Asymptotic behavior for inhomogeneous nonlinear Schrödinger equations

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The Plan

Introduction

We consider the following Damped Inhomogeneous Nonlinear Schrödinger Equation (DINLS)

$$\begin{cases} i\partial_t u + \Delta u + iau = K(x)|u|^{\alpha}u, \\ u(0,.) = u_0 \in H^1(\mathbb{R}^N), \end{cases} \tag{1}$$

where $u = u(t,x) \in \mathbb{C}$, $t > 0$, $x \in \mathbb{R}^N$, $N \geq 1$ and $\alpha > 0$.

- K is a real (complex) valued function potential, or coefficient.
- $a \geq 0$ is a damping term.
- $a = 0$ and $K = \mu$: constant $\rightarrow$ corresponds to the standard (NLS):

$$i\partial_t u + \Delta u = \mu|u|^{\alpha}u$$

- $a = 0 \rightarrow$ (INLS) : $i\partial_t u + \Delta u = K(x)|u|^{\alpha}u$. Examples:

$$K(x) = \mu(1+|x|^2)^{-\sigma}, \text{regular} \ , \ K(x) = \mu|x|^{-b}, \text{singular}$$

$$K(x) = \mu|x|^{-b_1}(1+|x|^2)^{-\frac{b_2}{2}}, \quad \sigma,\ b,\ b_1, b_2 \geq 0,\ \mu \in \mathbb{C}.$$

- Merle, Merle-Raphaël-Szeftel, Non-constant bounded potential $0 < K_1 \leq K(x) \leq K_2 < \infty$
- Fibich-Wang-Liu, $K \in C^4 \cap L^\infty$.
- $a > 0$, $K = \mu \to$ (DNLS) $i\partial_t u + \Delta u + iau = \mu|u|^\alpha u$, M. Tsutsumi (SIAM J. Math. Anal., 15 (1984))
- This kind of equations appears in diverse branches of physics such as Nonlinear Optics: Propagation of a laser beam, plasma physics and fluid mechanics.
    1. L. Bergé, Solition stability versus collapse, Phys. Rev. E, (2000).
    2. F. Genoud, Bifurcation and stability of travelling waves in self-focusing planar waveguides, Adv. Nonlinear Stud., (2010).
    3. G. Fibich, Self-focusing in the damped nonlinear Schrödinger equation, SIAM J. Appl. Math., (2001): The damping (absorption) term plays an important effect in the physical model and it is better to not be neglected.
    4. The potential $K(x)$ accounts for the inhomogeneity of the medium.

Aim

1. Study the impacts of the potential and the damping on:
   - The local existence.
   - The global/blow up existence
   - Asymptotic behavior of the global solutions : the scattering
2. Unify the theories for (NLS) and (DINLS).

Local well-posedness: standard NLS equation, $K(x) = \mu$

T. Cazenave, Semilinear Schrödinger Equations, Courant Lect. (2003).

$$(NLS) \quad i\partial_t u + \Delta u = \mu |u|^\alpha u, \quad \mu \in \mathbb{C}.$$

For $0 < \alpha \leq 4/(N-2)$, the problem (NLS) is well posed in $H^1(\mathbb{R}^N)$:
$\forall u_0 \in H^1$ $\exists!$ solution of (NLS) $u \in C([0, T_{max}(u_0)), H^1)$ and
$u \in C^1([0, T_{max}(u_0)), H^{-1})$.

$$u(t) = e^{it\Delta} u_0 - i\mu \int_0^t e^{i(t-\sigma)\Delta}(|u|^\alpha u(\sigma))d\sigma,$$

$e^{it\Delta}$ is the free Schrödinger group $e^{it\Delta} u_0 = \frac{1}{(4\pi i t)^{\frac{N}{2}}} e^{\frac{i|.|^2}{4t}} * u_0$.

INLS equation: homogeneous case $K = \mu|x|^{-b}$

$(INLS)_S \quad i\partial_t u + \Delta u = \mu|x|^{-b}|u|^\alpha u, \quad \mu \in \mathbb{C}$, $()_S$for this singular potential)

- Scaling: If u is a solution of $(INLS)_S$ then, for all $\lambda > 0$,

$$u_\lambda(t,x) = \lambda^{\frac{2-b}{\alpha}} u(\lambda^2 t, \lambda x)$$

  is also a solution. Moreover,

$$\|\lambda^{\frac{2-b}{\alpha}} u_0(\lambda \cdot)\|_{\dot{H}^s(\mathbb{R}^N)} = \lambda^{s-s_c}\|u_0\|_{\dot{H}^s(\mathbb{R}^N)}, \; \forall \, \lambda > 0,$$

  where the critical Sobolev index : $s_c := \frac{N}{2} - \frac{2-b}{\alpha}$.
- We are interested to initial value in $H^s(\mathbb{R}^N)$ with $s \geq 0$, $s_c \leq s < \frac{N}{2}$:

$$0 \leq s < \frac{N}{2}, \quad 0 < b < 2, \; b < N$$

$$0 < \alpha \leq \frac{4-2b}{N-2s}.$$

How to construct the solution.

The construction of a solution is reduced to find a fixed point for

$$\Phi(u)(t) = e^{it\Delta}u_0 - i\int_0^t e^{i(t-\sigma)\Delta}(K|u|^\alpha u(\sigma))d\sigma.$$

Problem: The linear part can be controlled in $L^\infty(0, T, H^1(\mathbb{R}^N))$:

$$\|e^{it\Delta}u_0\|_{H^1(\mathbb{R}^N)} = \|u_0\|_{H^1(\mathbb{R}^N)}$$

$\longrightarrow$ Choose an adequate functional framework for the nonlinear term.

Idea : Intercept $L^\infty(H^1)$ with another space :

$$X = \{u, \nabla u \in L^\infty(0, T, L^2) \cap L^q(0, T, L^r)\}.$$

Question :

For what value of (q, r), we have

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^N))} \lesssim \|u_0\|_{L^2(\mathbb{R}^N)}?$$

Necessary condition:

By scaling,

$$\frac{2}{q} + \frac{N}{r} = \frac{N}{2}.$$

Sufficient condition: Strichartz estimates

Let (q, r) admissible pair :

$2 \leq q, r \leq \infty$; $\frac{2}{q} + \frac{N}{r} = \frac{N}{2}$ and $(q, r, N) \neq (2, \infty, 2)$. Then

$$\|e^{it\Delta}u_0\|_{L^q(\mathbb{R},L^r(\mathbb{R}^N))} \lesssim \|u_0\|_{L^2(\mathbb{R}^N)}$$

$\longrightarrow$ Strichartz estimates with respect to the second member

$$\left\|\int_0^t e^{i(t-s)\Delta} f(s,.)ds\right\|_{L^{r_1}(\mathbb{R},L^{p_1}(\mathbb{R}^N))} \leq C\|f\|_{L^{r_2'}(\mathbb{R},L^{p_2'}(\mathbb{R}^N))}.$$

where (r_i, p_i) is an admissible pair.

- Case $q = r$: Strichartz (1977)
- Case $(q, r) \neq (2, \frac{2N}{N-2})$: Ginibre-Velo (1989)
- Case End-point $(q, r) = (2, \frac{2N}{N-2})$: Keel-Tao (1998)

For K bounded

$$\left\|\int_0^t e^{i(t-s)\Delta} K|u|^\alpha u(s,.)ds\right\|_{L^\gamma(\mathbb{R},L^\varrho(\mathbb{R}^N))} \leq C\|u\|^{\alpha+1}_{L^{(\alpha+1)\gamma'}(\mathbb{R},L^{(\alpha+1)\varrho'}(\mathbb{R}^N))}.$$

Problem : For (INLS) we can not use the same framework

In fact the potential $K(x) = |x|^{-b}$ can not be estimated in Lebesgue spaces.

(1) **Genoud-Stuart, DCDS (2008)**: $0 < \alpha < \frac{4-2b}{N-2}$
$|x|^{-b} = \chi_{B(0,1)}|x|^{-b} + \chi_{^cB(0,1)}|x|^{-b}$

- Applying an abstract result of Cazenave.
- The critical case is not treated.
- No results of regularity.

(2) **Guzman, N. Anal. (2017)**: $N \geq 3$, $0 < \alpha < \frac{4-2b}{N-2}$.

- Based on the Strichartz estimates.
- The critical case is not treated

(3) **Dinh, J. Evol. Equ. (2019)** improved the result of Guzman, $N = 2$, $0 < b < 1$, $\alpha > 0$.

(4) **Kim-Lee-Seo** : JDE (2021) H^s-theory, $0 \leq s < \frac{1}{3}$. **Lee and Seo** : Arch. Math. (2021) H^1-theory.

- Based on Strichartz estimates with weighted Lebesgue spaces
- The critical case is treated $\alpha = \frac{4-2b}{N-2}$
- No blow-up criterium
- No continuous dependence result
- No unconditional uniqueness result

$\longrightarrow$ The proofs are based on Strichartz estimates with

weighted Lebesgue spaces

$\longrightarrow$ These estimates are obtained by interpolating Strichartz estimates with smoothing effect of Kato and Yajima, using the optimal range obtained by Sugimoto, Vilela, Watanabe

$\longrightarrow$ For $u_0 \in H^1$, they constructed the local solution in

$$C([0,T],H^1) \cap L^q([0,T],W^{1,r}(|x|^{-r\gamma}dx))$$

$$\frac{2}{q}+\frac{N}{r}-\gamma=\frac{N}{2}, \gamma\text{-Schrödinger admissible pair } (q,r)$$

$$\max(0,\frac{b-1-\alpha}{\alpha+1})<\gamma<\min(\frac{N-2}{2},\frac{b}{\alpha+1}).$$

Problem

- There is no unified theory with the case $b = 0$.
- There is no complete local theory, and getting properties useful to study the asymptotic behavior of solutions.

$\longrightarrow$ The previous frameworks seem not appropriate to develop the local and global theories.

Different approach (Aloui-T.)

To handle the singular potential, we use Lorentz spaces. By real interpolation $L^{p,q} = [L^1, L^\infty]_{1-\frac{1}{p},q}, \quad p > 1, q \geq 1.$

- $|x|^{-b} \in L^{\frac{N}{b},\infty}$, $0 < b < N$.
- Refinement of the Lebesgue spaces : $q_1 \leq p \leq q_2$
  $L^{p,1} \subset \cdots \subset L^{p,q_1} \subset L^{p,p} = L^p \subset L^{p,q_2} \subset \cdots L^{p,\infty}$.
- The Hölder and Young inequalities hold in Lorentz spaces.
- Sobolev-Lorentz embedding:

$$\dot{W}_q^{s,p}(\mathbb{R}^N) = \{f \in \mathcal{S}'(\mathbb{R}^N),\ (-\Delta)^{s/2} f \in L^{p,q}\}.$$

$$\dot{W}_q^{s,p}(\mathbb{R}^N) \subset L^{\tilde{p},q}, \quad \frac{1}{\tilde{p}} = \frac{1}{p} - \frac{s}{N}, 0 < s < \frac{N}{p}.$$

- Sobolev-Lorentz embedding: $\dot{H}^1 \subset L^{\frac{2N}{N-2},2} \subset L^{\frac{2N}{N-2}}$
- $\|f(\lambda\cdot)\|_{L^{p,q}} = \lambda^{-\frac{N}{p}}\|f\|_{L^{p,q}},\ \lambda > 0$. The second index q does not intervene, Lorentz spaces has the same scaling as Lebesgue spaces.
- Strichartz in Lorentz spaces: Keel-Tao (1998)
  (i) Let (r, p) be an admissible pair. Then

$$\left\|e^{it\Delta}\varphi\right\|_{L^r(\mathbb{R},L^{p,2})} \leq C\|\varphi\|_{L^2}.$$

  (ii) Let (r_i, p_i); $i = 1, 2$ be two admissible pairs. Then

$$\left\|\int_0^t e^{i(t-s)\Delta} f(s,.)ds\right\|_{L^{r_1}(\mathbb{R},L^{p_1,2})} \leq C\|f\|_{L^{r_2'}(\mathbb{R},L^{p_2',2})}.$$

We choose ρ such that $u \mapsto |x|^{-b}|u|^{\alpha}u$ applies $\dot{W}_2^{s,\rho}(\mathbb{R}^N)$ into $\dot{W}_2^{s,\rho'}(\mathbb{R}^N)$.

$$\rho = \frac{N(\alpha+2)}{N+\alpha s - b}, \qquad \gamma = \frac{4(\alpha+2)}{\alpha(N-2s)+2b}.$$

Theorem (Aloui-T.)

Let $N \geq 1$, $\mu \in \mathbb{C}$, $0 \leq s \leq 1$ and $s < N/2$. Assume that $0 < b < \min(2, N-2s)$ and

$$0 < \alpha \leq \frac{4-2b}{N-2s}.$$

Then for every $u_0 \in H^s(\mathbb{R}^N)$ there exist $T_{\max}(u_0) > 0$ and a unique solution $u \in C\left([0, T_{\max}(u_0)), H^s(\mathbb{R}^N)\right) \cap L^{\gamma}_{\mathrm{loc}}\left([0, T_{\max}(u_0)), W_2^{s,\rho}(\mathbb{R}^N)\right)$.

Theorem (continuation)

(i) u *is unique in* $L^\gamma\left(0, T; W_2^{s,\rho}(\mathbb{R}^N)\right)$; $0 < T < T_{\max}(u_0)$.

(ii) $u \in L^r_{loc}\left(0, T_{\max}(u_0), W_2^{s,p}(\mathbb{R}^N)\right)$, (r, p) *is an admissible pair* .

(iii) *Let* $R > 0$ *and* $\mathcal{K}$ *be a compact of* $H^s(\mathbb{R}^N)$. *If* $\alpha < (4-2b)/(N-2s)$ *(respectively,* $\alpha = (4-2b)/(N-2s)$*), then there exists* $T = T(R) > 0$ *(respectively* $T = T(\mathcal{K}) > 0$*), such that for all* $u_0 \in B_R(H^s)$ *(respectively,* $u_0 \in \mathcal{K}$*),* $(INLS)$ *has a unique solution* u *in* $C\left([0, T], H^s(\mathbb{R}^N)\right) \cap L^\gamma(0, T; W_2^{s,\rho}(\mathbb{R}^N))$.

(iv) *(Blow-up criterion) If* $T_{\max}(u_0) < \infty$*, then* $\|u\|_{L^\gamma\left(0, T_{\max}(u_0); W_2^{s,\rho}(\mathbb{R}^N)\right)} = \infty$.

Theorem (continuation)

(v) *(Global existence for small data)*
If $\alpha = (4-2b)/(N-2s)$ *and* $\|(-\Delta)^{s/2}u_0\|_{L^2(\mathbb{R}^N)}$ *is sufficiently small, then* $T_{\max}(u_0) = \infty$.

(vi) *(Lower estimate of the blow-up rate)*
If $\alpha < (4-2b)/(N-2s)$, *then for* $0 < t < T_{\max}(u_0)$,

$$\|u(t)\|_{\dot{H}^s(\mathbb{R}^N)} \geq C(T_{\max}(u_0) - t)^{-\frac{4-2b-(N-2s)\alpha}{4\alpha}}.$$

In particular, if $T_{\max}(u_0) < \infty$, *then* $\lim_{t \to T_{\max}(u_0)} \|u(t)\|_{\dot{H}^s(\mathbb{R}^N)} = \infty$.

(vii) *(Lower estimate of the life-span)*
Let $\lambda > 0$ *and* $\alpha < (4-2b)/(N-2s)$,

$$T_{max}(\lambda u_0) \geq C\lambda^{-\frac{4\alpha}{4-2b-(N-2s)\alpha}}.$$

Comments

- Our result and its proof remain valid for the case $b = 0$. In this context, it unifies the results for $b = 0$ and $b > 0$.
- We obtain more precise regularity results with respect to the known ones.

$$L^q(L^{r_1,2}) \subset L^q(L^{r_1}), \quad r_1 \geq 2$$

- We give a different and simple proof compared with the known ones
- Lower estimates of the blowup rate
- unconditional uniqueness
- properties of T_{max} (l.s.c.)
- Aloui-T. if $N = 3$, $s = 1$ LWP holds also for $1 < b < 3/2$ (preprint). An and Kim extend the values of s and b using the same frame work (Strichartz estimates in Sobolev-Lorentz spaces) Evolution Equations & Control Theory, 2022.

Strategy to construct the local solution for $K \in L^{\frac{N}{b},\infty}$, $\nabla K \in L^{\frac{N}{b+1},\infty}$

$\rightarrow$ Choose an admissible pair (γ, ρ) such that the map $u \mapsto K|u|^{\alpha}u$ applies $L^{\rho,2}(\mathbb{R}^N)$ into $L^{\rho',2}(\mathbb{R}^N)$.

$\rightarrow$ Apply Strichartz estimates to check the hypothesis of the fixed point theorem.

$\Longrightarrow$ We construct a local solution
$u \in L^{\infty}\left(0, T, H^1(\mathbb{R}^N)\right) \cap L^{\gamma}\left(0, T, W_2^{1,\rho}(\mathbb{R}^N)\right)$

$\rightarrow$ Apply Strichartz to prove

1. the uniqueness in $L^{\infty}\left(0, T, H^1(\mathbb{R}^N)\right)$
2. $u \in C\left([0, T], H^1(\mathbb{R}^N)\right)$

By construction $(s=1)$

1. $0<\alpha<\frac{4-2b}{N} \Longrightarrow T_{max}(u_0)$ depends on $\|u_0\|_{L^2}$
2. $\frac{4-2b}{N}\leq\alpha<\frac{4-2b}{N-2} \Longrightarrow T_{max}(u_0)$ depends on $\|\nabla u_0\|_{L^2}$

For $K\in\mathbb{R}$

- Conservation of the mass $\|u(t)\|_{L^2}=\|u(0)\|_{L^2}$
  $0<\alpha<\frac{4-2b}{N} \Longrightarrow T_{max}(u_0)=+\infty$: Global existence
- Conservation of the energy $E(u)(t)=E(u)(0)$

$$E(u)(t)=\frac{1}{2}\|\nabla u(t)\|^2_{L^2(\mathbb{R}^N)}+\frac{1}{\alpha+2}\int_{\mathbb{R}^N}K(x)|u(t,x)|^{\alpha+2}dx$$

K real valued, we distinguish two cases

1. Defocusing case : $K>0$
2. Focusing case : $K<0$

We denote $\mathcal{G} = \{u_0; T_{max}(u_0) = +\infty\}$

	$0 < \alpha < \frac{4-2b}{N}$	$\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2}$
Defocusing	$\mathcal{G} = H^1$	$\mathcal{G} = H^1$
Focusing	$\mathcal{G} = H^1$	???

Focusing with $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$

- $\exists$ global solution : $\|u_0\|_{H^1} << 1$
- $\exists$ Blow-up solutions: $u_0 \in H^1$, $xu_0 \in L^2$ with $E(u_0) < 0$.

See for the complex coefficients

Global existence for oscillating data

For $K(x) = \mu \in \mathbb{C}$, Cazenave-Weissler (Commun. Math. Phys. 1992) introduced another type of initial data : Oscillating Data

$$u_0 = e^{id|x|^2}\varphi$$

($d > 0$ large)$\Longrightarrow$ (global existence)

They introduced the notion of rapidly decay solution:

$$u \in L^a([0,\infty), L^{\alpha+2}), \quad a = \frac{2\alpha(\alpha+2)}{4-\alpha(N-2)}.$$

$$\begin{cases} \|e^{it\Delta}u_0\|_{L^a([0,\infty),L^{\alpha+2})} \text{ small} \\ \alpha_0 < \alpha < \dfrac{4}{N-2} \end{cases} \Longrightarrow \text{global existence.}$$

α_0 is the positive root of $N\alpha^2 + (N-2)\alpha - 4 = 0$

$$\frac{4}{N+2} < \alpha_0 < \frac{4}{N}.$$

Y. Tsutsumi (IHP 1985)

We introduce the following positive real numbers

$$\varrho = \frac{N(\alpha+2)}{N-b}, \; a = \frac{2\alpha(\alpha+2)}{4-2b-\alpha(N-2)}.$$

$\longrightarrow$ The value of ϱ allows the map $f \to |x|^{-b}|f|^{\alpha}f$ to apply $L^{\varrho,q}$ into $L^{\varrho',\frac{q}{\alpha+1}}$.

$\longrightarrow$ $\|e^{it\Delta}\lambda^{\frac{2-b}{\alpha}}f(\lambda.)\|_{L^a(0,\infty;L^{\varrho,q})} = \lambda^{\frac{2-b}{\alpha}-\frac{N}{\varrho}-\frac{2}{a}}\|e^{it\Delta}f\|_{L^a(0,\infty;L^{\varrho,q})}$.

The value of a guarantees (independent of λ)

$$\frac{2}{a} + \frac{N}{\varrho} = \frac{2-b}{\alpha}.$$

We will use Strichartz estimates for non-admissible pairs, so need

$$\frac{1}{a} < \frac{2}{\zeta} = \frac{N}{2} - \frac{N}{\varrho} \Longleftrightarrow \alpha > \alpha_0(b)$$

$\alpha_0(b)$ is the positive root of the equation

$$N\alpha^2 + (N - 2 + 2b)\alpha - 4 + 2b = 0.$$

$$\frac{4 - 2b}{N + 2} < \alpha_0(b) < \frac{4 - 2b}{N},$$

$\longrightarrow \alpha_0(0) = \alpha_0$ Strauss exponent
$\longrightarrow \alpha_0(2) = 0.$

Theorem (Aloui-T.)

Let $N \geq 4$, $q \geq 1$, $u_0 \in H^1(\mathbb{R}^N)$. *Assume that*

$$\alpha_0(b) < \alpha < \frac{4-2b}{N-2}.$$

Then there exists $\varepsilon > 0$ *such that if*

$$\|e^{it\Delta}u_0\|_{L^a(0,\infty;L^{\varrho,q}(\mathbb{R}^N))} \leq \varepsilon,$$

then u is global. Moreover, there exists a constant $C > 0$, *s. t.*

$$\|u\|_{L^a(0,\infty;L^{\varrho,q}(\mathbb{R}^N))} \leq 2\|e^{it\Delta}u_0\|_{L^a(0,\infty;L^{\varrho,q}(\mathbb{R}^N))},$$

$$\|u\|_{L^r(0,\infty;W_2^{s,p}(\mathbb{R}^N))} \leq C\|u_0\|_{H^s(\mathbb{R}^N)},$$

for every admissible pair (r, p)

Sketch of the proof:

- Let ς; s. t. (ς, ϱ) is an admissible pair

$$\alpha > \alpha_0(b) \Longleftrightarrow a > \frac{\varsigma}{2}$$

Strichartz non-admissible: If $a > \varsigma/2$ and $\frac{1}{\tilde{a}} + \frac{1}{a} = \frac{2}{\varsigma}$, then

$$\left\| \int_0^t e^{i(t-s)\Delta} f(s,\cdot)ds \right\|_{L^a(0,T;L^{\varrho,q}(\mathbb{R}^N))} \leq C\|f\|_{L^{\tilde{a}'}(0,T;L^{\varrho',q}(\mathbb{R}^N))}$$

$$\|u\|_{L^a(0,T;L^{\varrho,q}(\mathbb{R}^N)} \leq \varepsilon + C\|u\|^{\alpha+1}_{L^a(0,T;L^{\varrho,q}(\mathbb{R}^N)},$$

$$\begin{aligned}\|u\|_{L^\varsigma(0,T;W_2^{1,\varrho}(\mathbb{R}^N))} \leq & C\|u_0\|_{H^1(\mathbb{R}^N)} \\ & + C\|u\|^{\alpha}_{L^a(0,T;L^{\varrho,q}(\mathbb{R}^N))}\|u\|_{L^\varsigma(0,T;W_2^{1,\varrho}(\mathbb{R}^N))}.\end{aligned}$$

Comments

- Our result improves the one of Guzmán, Nonlinear Anal., (2017) which proved the global existence under the conditions:
  1. $\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2}$
  2. Smallness of $\|e^{it\Delta}u_0\|_{L^\sigma(0,\infty;L^p(\mathbb{R}^N))}$, for all (σ, p);

$$\frac{2}{\sigma} + \frac{N}{p} = \frac{2-b}{\alpha}.$$

(the pair (a, ϱ) satisfies this equality)

Examples

(1) Let $\varphi \in H^1(\mathbb{R}^N)$, $\|\varphi\|_{H^1} << 1$ and $\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2}$
$\Longrightarrow$ Global existence in $\mathbb{R}$:

- Sobolev embedding $\Longrightarrow e^{it\Delta}\varphi \in L^\infty L^{\varrho,2}(\mathbb{R}^N)$.
- Strichartz $\Longrightarrow e^{it\Delta}\varphi \in L^\varsigma L^{\varrho,2}(\mathbb{R}^N)$, (ς, ϱ) is an admissible pair.

$$\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2} \Longrightarrow a \geq \varsigma$$

Interpolation $\Longrightarrow \|e^{it\Delta}\varphi\|_{L^a(\mathbb{R};L^{\varrho,2})} \leq C\|\varphi\|_{H^1}$.

(2) Let $\varphi \in H^1 \cap L^{\varrho',\infty}$, $\|\varphi\|_{H^1} + \|\varphi\|_{L^{\rho',\infty}} << 1$ and $\alpha_0(b) < \alpha < \frac{4-2b}{N-2}$.
$\Longrightarrow$ Global existence in $\mathbb{R}$:

- $\varphi \in L^{\varrho',\infty} \Longrightarrow \|e^{it\Delta}\varphi\|_{L^{\varrho,\infty}(\mathbb{R}^N)} \leq C|t|^{-N(\frac{1}{2}-\frac{1}{\varrho})}\|\varphi\|_{L^{\rho',\infty}(\mathbb{R}^N)}$.

$$\alpha > \alpha_0(b) \Longleftrightarrow Na(\frac{1}{2} - \frac{1}{\varrho}) = a\frac{2}{\varsigma} > 1$$

$$\|e^{it\Delta}\varphi\|_{L^a(\mathbb{R};L^{\varrho,\infty})} \leq C\left(\|\varphi\|_{H^1} + \|\varphi\|_{L^{\rho',\infty}}\right).$$

(3) Let $\varphi \in \Sigma := \{\varphi \in H^1, |x|\varphi \in L^2\}$.
By Hölder estimate in Lorentz spaces and interpolation
$\Longrightarrow \Sigma \subset H^1 \cap L^{\frac{2N}{N+2},2} \subset H^1 \cap L^{\varrho',2}$. (2), $\|\varphi\|_\Sigma << 1, \alpha_0(b) < \alpha \Longrightarrow$ global existence

(4) Let $\varphi \in \Sigma$. Set

$$u_0(x) := e^{\frac{id|x|^2}{4}}\varphi(x) \in H^1.$$

$$d >> 1,\ \alpha_0(b) < \alpha < \frac{4-2b}{N-2} \Longrightarrow T_{\max}(u_0) = \infty.$$

$$\|e^{it\Delta}u_0\|^a_{L^a(0,\infty;L^{\varrho,2})} \leq C\|\varphi\|^a_{H^1(\mathbb{R}^N)} \int_0^{\frac{1}{d}} (1-d\tau)^{-2(\varsigma-a)/\varsigma} d\tau < \infty,$$

$$\alpha > \alpha_0(b) \Rightarrow 2(\varsigma - a)/\varsigma < 1.$$

$\longrightarrow$ In the particular case $K(x) = \mu|x|^{-b}$ with $\mu < 0$,

$$\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2}, \quad E(\varphi) < 0$$

$T_{\min}(u_0) < \infty$.

A global solution u of (INLS) scatters in X if there exists $\varphi^+ \in X$ such that

$$\lim_{t\to\infty} \|e^{-it\Delta}u(t) - \varphi^+\|_X = 0.$$

$X = H^1$, we may write $\lim_{t\to\infty} \|u(t) - e^{it\Delta}\varphi^+\|_X = 0$.

$$z(t) := e^{-it\Delta}u(t) = u_0 - i\int_0^t e^{-is\Delta}(K|u|^\alpha u(s))ds.$$

$$\|z(t) - z(\tau)\|_{H^1} \leq \||x|^{-\frac{b}{\alpha+2}}u\|^\alpha_{L^a((t,\tau),L^{\alpha+2,\infty})}\|u\|_{L^\varsigma(0;\infty,W_2^{1,\varrho})}$$

$$\|u\|_{L^\varsigma([T,t],W_2^{1,\varrho})} \lesssim \|u(T)\|_{H^1} + \||x|^{-\frac{b}{\alpha+2}}u\|^\alpha_{L^a([T,t],L^{\alpha+2,\infty})}\|u\|_{L^\varsigma([T,t],W_2^{1,\varrho})}.$$

H^1 and Σ-Scattering criterion

Assume $N \geq 4$, $0 < b < 2$,

$$0 < \alpha < \frac{4-2b}{N-2}$$

$$|K(x)| \lesssim |x|^{-b}, \qquad |\nabla K(x)| \lesssim |x|^{-b-1},$$

Theorem (Aloui-T., Scattering criterion)

Let $u \in C([0,\infty), H^1)$ be a global solution of $(INLS)$. If $|x|^{-\frac{b}{\alpha+2}} u \in L^a(0,\infty; L^{\alpha+2,\infty}(\mathbb{R}^N))$ then

- *$u \in L^r(0,\infty; W_2^{1,p}(\mathbb{R}^N))$, for any admissible pair (r,p).*
- *u scatters in $H^1(\mathbb{R}^N)$.*
- *If $u_0 \in \Sigma$ then $u \in C([0,\infty),\Sigma)$ and it scatters in Σ.*

H^1 and Σ-Scattering criterion

Let $u \in C([0,\infty), H^1)$ be a global solution of $(INLS)$. If

(i) $N \geq 1$, $0 < b < \min(N,2)$, $K, \nabla K \in L^{\frac{N}{b},\infty}$ and $u \in L^a(0,\infty; L^{\varrho,\infty}(\mathbb{R}^N))$.

(ii) $N \geq 4$, $0 < b < 2$, $K \in L^{\frac{N}{b},\infty}$, $\nabla K \in L^{\frac{N}{b+1},\infty}$ and $u \in L^a(0,\infty; L^{\varrho,\infty}(\mathbb{R}^N))$.

$$0 < \alpha < \frac{4-2b}{N-2}$$

then

- $u \in L^r(0,\infty; W_2^{1,p}(\mathbb{R}^N))$, for any admissible pair (r,p).
- u scatters in $H^1(\mathbb{R}^N)$.
- If $u_0 \in \Sigma$ then $u \in C([0,\infty),\Sigma)$ and it scatters in Σ.

Scattering for oscillating data

Theorem

Let $N \geq 4$, $u_0 \in H^1(\mathbb{R}^N)$. *Assume that*

$$\alpha_0(b) < \alpha < \frac{4-2b}{N-2}.$$

Then there exists $\varepsilon > 0$ *such that if*

$$\|e^{it\Delta}u_0\|_{L^a(0,\infty;L^{\varrho,\infty}(\mathbb{R}^N))} \leq \varepsilon, \tag{2}$$

then u *is global and scatters*

Applications: Scattering result for regular potential decaying at infinity

Let

$$K(x) = \frac{1}{1+|x|^2} \in L^\infty$$

It is known that for bounded potential, the scattering for small initial data in H^1 holds if

$$\frac{4}{N} < \alpha < \frac{4}{N-2}$$

and in Σ if

$$\frac{4}{N+2} < \alpha < \frac{4}{N-2}.$$

Using our results, we can prove the scattering for small initial data in H^1 and Σ for

$$0 < \alpha < \frac{4}{N-2}.$$

Scattering result for regular potential decaying at infinity

The following result shows that if the potential is regular and decays at infinity faster than $|x|^{-2}$ then the scattering occurs for the whole range $0 < \alpha < 4/(N-2)$ if the initial data is small in $H^1(\mathbb{R}^N)$, small in Σ or oscillating.

Corollary

Let $N \geq 1$, $K(x) = \mu(1+|x|^2)^{-\sigma/2}$ with $\mu \in \mathbb{C}$, $\sigma > 0$ and $0 < \alpha < \frac{4}{N-2}$. Then the following hold.

(i) *If $\alpha > \frac{4-2\sigma}{N}$ and u_0 is small in $H^1(\mathbb{R}^N)$, then u is global and scatters in $H^1(\mathbb{R}^N)$.*

(ii) *If $\alpha > \alpha_0(\min(\sigma, 2))$ and u_0 is small in Σ, then u is global and scatters in Σ.*

(iii) *If $\alpha > \alpha_0(\min(\sigma, 2))$ and $u_0(x) = e^{\frac{id|x|^2}{4}}\varphi(x)$ where $\varphi \in \Sigma$, then there exists $d_0 < \infty$ such that for every $d \geq d_0$, the solution u is global. Furthermore $u \in L^a\left(0, \infty; L^{\varrho,2}(\mathbb{R}^N)\right)$ and it scatters in Σ as $t \to \infty$.*

Scattering result for regular potential decaying at infinity.

Sketch of the proof

$$K = \frac{1}{1+|x|^2},$$

$$K, \nabla K \in L^{\frac{N}{b},\infty}, \forall\, 0 \leq b \leq 2$$

Fix $0 < \alpha < \frac{4}{N-2}$.

- We choose b such that $\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}$
- We apply our result for this b, we obtain the scattering for small data.

Scattering for potentials with different powers for singularity and decay

Corollary

Let $0 \leq b_1 < \min(2, N)$, $b_2 > b_1$ *and* K *satisfying*

$$|K(x)| \lesssim |x|^{-b_1}(1+|x|^2)^{-(b_2-b_1)/2},$$

$$|\nabla K(x)| \lesssim \begin{cases} |x|^{-b_1-1}(1+|x|^2)^{-(b_2-b_1)/2} & \text{if } b_1 > 0, \\ (1+|x|^2)^{-(b_2+1)/2} & \text{if } b_1 = 0. \end{cases}$$

Let $\alpha_0(\min(2, b_2)) < \alpha < \frac{4-2b_1}{N-2}$ *and* b *be such that*

$$\max\left(b_1, \frac{4-N\alpha^2-\alpha(N-2)}{2(\alpha+1)}\right) \leq b \leq \min\left(b_2, \frac{4-(N-2)\alpha}{2}\right).$$

Corollary

Let $u_0 \in H^1$. *Then there exists* $\varepsilon = \varepsilon(b_1, b_2, \alpha) > 0$, *such that if*

$$\|e^{it\Delta}u_0\|_{L^a(0,\infty;L^{\varrho,\infty}(\mathbb{R}^N))} \leq \varepsilon,$$

the solution of (INLS) is global and scatters in $H^1(\mathbb{R}^N)$. *Moreover, if* $u_0 \in \Sigma$ *then* u *scatters in* Σ *as* $t \to \infty$.

Sketch of the proof Since $b \in [0, 2) \to \alpha_0(b) \in (0, \alpha_0(0)]$ is decreasing then

More the potential decreases

$$\Downarrow$$

More the range of allowed α giving scattering is wider.

Defocusing case: Σ-Scattering

Theorem (Aloui-T.)

Let $u_0 \in \Sigma$. *Assume that* $K(x) = |x|^{-b}$

$$\alpha_0(b) < \alpha < \frac{4-2b}{N-2}.$$

Then $|x|^{-\frac{b}{\alpha+2}} u \in L^a(0,\infty, L^{\alpha+2,\infty}(\mathbb{R}^N))$ *and hence* u *scatters in* Σ.

Sketch of the proof

- Decay estimates: $\||x|^{-\frac{b}{\alpha+2}} u(t)\|_{L^{\alpha+2,2}(\mathbb{R}^N)} \le C|t|^{-N\left(\frac{1}{2}-\frac{1}{\varrho}\right)}$.
  This result improves the one of Dinh $\left(\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}\right)$
- $\left(\alpha > \alpha_0(b) \Rightarrow Na\left(\frac{1}{2}-\frac{1}{\varrho}\right) > 1\right) + \left(u \in C([0,\infty), H^1(\mathbb{R}^N))\right)$

$$\Longrightarrow |x|^{-\frac{b}{\alpha+2}} u \in L^a(0,\infty, L^{\alpha+2,\infty}(\mathbb{R}^N)).$$

Remarks

Aloui-Grira-T., we prove ($K(x) = |x|^{-b}$, $N \geq 3$)

- Defocusing :The scattering in Σ for $\alpha = \alpha_0(b)$
- The scattering for small data in Σ and $\frac{4-2b}{N+2} < \alpha < \frac{4-2b}{N-2}$
- The scattering for small data in Σ_s and $\frac{4-2b}{N+2+2s} < \alpha < \frac{4-2b}{N-2}$

$$\Sigma_s := \{v \in \Sigma; v,\ \nabla v,\ |.|v\ \in \dot{H}^s\}.$$

$s \in (0, s_0]$, s_0 the positive root of $2s^2 + (N+4-2b)s + 2 - 2b - N = 0$. the smallest α is $\tilde{\alpha}(b) := \frac{4-2b}{N+2+2s_0}$ is the positive root of the following equation $N\alpha^2 + (N+2b)\alpha + 2b - 4 = 0$, which is given by

$$\tilde{\alpha}(b) = \frac{8-4b}{(N+2b) + \sqrt{N^2 + 16N + 4b^2 - 4bN}} (> \frac{2-b}{N} > \frac{2-2b}{N}).$$

- Self-similar and asymptotically self-similar large time behavior: Aloui-BenMosbah-T.

Global existence and scattering for (DINLS)

We consider now the (DINLS)

$$\begin{cases} i\partial_t u + \Delta u + iau = \mu|x|^{-b}|u|^{\alpha}u, \\ u(0,.) = u_0 \in H^1(\mathbb{R}^N), \end{cases}$$

We recall that for $a = 0$ we have

1. For the focusing case ($\mu < 0$) and $\frac{4-2b}{N} \leq \alpha < \frac{4-2b}{N-2}$: $\exists$ Blow-up solutions. Ginibre-Velo, Glassey, M. Tsutsumi
2. Even for the defocusing case ($\mu > 0$) and $0 < \alpha < \frac{4-2b}{N}$: $\exists$ global solutions which do not scatter. J. Barab

Question : What is the impact of the damping term on the global and scattering results?

We define the following quantities.

$$L^2\text{-norme}: \quad M(f) := \|f\|^2_{L^2(\mathbb{R}^N)}. \tag{3}$$

$$\text{Energy}: \quad E(f) := \frac{1}{2}\|\nabla f\|^2_{L^2(\mathbb{R}^N)} + \frac{\mu}{\alpha+2}\||x|^{-b}|f|^{\alpha+2}\|_{L^1(\mathbb{R}^N)}. \tag{4}$$

For real potential ($\mu \in \mathbb{R}$), we have

$$\partial_t \|u(t)\|^2_{L^2(\mathbb{R}^N)} = -2a\|u(t)\|^2_{L^2(\mathbb{R}^N)},$$
$$\partial_t E(u(t)) =: -aK(u(t)),$$

where

$$K(f) = \|\nabla f\|^2_{L^2(\mathbb{R}^N)} + \mu \int_{\mathbb{R}^N} |x|^{-b}|f|^{\alpha+2}dx.$$

- $M, \; E$ are no longer conserved,
- $E(0) \leq 0 \nRightarrow E(t) \leq 0$
- the L^2-norm of solutions decays exponentially if solutions exist globally.

Then naturally arises the question whether or not the damping term prevents blowing up of solutions $\Longrightarrow$ Intuitively the damping term will arrest the blow-up

Focusing case $\mu < 0$: Blow-up solutions

$$\frac{4-2b}{N} < \alpha < \frac{4-2b}{N-2}, \; u_0 \in H^1, \; xu_0 \in L^2$$

Set $V_0 = \Im m \int_{\mathbb{R}^N} x.\nabla u_0(x)\overline{u_0}(x)dx$ and $I_0 := \| | \cdot | u_0\|^2_{L^2(\mathbb{R}^N)}$.
Assume that one of the following holds

(i) $E(u_0) < 0$,

(ii) $E(u_0) = 0$ and $V_0 < 0$

(iii) $E(u_0) > 0$ and $V_0 < -\sqrt{2E(u_0)I_0}$.

1. $a = b = 0$, R. T. Glassey, J. Math. Phys., (1977): $T_{\max}(u_0) < \infty$.
2. $a > 0$, $b = 0$, M. Tsutsumi, SIAM J. Math. Anal. (1984): (i) or (ii) Then there exists $a_\star(u_0)$ such that for $a \in [0, a_\star(u_0))$

$$T_{\max}(u_0) < \infty.$$

1. $a>0$ $b=0$, M. Ohta and G. Todorova, Discrete Contin. Dyn. Syst. (2009):
   - There exists $a_\star(u_0)>0$ such that if $a<a_\star(u_0)$, (iii) holds then $T_{\max}(u_0)<\infty$.
   - There exists $a^\star(\|u_0\|_{H^1})>0$ such that if $a>a^\star(\|u_0\|_{H^1})$ then $T_{\max}(u_0)=\infty$.
2. $a>0$, $b>0$, Aloui-Jbari-T.: (i) or (ii) or (iii) We obtain a lower bound of $a_\star(u_0)$: For instance, for (i) and (ii), we have

$$a_\star(u_0)\geq\frac{(N\alpha-4+2b)}{2\alpha}\frac{\sqrt{V_0^2-4E(u_0)I_0}-V_0}{I_0}$$

For (i) $\tilde{a}=\frac{4\alpha}{N\alpha-4+2b}a\in\left[0,\frac{\sqrt{V_0^2-4E_0I_0}-V_0}{2I_0}\right)$ then

$$T_{\max}(u_0)\leq\frac{2I_0}{\sqrt{16(V_0^2-2I_0E_0)-I_0^2\tilde{a}^2}-I_0\tilde{a}-4V_0},$$

For (ii)

$$T_{\max}(u_0)\leq\frac{1}{\tilde{a}}\log\left(1+\frac{I_0\tilde{a}}{|I_0\tilde{a}+4V_0|}\right).$$

We shows that the more negative the value of the variance becomes, the wider the range of damping values giving rise to blow up.

Numerical study

G. Fibich , SIAM J. APPL. MATH. (2001):

$$N = 2,\ b = 0,\ \alpha = \frac{4-2b}{N} = 2, \mu = -1,\ \text{Focusing case}.$$

$$i\partial_t u + \Delta u + iau = -|u|^2 u, \quad in \quad \mathbb{R}^2.$$

For $u_0 \in H^1(\mathbb{R}^2)$ that leads to blowup in the undamped NLS there exists $a\star(u_0)$ such that

- For $0 \leq a < A\star$, we have blow-up
- For $a > A\star$, we have global solution

But there is no definite answer to this question based on rigorous analysis.
Conjecture (Fibich) : For a given initial condition u_0, $a \to T_{\max}(u_0)$ is monotonically increasing in $a < A\star$.

Some references:

- M. Cardoso and L. G. Farah, Blow-up of non-radial solutions for the L2 critical inhomogeneous NLS equation, Nonlinearity, 35 (2022), 4426-4436.

- R. Bai and B. Li, *Blow-up for the inhomogeneous nonlinear Schrödinger equation*, Nonlinear Analysis 232 (2023), 113266.

- C. Besse, R. Carles, N. Mauser and H.-P. Stimming, *Monotonicity properties of blow-up time for nonlinear Schrödinger equation: numerical tests*, Discrete Cont. Dyna. Syst.-B 9 (2008), 11–36.

- G. Fibich, *The nonlinear Schrödinger equation. Singular solutions and optical collapse*, Applied Mathematical Sciences Vol. 192 (2014), Springer

Focusing case $\mu < 0$: Global solutions

T. Inui, Proceedings of the AMS, (2019) $b = 0$
Aloui-Jbari-T. $b > 0$: Assume that

- $0 < \alpha < \frac{4-2b}{N}$, or
- $\alpha = \frac{4-2b}{N}$ and $\|u_0\|_2 < \|Q\|_2$ where Q is the ground state

$$\Delta Q - Q + |x|^{-b}|Q|^{\frac{4-2b}{N}}Q = 0.$$

Then we have global existence and exponential scattering:

$$\lim_{t\to\infty} e^{at}\|u(t) - e^{t(i\Delta - a)}\varphi^+\|_{H^1} = 0.$$

Defocusing case $\mu > 0 \Longrightarrow$ Global existence

- T. Inui, Proceedings of the AMS, (2019):

$$b = 0, \ \mu > 0, \ 0 < \alpha < \frac{4}{N-2}$$

For all $a > 0$, every solution is global and exponentially scatters:

$$\lim_{t \to \infty} e^{at} \| u(t) - e^{t(i\Delta - a)} \varphi^+ \|_{H^1} = 0.$$

- Aloui-Jbari-T.:

$$b > 0, \ \mu > 0, \ 0 < \alpha < \frac{4-2b}{N-2}.$$

For all $a > 0$, every solution is global and exponentially scatters

Also we consider the two questions :

- Improve the value of $a^\star$ even for $\mu \in \mathbb{C}$.
- Decay order of the scattering: A question razed open by Inui: Additional time decay $e^{-\alpha a t}$ when $\alpha < 4/N$?

Recently, Aloui-Jbari-T.: $b \geq 0$ Local existence: $\forall u_0 \in H^1$, $\exists!$ solution $u \in C([0, T_{max}(u_0)), H^1)$

$$T_{\max}(u_0) \geq \frac{C_1}{a} \log\left(\frac{C_2 \|u_0\|^{\alpha\theta}_{\dot{H}^1(\mathbb{R}^N)}}{\left(C_3 \|u_0\|^{\alpha\theta}_{\dot{H}^1(\mathbb{R}^N)} - a\right)_+} \right).$$

where

$$\theta = \frac{4}{4 - 2b - \alpha(N-2)}$$

In particular, if $a \geq C_3 \|u_0\|^{\alpha\theta}_{\dot{H}^1(\mathbb{R}^N)}$, then the solution exists globally in time. Exponentially scatters with decay order.

$$\lim_{t\to\infty} e^{(\alpha+1)at} \|u(t) - e^{t(i\Delta - a)}\varphi_+\|_{H^1(\mathbb{R}^N)} = 0.$$

Sketch of the proof

We consider the following transformation :

$$v(t,x) := e^{at}u(t,x), \tag{5}$$

It follows that u is a solution of (DINLS) if and only if v is a solution of

$$i\partial_t v + \Delta v + \mu h(t)|x|^{-b}|v|^\alpha v = 0, \quad v(0) = v_0 = u_0,$$

where $h(t) = \mathrm{e}^{-\alpha at}$. We study the previous equation via its Duhamel formulation

$$v(t) = \mathrm{e}^{it\Delta}\, u_0 + i\mu \int_0^t \mathrm{e}^{i(t-s)\Delta}[h(s)|x|^{-b}|v(s)|^\alpha v(s)]ds. \tag{6}$$

$$\phi(v)(t) = \mathrm{e}^{it\Delta}\, u_0 + i\mu \int_0^t \mathrm{e}^{i(t-s)\Delta}[h(s)|x|^{-b}|v(s)|^\alpha v(s)]ds \tag{7}$$

We use the Banach fixed point theorem. Let $M, T > 0$ to be chosen later and (γ, ρ) be an appropriate admissible pair. Consider the set

$$E = \{v \in L^{\gamma}([0, T] : W_2^{s,\rho}), \|v\|_{L^{\gamma}([0,T]:\dot{W}_2^{s,\rho})} \leq M\}$$

equipped with the distance

$$d(v, w) = \|v - w\|_{L^{\gamma}([0,T]:L^{\rho,2})}.$$

$$\|\phi(v)\|_{L^p([0,T]:\dot{W}_2^{s,l})} \lesssim \|u_0\|_{\dot{H}^s} + \|h\|_{L^{\theta}([0,T])}\|v\|^{\alpha+1}_{L^{\gamma}([0,T]:\dot{W}_2^{s,\rho})}.$$

$$\|h\|^{\theta}_{L^{\theta}([0,T])} = \frac{1}{\alpha a \theta}[1 - \mathrm{e}^{-\alpha a \theta T}].$$

we need to choose $M > 0$ and $T > 0$ such that

$$\begin{cases} K\|u_0\|_{\dot{H}^s} + KM^{\alpha+1}\|h\|_{L^{\theta}([0,T])} \leq M, \\ KM^{\alpha}\|h\|_{L^{\theta}([0,T])} < 1. \end{cases}$$

For this we consider $M = 2K\|u_0\|_{\dot{H}^s}$ and T be such that

$$2^{\alpha}K^{\alpha+1}\|u_0\|^{\alpha}_{\dot{H}^s}\|h\|_{L^{\theta}([0,T])} \leq \frac{1}{2}. \tag{8}$$

We note that

$$\|h\|^{\theta}_{L^{\theta}([0,T])} = \frac{1}{\alpha a \theta}[1 - \mathrm{e}^{-\alpha a \theta T}].$$

A simple calculation shows that if $a < \frac{(2^{\alpha+1}K^{\alpha+1}\|u_0\|^{\alpha}_{\dot{H}^s})^{\theta}}{\theta\alpha}$, then

$$T_{\max}(u_0) > \frac{1}{\alpha\theta a}\log\left(\frac{(2^{\alpha+1}K^{\alpha+1}\|u_0\|^{\alpha}_{\dot{H}^s})^{\theta}}{(2^{\alpha+1}K^{\alpha+1}\|u_0\|^{\alpha}_{\dot{H}^s})^{\theta} - \alpha\theta a}\right).$$

On the other hand, if $a \geq \frac{(2^{\alpha+1}K^{\alpha+1}\|u_0\|^{\alpha}_{\dot{H}^s})^{\theta}}{\theta\alpha}$, then $T_{\max}(u_0) = \infty$.

Sketch of the proof

Exponentially scatters with decay order.

$$\lim_{t\to\infty} e^{(\alpha+1)at}\|u(t)-e^{t(i\Delta-a)}\varphi_+\|_{H^1(\mathbb{R}^N)}=0.$$

By Duhamel's formula, we have

$$v(t)=\mathrm{e}^{it\Delta}\,v_+-i\mu\int_t^\infty \mathrm{e}^{i(t-s)\Delta}\{h(s)|x|^{-b}|v|^\alpha v(s)\}ds.$$

Using the Strichartz estimates, we obtain

$$\begin{aligned}\|v(t)-\mathrm{e}^{it\Delta}\,v_+\|_{H^1} &= |\mu|\left\|\int_t^\infty \mathrm{e}^{i(t-s)\Delta}\{h(s)|x|^{-b}|v|^\alpha v\}ds\right\|_{H^1}\\ &\lesssim \|\,\mathrm{e}^{-a\alpha s}\,\|_{L^\theta((t,\infty))}\|v\|^{\alpha+1}_{L^\gamma((t,\infty):W_2^{1,\rho})}\\ &\lesssim \mathrm{e}^{-a\alpha t}\,\|v\|^{\alpha+1}_{L^\gamma((t,\infty):W_2^{1,\rho})}.\end{aligned}$$

By the hypothesis v scatters then using: If v scatters in H^1 then $v\in L^q((0,\infty);W_2^{1,r})$ for any admissible pair (q,r).
$\|v\|_{L^\gamma((0,\infty):W_2^{1,\rho})}<\infty$ hence $\|v\|_{L^\gamma((t,\infty);W_2^{1,\rho})}\to 0$ as $t\to\infty$.

Open Problems

1. Scattering: optimality of the decay rate
2. Blow up: take advantage of the decay of the potential: (for example INLS, in H^1 substitution for the radial assumption)
3. $A\star : a_\star = a^\star = A\star$ (only numerical)
4. Lifespan: lower and upper optimal estimates, dependence on a when the initial data are fixed: only Numerical results: No monotonicity

Thank you