $\sigma\left(\tilde{A}(t,x,\xi)\right) \subset \Sigma_\zeta \setminus \{0\}$,

for $(t,x) \in [0,T] \times \mathbb{R}^n$, and $\xi \in \mathbb{R}^n$ with $|\xi| = 1$. Here, $\sigma(B)$ refers to the spectrum of the operator B, and $\Sigma_\zeta := \{z \in \mathbb{C} : |\arg z| \leq \zeta\}$ is a sector in the right half of the complex plane.

Maximal parabolic regularity

Lemma 5.3 (Maximal parabolic regularity)

Let X be a reflexive Banach space and $A : X_1 \mapsto X$ be a (K, ζ)-elliptic operator defined on $D(A) = X_1 \subset X$. For $T > 0$, consider the initial value problem

$$\partial_t u(t) + Au(t) = f(t)$$
$$u(0) = u_0,$$

where $f \in L^r([0,T]; X)$ with $1 < r < \infty$ and $u_0 \in X$. If it is known that u_0 belongs to $Y := (X, X_1)_{1-\frac{1}{r},r}$, then there exists a unique solution $u \in W^{1,r}([0,T]; X) \cap L^r([0,T]; X_1)$ satisfying

$$\|u\|_{L^r([0,T];X)} + \|u\|_{L^r([0,T];X_1)} + \|\partial_t u\|_{L^r([0,T];X)} \leq C_r \left(\|u_0\|_Y + \|f\|_{L^r([0,T];X)} \right).$$

Verifying applicability of MPR

- $\partial_t \psi - \frac{\lambda+i}{2}\Delta\psi = -\frac{\lambda}{2}|u|^2\psi - i\lambda u\cdot\nabla\psi - \mu(\lambda+i)|\psi|^p\psi$
- $A = -\frac{\lambda+i}{2}\Delta$
    - Bounded coefficients
    - $\tilde{A}(\xi) = \frac{\lambda+i}{2}|\xi|^2 \Rightarrow$ inverse is bounded for all $|\xi| = 1$
    - The spectrum $\sigma\left(\frac{\lambda+i}{2}|\xi|^2\right)$ belongs to the sector Σ_{ζ_0} for $\tan\zeta_0 > \lambda^{-1} > 0$
- A is uniformly (K,ζ_0)-elliptic for $K = \max\{\frac{\sqrt{1+\lambda^2}}{2}, \frac{2}{\sqrt{1+\lambda^2}}\}$

A priori estimate for ψ

- Act upon NLS by $(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}$; use MPR with $X = L^2, X_1 = H^2$
-

$$\|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi\|_{L^r_t H^2_x} \lesssim \|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi_0\|_{(L^2_x, H^2_x)_{1-\frac{1}{r},r}} + \||u|^2\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}} + \|u\cdot\nabla\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}} + \||\psi|^p\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}},$$

for $r = 1+\delta$ with $\delta \in (0, \frac{1}{3})$

A priori estimate for ψ

- Act upon NLS by $(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}$; use MPR with $X = L^2, X_1 = H^2$
- $$\|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi\|_{L^r_t H^2_x} \lesssim \|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi_0\|_{(L^2_x, H^2_x)_{1-\frac{1}{r},r}} + \||u|^2\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}} + \|u\cdot\nabla\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}} + \||\psi|^p\psi\|_{L^r_t H_x^{\frac{3}{2}+\delta_1}},$$
  for $r = 1+\delta$ with $\delta \in (0, \frac{1}{3})$
- $$\left\|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi_0\right\|_{(L^2_x, H^2_x)_{1-\frac{1}{r},r}} = \left\|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi_0\right\|_{B^{\frac{2\delta}{1+\delta}}_{2,1+\delta}} \lesssim \left\|(-\Delta)^{\frac{3}{4}+\frac{\delta_1}{2}}\psi_0\right\|_{H^{\frac{2\delta}{1+\delta}+\delta_2}} \leq \|\psi_0\|_{H^{\frac{3}{2}+\frac{2\delta}{1+\delta}+\delta_1+\delta_2}} = \|\psi_0\|_{H^2}$$
- Enough control to ensure $\int_0^T \|\psi\|_{H^2}\|B\psi\|_{H^{\frac{3}{2}+\delta_1}} < f(\text{data})$

What next?

- Compressible (barotropic) system . . .
- Uniqueness . . .
- Strong solutions . . .
- Large initial data (?)
- Allowing vacuum/extending to $\mathbb{R}^3$ (?)
- Including temperature effects (?)

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- Macro-scale model — global-in-time regular in 2D (viscous), local-in-time analytic in 3D (inviscid)
- Macro-scale inviscid (?)

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