Layer separation and energy dissipation for 3D NSE at high Reynolds number

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The University of Chicago · Joint work with ***Alexis F. Vasseur*** *(UT Austin)*

Incompressible fluid at high Reynolds number may exhibit turbulence and layer separation, which differs from ideal fluids.

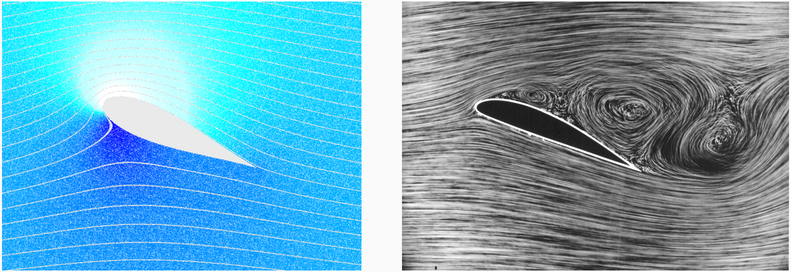

Figure 1: Euler vs Navier–Stokes: airfoil

The (limit) difference between u^ν and $\bar{u}$: **boundary layer separation**

Goal of today's talk: **estimate the energy** of layer separation for NSE

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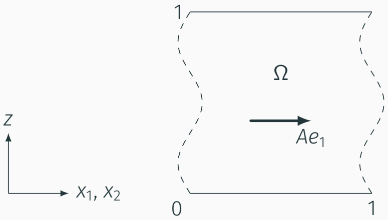
The inviscid limit problem

Consider the incompressible **Euler equation** and **Navier–Stokes equation** during time $[0, T]$ in a smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t \bar{u} + \bar{u} \cdot \nabla \bar{u} + \nabla \bar{P} = \bar{f} \\ \operatorname{div} \bar{u} = 0 \\ \bar{u}\big|_{\partial\Omega} \cdot n = 0 \\ \bar{u}\big|_{t=0} = \bar{u}_0 \end{cases} \qquad \begin{cases} \partial_t u^\nu + u^\nu \cdot \nabla u^\nu + \nabla P^\nu = \nu \Delta u^\nu + f^\nu \\ \operatorname{div} u^\nu = 0 \\ u^\nu\big|_{\partial\Omega} = 0 \\ u^\nu\big|_{t=0} = u_0^\nu \end{cases} \tag{$*$}$$

We are interested in the inviscid limit $\nu \to 0$ under the condition that u_0^ν converges to $\bar{u}_0$ in $L^2(\Omega)$ and f^ν converges to $\bar{f}$ in $L^1(0, T; L^2(\Omega))$.



Example: plug flow $\bar{u} \equiv Ae_1$, $\bar{f} \equiv 0$
in periodic channel $\Omega = \mathbb{T}^2 \times (0, 1)$
or in periodic tube $\Omega = \mathbb{T} \times B_1$

For the Navier–Stokes equation, even though the existence of a global classical solution is open, Leray and Hopf established the global existence of weak solutions: for divergence-free $u_0^\nu \in L^2(\Omega)$ with $f^\nu \in L^1(0,T;L^2(\Omega))$, there exists a weak solution in

$$u^\nu \in C_{\mathrm{w}}(0,T;L^2(\Omega)) \cap L^2(0,T;\dot{H}^1_0(\Omega))$$

with energy inequality: for a.e. $t \in [0,T]$, it holds

$$\frac{1}{2}\|u^\nu\|^2_{L^2(\Omega)}(t) + \nu \int_0^t \|\nabla u^\nu\|^2_{L^2(\Omega)}(s)\,\mathrm{d}s \leq \frac{1}{2}\|u_0^\nu\|^2_{L^2(\Omega)}.$$

We want to investigate how *stable* the weak solutions are in this natural energy space.

Asymptotic Limit

- It is a major open problem to know whether the limit of u^ν converges to $\bar{u}$, even in dimension 2.
- For the example of plug flow, if $u_0^\nu = Ae_1$, then u^ν corresponds to the poiseuille flow in periodic pipes or **Prandtl layer** in periodic channels.
- Conditional results exist: the **Kato's criterion** (1984) states that if, when $\nu \to 0$, $u_0^\nu \to \bar{u}_0$ in $L^2(\Omega)$ and $f^\nu \to \bar{f}$ in $L^1(0,T;L^2(\Omega))$:

$$\int_0^T \int_{\mathcal{U}_\delta(\partial\Omega,\Omega)} \nu |\nabla u^\nu|^2 \,\mathrm{d}x\,\mathrm{d}t \to 0,$$

where $\mathcal{U}_\delta(\partial\Omega,\Omega) = \{x \in \Omega : \mathrm{dist}(x,\partial\Omega) < \delta\}$ is a thin region near the boundary with width $\delta = c\nu$, then

$$u^\nu \to \bar{u}, \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Asymptotic Limit

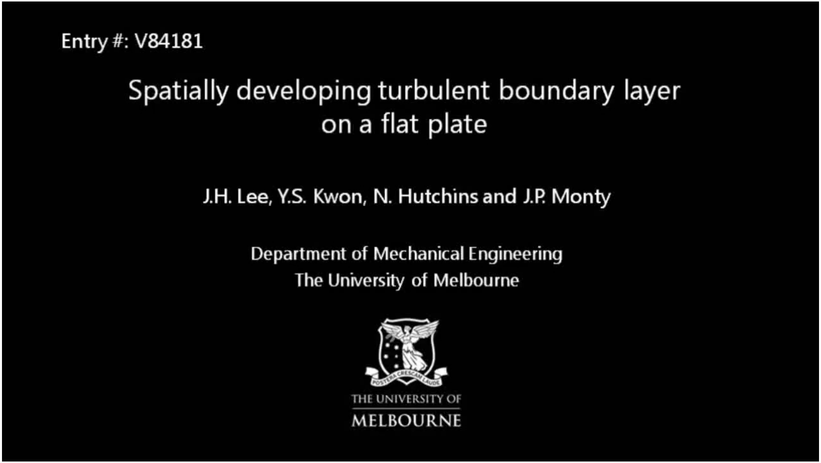
Conditional results

1. Sharpening of Kato's result: Temam–Wang (1997), Wang (2001), Kelliher (2008, 2017)
2. Other conditional results: Bardos–Titi–Wiedemann (2012), Constantin–Kukavica–Vicol (2015), Constantin–Vicol (2018)

Unconditional results

1. Analyticity/near boundary: Sammartino–Caflisch (1998), Maekawa (2014), Fei–Tao–Zhang (2018), Kukavica–Vicol–Wang (2018, 2022), Wang (2020)
2. Symmetry: Mazzucato–Taylor (2008), Lopes Filho–Mazzucato–Nussenzveig Lopes–Taylor (2008)
3. Anisotropic dissipation: Masmoudi (1998)

Turbulence and Layer Separation



Spatially developing turbulent boundary layer on a flat plate

A Simple Computation

The growth rate of the boundary layer is controlled by (no force)

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)} + \frac{\nu}{2}\int_\Omega |\nabla u^\nu|^2 - |\nabla \bar{u}|^2 \,\mathrm{d}x \\ \leq \|u^\nu - \bar{u}\|^2_{L^2(\Omega)}\|D\bar{u}\|_{L^\infty(\Omega)} - \nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n}\cdot \bar{u}\,\mathrm{d}S.\end{aligned}$$

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$$\leq \underbrace{\|u^\nu - \bar{u}\|^2_{L^2(\Omega)}\|D\bar{u}\|_{L^\infty(\Omega)}}_{\text{can be Grönwall-ized}} - \nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n}\cdot \bar{u}\,\mathrm{d}S.$$

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- $\|u^\nu - \bar{u}\|^2_{L^2}$: boundary layer separation

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$$\leq \|u^\nu - \bar{u}\|^2_{L^2(\Omega)}\|D\bar{u}\|_{L^\infty(\Omega)} - \nu\int_{\partial\Omega}\frac{\partial u^\nu}{\partial n}\cdot \bar{u}\,\mathrm{d}S.$$

- $\|u^\nu - \bar{u}\|^2_{L^2}$: boundary layer separation
- $\nu\|\nabla u^\nu\|^2_{L^2}$: energy dissipation

A Simple Computation

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$$\leq \|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 \|D\bar{u}\|_{L^\infty(\Omega)} - \nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n} \cdot \bar{u} \,\mathrm{d}S.$$

- $\|u^\nu - \bar{u}\|_{L^2}^2$: boundary layer separation
- $\nu\|\nabla u^\nu\|_{L^2}^2$: energy dissipation
- $\int_{\partial\Omega} \tau n \cdot \bar{u}$: power of boundary stress (friction) exerted on $\bar{u}$

A Simple Computation

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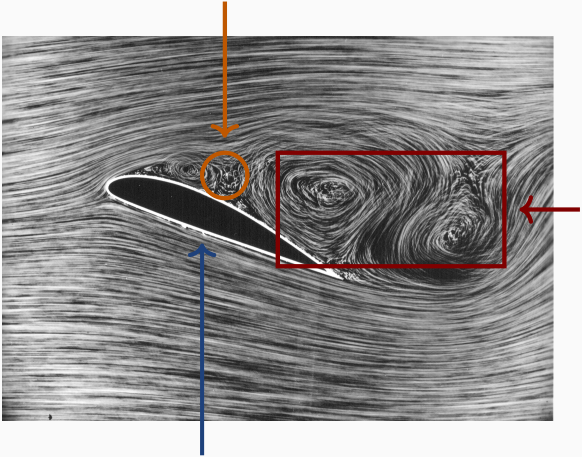
$$\leq \|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 \|D\bar{u}\|_{L^\infty(\Omega)} - \nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n} \cdot \bar{u} \,\mathrm{d}S.$$

- $\|u^\nu - \bar{u}\|_{L^2}^2$: boundary layer separation
- $\nu\|\nabla u^\nu\|_{L^2}^2$: energy dissipation
- $\int_{\partial\Omega} \tau n \cdot \bar{u}$: power of boundary stress (friction) exerted on $\bar{u}$
  $\tau = -p\,\mathrm{Id} + \nu \nabla u^\nu$ is the stress tensor

Airfoil



Layer Separation, Anomalous Dissipation, Work of Friction

For a regular Euler solution $\bar{u}$, define

- Layer separation:

$$\mathrm{LS}(\bar{u}) := \limsup_{\nu\to 0}\left\{\|u^\nu - \bar{u}\|_{L^2(\Omega)}^2(T) : \begin{array}{l} u_0^\nu \to \bar{u}_0 \text{ in } L_x^2 \\ f^\nu \to \bar{f} \text{ in } L_t^1 L_x^2 \end{array}\right\}.$$

- Anomalous dissipation:

$$\mathrm{AD}(\bar{u}) := \limsup_{\nu\to 0}\left\{\int_0^T \nu\|\nabla u^\nu\|_{L^2(\Omega)}^2(t)\,\mathrm{d}t : \begin{array}{l} u_0^\nu \to \bar{u}_0 \text{ in } L_x^2 \\ f^\nu \to \bar{f} \text{ in } L_t^1 L_x^2 \end{array}\right\}.$$

- Total work of friction:

$$\mathrm{W}_{\mathrm{fric}}(\bar{u}) := \limsup_{\nu\to 0}\left\{\left|\int_0^T \int_{\partial\Omega} \nu\partial_n u^\nu \cdot \bar{u}\,\mathrm{d}x'\,\mathrm{d}t\right| : \begin{array}{l} u_0^\nu \to \bar{u}_0 \text{ in } L_x^2 \\ f^\nu \to \bar{f} \text{ in } L_t^1 L_x^2 \end{array}\right\}.$$

Theorem (Vasseur–Y., 2024, Comm. PDE)

$$\mathrm{LS}(\bar{u}) + \mathrm{AD}(\bar{u}) + \mathrm{W}_{\mathrm{fric}}(\bar{u}) \leq CA^3T|\partial\Omega| \exp\left(2\int_0^T \|D\bar{u}(t)\|_{L^\infty(\Omega)}\,\mathrm{d}t\right).$$

- $A = \|\bar{u}\|_{L^\infty((0,T)\times\partial\Omega)}$ is the maximum **boundary velocity**
- $D\bar{u} = \frac{1}{2}(\nabla\bar{u} + \nabla\bar{u}^\top)$ is the symmetric velocity gradient
- $L^\infty(\Omega)$ measures the largest absolute eigenvalue
- constant C is a universal constant independent of Ω

Relation with Nonuniqueness Results

Euler nonuniqueness:

- The method of convex integration shows that in $\Omega \in \mathbb{T}^2 \times [0,1]$, the constant shear solution $\bar{u}(t,x) = Ae_1$ of $(*)$ is not unique (see Székelyhidi, 2011). For every constant $C < 2$, there exists a spurious Euler weak solution u with layer separation for $T < 1/A$:

$$\|u(T) - Ae_1\|_{L^2(\Omega)}^2 = CA^3T.$$

Navier–Stokes nonuniqueness:

- Nonuniqueness of weak solutions in $\mathbb{T}^3$: via *convex integration*, Buckmaster–Vicol (2019)
- Nonuniqueness for forced Leray–Hopf solution in $\mathbb{R}^3$, $\mathbb{T}^3$ or in Ω: using *self-semilar solution*, Albritton–Brué–Colombo (2022)

Non-uniqueness and Pattern Predictability

- In general, non-uniqueness result by convex integration raised the question of predictability: Why can we observe patterns?
- If the amplitude of the Euler solution is $|\bar{u}(t,x)| \sim A$, then the kinetic energy of $\bar{u}$ is

$$\frac{1}{2}\int_{\Omega} |\bar{u}|^2 \, \mathrm{d}x \sim A^2|\Omega|.$$

- We prove that the layer separation has an energy of at most $CA^3T|\partial\Omega|$ at time T (leading term)
- Therefore, the perturbation stays negligible on a time span $T \ll_{\Omega} 1/A$. This is a large time for A small (small pattern)
- It predicts the lapse of time where the pattern stays predictable

Wake

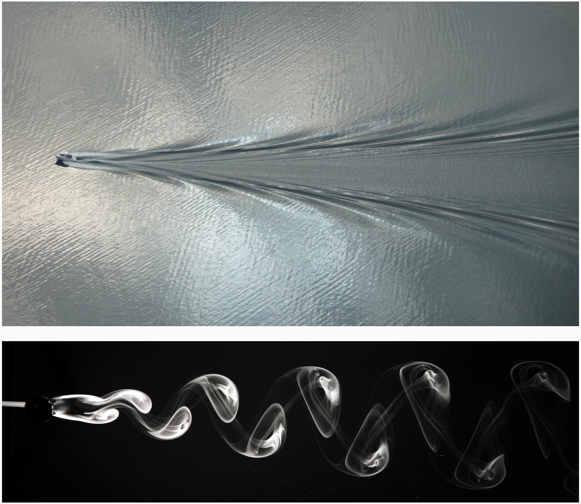


Figure 2: Wake and von Kármán vortex street

Brué and De Lellis (2022) constructed forced *classical* solutions u^ν with u_0^ν and f^ν uniformly bounded and

$$\lim_{\nu\to 0} \nu \|\nabla u^\nu\|^2_{L^2((0,T)\times\mathbb{T}^3)} > 0.$$

Note that neither anomalous dissipation nor layer separation can exist near a **regular** Euler solution if **without boundary**, so [ABC2022] and [BDL2022] are due to a different mechanism.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u^\nu - \bar{u}\|^2_{L^2(\Omega)} &+ \frac{\nu}{2}\int_\Omega |\nabla u^\nu|^2 - |\nabla \bar{u}|^2 \,\mathrm{d}x \\ &\le \|u^\nu - \bar{u}\|^2_{L^2(\Omega)}\|D\bar{u}\|_{L^\infty(\Omega)} - \nu\int_{\partial\Omega}\frac{\partial u^\nu}{\partial n}\cdot \bar{u}\,\mathrm{d}S.\end{aligned}$$

Long-time average of energy dissipation: Foias and Doering (2002)

$$\frac{\epsilon \ell}{U^3} \le C_1 + C_2 \mathrm{Re}^{-1}$$

The bound is sharp (Cheskidov, 2023)

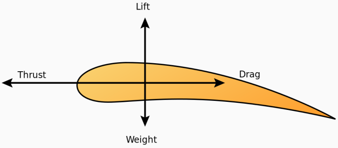
Drag, Lift, and Work of Boundary Stress

By dimensional analysis, the **drag** and **lift** experienced by an aircraft depend on the surface area, airspeed, shape of the airfoil, angle of attack, density and viscosity of the fluid, by following empirical formulae:

$$F_{\text{drag}} = \frac{1}{2}\rho\, c_D(\text{Re}) U^2 S$$

$$F_{\text{lift}} = \frac{1}{2}\rho\, c_L(\text{Re}) U^2 S$$

- $\rho = 1$ is the density
- $U \approx A$ is the airspeed
- $S = |\partial\Omega|$ is the surface area
- c_D and c_L are dimensionless



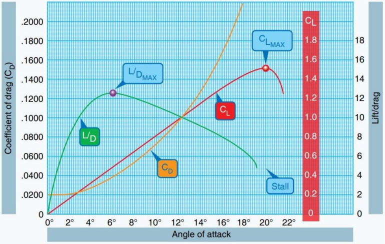


Figure 3: Drag and Lift

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The work of drag force on object

$$\mathrm{W}_{\text{body}} = F_{\text{drag}} U T \le C U^3 T |\partial\Omega|$$

When the wing is a thin plate:

$\mathrm{W}_{\text{fric}} = \mathrm{W}_{\text{body}}$, $c_D(\text{Re}) < C$ is bounded in the inviscid limit $\text{Re} \to \infty$

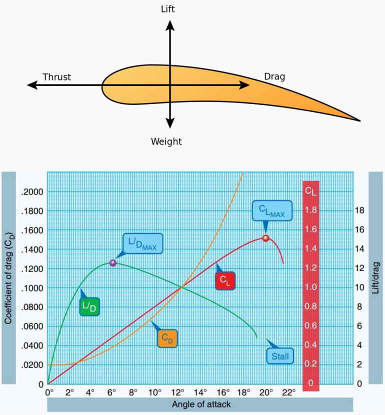


Figure 3: Drag and Lift

Boundary vorticity

Recall the estimate

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\int_\Omega |\nabla u^\nu|^2 - |\nabla \bar{u}|^2 \,\mathrm{d}x \leq \|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 \|D\bar{u}\|_{L^\infty(\Omega)} - \nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n} \cdot \bar{u} \,\mathrm{d}S.$$

Integrate between 0 and T, then send $\nu \to 0$:

$$\mathrm{LS}(\bar{u}) + \mathrm{AD}(\bar{u}) \leq \mathrm{W}_{\mathrm{fric}}(\bar{u}) \exp\left(2\int_0^T \|D\bar{u}(t)\|_{L^\infty(\Omega)} \,\mathrm{d}t\right).$$

So the key is to bound the work of friction:

$$\nu \int_{\partial\Omega} \frac{\partial u^\nu}{\partial n} \cdot \bar{u} \,\mathrm{d}S = \nu \int_{\partial\Omega} \omega^\nu \cdot (n \times \bar{u}) \,\mathrm{d}S$$

- $\omega^\nu = \mathrm{curl}\, u^\nu$ is the vorticity of u^ν
- Need to control (some weak norm of) $\nu\omega^\nu\big|_{\partial\Omega}$ uniform in ν

Boundary Vorticity Estimate for Navier–Stokes (Intuition)

If we treat $(*)$ as a linear Stokes system, then for $\nu = 1$, we have the estimate

$$\left\|\nabla^2 u\right\|_{L_t^{\frac{4}{3}} L_x^{\frac{6}{5}}} \lesssim \|u \cdot \nabla u\|_{L_t^{\frac{4}{3}} L_x^{\frac{6}{5}}} \lesssim \|u\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla u\|_{L_{t,x}^2}^{\frac{3}{2}}.$$

By Trace theorem and fractional Sobolev embedding,

$$\left\|\nabla u\big|_{\partial\Omega}\right\|_{L_{t,x'}^{\frac{4}{3}}} \lesssim \|u\|_{L_t^\infty L_x^2}^{\frac{1}{2}} \|\nabla u\|_{L_{t,x}^2}^{\frac{3}{2}} \quad (+\text{ l.o.t.}).$$

Now for any $\nu > 0$, we obtain via scaling

$$\int_{(0,T)\times\partial\Omega} |\nu \nabla u^\nu|^{\frac{4}{3}} \, \mathrm{d}x' \, \mathrm{d}t \lesssim \nu^{-1} \|u^\nu\|_{L_t^\infty L_x^2}^{\frac{2}{3}} \int_{(0,T)\times\Omega} \nu |\nabla u^\nu|^2 \, \mathrm{d}x \, \mathrm{d}t \quad (+\text{ l.o.t.})$$

This bad estimate is NOT UNIFORM in the inviscid limit $\nu \to 0$.

$$\begin{aligned} \mathrm{W}_{\text{fric}} &\leq \|\bar{u}\|_{L^4((0,T)\times\partial\Omega)} \|\nu \partial_n u^\nu\|_{L^{\frac{4}{3}}((0,T)\times\partial\Omega)} \\ &\leq \frac{1}{2} \int_{(0,T)\times\Omega} \nu |\nabla u^\nu|^2 \, \mathrm{d}x \, \mathrm{d}t + C\nu^{-3} \|u_0^\nu\|_{L^2(\Omega)}^2 A^4 T |\partial\Omega|. \end{aligned}$$

Boundary Vorticity Estimate for Navier–Stokes (Intuition)

Where is the problem? The *inviscid scaling* of the equation $u^\nu(t,x) = \nu u(\nu t, x)$ determines the physical unit.

Time $t \sim \mathrm{T}$, space $x \sim \mathrm{L}$, $u \sim \mathrm{L}\mathrm{T}^{-1}$, viscosity constant $\nu \sim \mathrm{L}^2\mathrm{T}^{-1}$.

The energy has unit (treating density as unitless)

$$\|u\|_{L_t^\infty L_x^2}^2, \int_{(0,T)\times\Omega} \nu|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t \sim \mathrm{L}^5\mathrm{T}^{-2}$$

The bad estimate has unit

$$\int_{(0,T)\times\partial\Omega} |\nu\nabla u^\nu|^{\frac{4}{3}} \,\mathrm{d}x'\,\mathrm{d}t \lesssim \nu^{-1}\|u\|_{L_t^\infty L_x^2}^{\frac{2}{3}} \int_{(0,T)\times\Omega} \nu|\nabla u|^2 \,\mathrm{d}x\,\mathrm{d}t$$

$$\mathrm{L}^2\mathrm{T}(\mathrm{L}^2\mathrm{T}^{-2})^{\frac{4}{3}} \qquad (\mathrm{L}^2\mathrm{T}^{-1})^{-1}(\mathrm{L}^5\mathrm{T}^{-2})^{\frac{4}{3}}.$$

To get rid of ν, the correct boundary norm should be $L^{\frac{3}{2}}$ instead of $L^{\frac{4}{3}}$.

$$\mathrm{L}^2\mathrm{T}\left(\mathrm{L}^2\mathrm{T}^{-2}\right)^{\frac{3}{2}} \sim (\mathrm{L}^2\mathrm{T}^{-1})^0(\mathrm{L}^5\mathrm{T}^{-2})^1.$$

Boundary Vorticity Estimate for Navier–Stokes (Intuition)

If we take the curl of $(*)$, we have the vorticity equation,

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \omega \cdot \nabla u.$$

Suppose we can ignore the transport term and the boundary effect, then the regularity we could expect for ω is at best

$$\nu^2 \left\| \nabla^2 \omega \right\|_{L^1((0,T)\times\Omega)} \lesssim \nu \|\omega \cdot \nabla u\|_{L^1((0,T)\times\Omega)} \leq \nu \|\nabla u^\nu\|^2_{L^2((0,T)\times\Omega)}.$$

(although parabolic regularization is false in L^1) By interpolation with $\nu \|\omega\|^2_{L^2((0,T)\times\Omega)} \leq \nu \|\nabla u\|^2_{L^2((0,T)\times\Omega)}$,

$$\nu^{\frac{3}{2}} \left\| \nabla^{\frac{2}{3}} \omega \right\|^{\frac{3}{2}}_{L^{\frac{3}{2}}((0,T)\times\Omega)} \lesssim \nu \|\nabla u\|^2_{L^2((0,T)\times\Omega)}.$$

Finally the (critical) trace theorem suggests that (cheating again)

$$\int_{(0,T)\times\partial\Omega} |\nu \nabla u^\nu|^{\frac{3}{2}} \,\mathrm{d}x' \,\mathrm{d}t \lesssim \int_{(0,T)\times\Omega} \nu |\nabla u|^2 \,\mathrm{d}x \,\mathrm{d}t.$$

Theorem (Boundary Regularity)

For any Leray-Hopf solution u^ν to $(*)$ in $(0,T)\times\Omega$, δ sufficiently small, there exists a partition $(0,T)\times\partial\Omega = \bigcup_i \bar{Q}^i$, such that the following is true. Define the piecewise average on boundary by

$$\tilde{\omega}^\nu(t,x) = \fint_{\bar{Q}_i} \omega^\nu \, \mathrm{d}x \, \mathrm{d}t, \qquad \textit{for } (t,x) \in \bar{Q}_i$$

Then we have

$$\left\| \nu\tilde{\omega}^\nu \mathbf{1}_{\left\{|\nu\tilde{\omega}^\nu| > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{\delta^2}\right\}\right\}} \right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)}^{\frac{3}{2}} \lesssim \int_{(0,T)\times\mathcal{U}_\delta(\partial\Omega,\Omega)} \nu|\nabla u^\nu|^2 \, \mathrm{d}x \, \mathrm{d}t.$$

We will be choosing $\delta = c\nu$, so Kato's condition will imply RHS $= 0$ in the inviscid limit.

- Recall the growth rate of the layer separation is controlled by

$$\frac{\mathrm{d}}{\mathrm{dt}}\frac{1}{2}\|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 + \frac{\nu}{2}\int_\Omega |\nabla u^\nu|^2 - |\nabla \bar{u}|^2 \,\mathrm{d}x$$
$$\leq \|u^\nu - \bar{u}\|_{L^2(\Omega)}^2 \|D\bar{u}\|_{L^\infty(\Omega)} - \int_{\partial\Omega} \nu\omega^\nu \cdot (n \times \bar{u}) \,\mathrm{d}S.$$

- Integrate in time

$$\int_0^T \int_{\partial\Omega} \nu\omega^\nu \cdot (n \times \bar{u}) \,\mathrm{d}S \,\mathrm{d}t$$
$$\leq \left\| \nu\tilde{\omega}^\nu \mathbf{1}_{\left\{|\nu\tilde{\omega}^\nu| > \max\left\{\frac{\nu}{t}, \frac{\nu^2}{\delta^2}\right\}\right\}} \right\|_{L^{\frac{3}{2},\infty}((0,T)\times\partial\Omega)} \|\bar{u}\|_{L^{3,1}((0,T)\times\partial\Omega)} + \text{l.o.t.}$$
$$\leq \frac{\nu}{4}\|\nabla u^\nu\|_{L^2_{t,x}}^2 + CA^3T|\partial\Omega| + \text{l.o.t.}$$

- $\Rightarrow \mathrm{W}_{\mathrm{fric}}(\bar{u}) \leq \frac{1}{2}\,\mathrm{AD}_\delta(\bar{u}) + CA^3T|\partial\Omega|$.
- $\Rightarrow \mathrm{LS}(\bar{u}) + \mathrm{AD}(\bar{u}) \leq CA^3T|\partial\Omega| \exp(2\|D\bar{u}\|_{L^1_tL^\infty_x})$.
- $\Rightarrow \mathrm{LS}(\bar{u}) = \mathrm{AD}(\bar{u}) = \mathrm{W}_{\mathrm{fric}}(\bar{u}) = 0$ with Kato's condition.

Blow-up on boundary

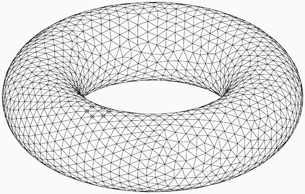
The Problems: Transport Term and Boundary

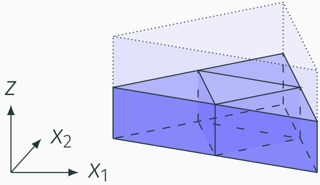
- We cannot control the transport term $u \cdot \nabla\omega$: $u \in L^{\frac{10}{3}}$ and $\nabla\omega \in L^{\frac{4}{3},q}$, $q > \frac{4}{3}$ (Vasseur–Y. 2021). $u \cdot \nabla\omega$ is less than L^1!
- Therefore we work on u and use a **blow-up method** introduced in Vasseur (2010) [see also Choi–Vasseur (2014)] to control higher derivatives, following the flow at the scale of the blow-up.
- Problem of boundary: without control on the pressure, the local Stokes regularity does *not* hold at the boundary.
- But it holds **AFTER** taking local mean value $\tilde{\omega}$.

Fix $\nu = 1$ from now on. To control the boundary vorticity, we use a blow-up argument based on the *canonical scaling* of the Navier–Stokes:

$$\tilde{u}(t,x) = \varepsilon u(\varepsilon^2 t, \varepsilon x).$$

We will make a Calderón–Zygmund style partition on $(0,T) \times \partial\Omega$. Since $\partial\Omega$ is non-flat, we need a **triangularization** to make sense of **dyadic decomposition**.





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The goal is to control the boundary vorticity. Note the scaling

$$x \sim \varepsilon \qquad t \sim \varepsilon^2 \qquad u \sim \varepsilon^{-1}$$
$$\omega, \nabla u \sim \varepsilon^{-2} \qquad \nabla^2 u \sim \varepsilon^{-3} \qquad \nabla P \sim \varepsilon^{-3}$$
$$\int_{\bar{Q}} |\omega|^{\frac{3}{2}}\, \mathrm{d}x\, \mathrm{d}t \sim \varepsilon \qquad \int_{Q} |\nabla u|^2\, \mathrm{d}x\, \mathrm{d}t \sim \varepsilon \qquad \int_{Q} |u|^{\frac{10}{3}}\, \mathrm{d}x\, \mathrm{d}t \sim \varepsilon^{\frac{5}{3}}$$

Boundary vorticity and the interior *energy dissipation* have the same scaling!

Proposition (linear Stokes boundary estimate)

Suppose $u \in L^2(-4,0;H^1(C_2))$ is a solution to the following Stokes system with forcing term $f \in L^1(-4,0;L^{\frac{6}{5}}(C_2))$:

$$\begin{cases} \partial_t u + \nabla P = \Delta u + f & \text{in } Q_2 \\ \operatorname{div} u = 0 & \text{in } Q_2 \\ u = 0 & \text{on } \bar{Q}_2 \end{cases}.$$

Then the average vorticity on the boundary is bounded by

$$\int_{T_1} \left| \int_{-1}^{0} \omega(t,x',0)\,\mathrm{d}t \right| \mathrm{d}x' \leq C \left(\|\nabla u\|_{L^2_t L^2_x(Q_2)} + \|f\|_{L^1_t L^{6/5}_x(Q_2)} \right).$$

$T_2 \subset \partial\Omega$: curved triangle $\quad C_2 \approx T_2 \times [0,2] \subset \Omega$: curved cylinder

$\bar{Q}_2 = (-4,0) \times T_2 \quad Q_2 = (-4,0) \times C_2$

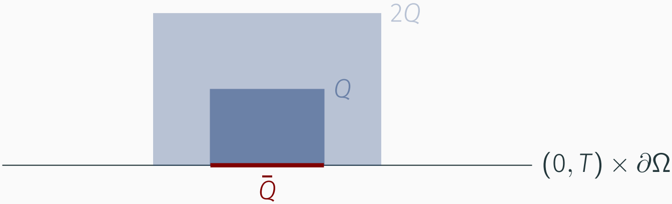
- We prove a local theorem: if Q has radius 2^{-k}, and in $2Q$

$$\fint_{2Q} |\nabla u|^2 \, dx \, dt \leq c_0 (2^{-k})^{-4}$$

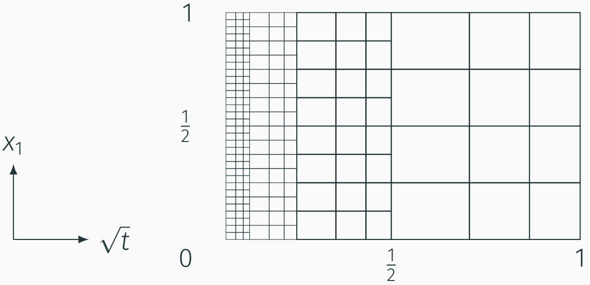
- then the average boundary vorticity on $\bar{Q} = Q \cap \partial\Omega$ is

$$\tilde{\omega} = \fint_{\bar{Q}} \omega \, dx' \, dt \leq c_1 (2^{-k})^{-2}$$

- This links the interior gradient and the boundary mean vorticity at a local level.



Calderón–Zygmund Decomposition



A parabolic cube Q of size $4^{-k} \times (2^{-k})^d$ is said to be suitable if it touches the boundary $\partial\Omega$ but not $\{t = 0\}$, and satisfies

$$\fint_{2Q} |\nabla u|^2 \, \mathrm{d}x \, \mathrm{d}t \leq c_0 (2^{-k})^{-4}$$

for some c_0. For each cube that is not suitable, we dyadically dissect it into smaller cubes till suitable. We prove the boundary vorticity estimate using the maximal function.

Open questions

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- Can we construct a **regular** Euler solution $\bar{u}$ and find Leray–Hopf solutions $\{u^\nu\}_{\nu \to 0}$ with $\mathsf{LS}(\bar{u}) > 0$ or $\mathsf{AD}(\bar{u}) > 0$ or $\mathsf{W}_{\text{body}}(\bar{u}) > 0$?
    - $\mathsf{LS}(\bar{u}) > 0$: invalidity of the inviscid limit
    - $\mathsf{AD}(\bar{u}) > 0$: zeroth law of turbulence
    - $\mathsf{W}_{\text{body}}(\bar{u}) > 0$: D'Alembert's paradox

Thank you for your listening!

https://arxiv.org/abs/2303.05236