General equilibrium theory
Lecture notes

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November 4, 2014

[1]These notes constitute the material for the second section of the first year graduate Micro course at NYU. The first section on *Decision theory*, is taught by Ariel Rubinstein. The notes owe much to the brilliant TA's Ariel and I had, Sevgi Yuksel, Bernard Herskovic, Andrew McClellan.

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Chapter 1

Introduction

The standard approach to graduate teaching of *general equilibrium theory* involves introducing a series of theorems on existence, characterization, and welfare properties of competitive equilibria under weaker and weaker assumptions in larger and larger (and sometimes weirder) commodity spaces. Such an approach introduces the students to precise rigorous mathematical analysis and invariably impresses them with the elegance of the theory. Various textbooks take this approach, in some form or another:

A. Mas-Colell, M. Whinston, and J. Green (1995): *Microeconomic Theory*, Oxford University Press, Part 4 - the main modern reference; it also contains a short introduction to two-period economies.

L. McKenzie (2002), *Classical General Equilibrium Theory*, MIT Press - a beautiful modern treatment of the classical theory.

K. Arrow and F. Hahn (1971): *General Competitive Analysis*, North Holland - the classical treatment of the classical theory.[1]

B. Ellickson (1994): *Competitive Equilibrium: Theory and Applications*, Cambridge University Press - somewhat heterodox in the choice of the main themes; it contains a useful chapter on non-convex economies.

[1]And so is G. Debreu (1972), *Theory of Value: An Axiomatic Analysis of Economic Equilibrium*, Cowles Foundation Monographs Series, Yale University Press, an invaluable little book for several generations of theorists.

M. Magill and M. Quinzii (1996): *Theory of Incomplete Markets*, MIT Press - a classical theory approach to financial market equilibrium in two-period economies.

The approach adopted in these notes aims instead at introducing general equilibrium theory as the canonical theoretical structure of economics in its application, e.g., general assignment problems in labor, industrial organization, social economics and Walrasian (competitive) equilibria as the main microfoundation for macroeconomics and finance. To this end, the standard theory of general equilibrium is introduced in its rigor and elegance, but only under restrictive assumptions, allowing some shortcuts in analysis and proofs. On the other hand, we shall be able to introduce financial market equilibria in two-period economies rather quickly, exposing students to fundamental conceptual notions like complete and incomplete markets, no-arbitrage pricing, constained efficiency, equilibria in moral hazard and adverse selection economies, and many more. We shall also be able to treat non-Walrasian economies like economies with assignment problems as well as monopolistically competitive economies. The course ends with a treatment of dynamic economies and recursive competitive equilibria. Pedagogically, from two-period to fully dynamic economies the step is rather short, so that we can concentrate on purely dynamic concepts, like bubbles.

1.1 Preliminaries

We denote identity equal by :=.

For any $x, y \in \mathbb{R}^N_+$, we say $x > y$ if $x_n \geq y_n$, for any $n = 1, 2, ..., N$, and $x_n > y_n$ for at least one n; we say instead $x >> y$ if $x_n > y_n$, for any $n = 1, 2, ..., N$.

For any map $f : \mathbb{R}^N \to \mathbb{R}$ we let the *gradient vector* be defined as $\nabla f = \left(\frac{\partial f}{\partial x_i}\right)_{i \in I}$.

Chapter 2

Abstract exchange economies

Consider an economy populated by agents with exogenously given preferences over and endowments of commodities. There is no production. Nonetheless agents do not necessarily consume their own endowments but rather participate in an allocation mechanism. We now formalize this structure.

The economy is populated by an infinity of agents. Agents are categorized in a finite set $I = \{1, .., I\}$ of types, with generic element i.[1] We also assume an infinite number of agents is of type i, for any $i \in I$. The *consumption set* X is the set of admissible levels of consumption of L existing commodities. We shall assume

$$X = \mathbb{R}^L_+, \text{ with generic element } x.$$

X is then a convex set, bounded below.

Each agent of type $i \in I$ has a utility function

$$U^i : X \to \mathbb{R}$$

which represents his preferences.[2] Each agent of type $i \in I$ also has an

[1] We are conscious of the notational abuse.

[2] In these notes we shall adopt the utility function as a primitive. That is, we shall assume that any agent's underlying preference ordering $\succsim$ on X (is complete, transitive, and continuous; so that it) can be represented by a utility function; see Rubinstein (2009). We shall in fact typically also assume struct convexity. A preference ordering $\succsim$ on X which (is not continuous and hence it) cannot be represented by a utility function is the Lexicographic ordering:

$$x \succsim y \text{ if } x_1 \geq y_1 \text{ or } x_1 = y_1 \text{ and } x_l \geq y_l, \text{ for } l = 2, ..., L.$$

endowment $\omega^i \in X$.

Definition 1 *An allocation for the economy is an array* $\mathbf{x} = (x^1, .., x^I) \in X^I := \mathbb{R}_+^{LI}$. *An allocation is feasible if it satisfies*

$$\sum_{i=1}^{I} x^i \leq \sum_{i=1}^{I} \omega^i.$$

In the special case in which $L = 2$, $I = 2$, feasible allocations can be graphically represented using the *Edgeworth Box.*

2.1 Pareto efficiency

An interesting possible property of an allocation is Pareto efficiency.

Definition 2 *An allocation* $\mathbf{x} \in X^I$ *is* Pareto efficient *if it is feasible and there is no other feasible allocation* $y \in X^I$ *which* Pareto dominates *it; that is, if there is no* $y \in$ *such that*

$$U^i(y^i) \geq U^i(x^i) \text{ for all } i \in I, \; > \text{ for at least one } i.$$

Pareto dominance defines a (social) preference relation over the set of feasible allocations in X^I:

$$y \text{ Pareto dominates } x \text{ if } U^i(y^i) \geq U^i(x^i) \text{ for all } i \in I, \; > \text{ for at least one } i.$$

This preference relations is however incomplete, in the sense that given two allocations $\mathbf{x}, \mathbf{y} \in X^I$, it might very well be that neither $\mathbf{x}$ Pareto dominates $\mathbf{y}$, nor viceversa.

It is useful to impose the following strong (but not outrageous) assumptions on the economic environment.

Assumption 1 $U^i : X \to \mathbb{R}$ *is* $\mathbb{C}^2$ *in any open subset of* X, *strongly monotonic increasing, and strictly quasi-concave. Furthermore,* $\omega \in \mathbb{R}_{++}^{LI}$.[3]

[3] The strong monotonicity requirement can be formally written as $\nabla U^i >> 0$; while strict quasi-concavity requires that, for any $x, y \in$, $x \neq y$,

$$U^i(\lambda x + (1-\lambda)y) > \lambda U^i(x) + (1-\lambda)U^i(y) \geq \min\{U^i(x), U^i(y)\} \text{ for all } \lambda \in (0,1).$$

Pareto Efficient allocations are solutions of the following problem (convince yourself this is just a formal translation of the definition):

$$\max_{\mathbf{x}\in\mathbf{X}} U^i(x^i) \tag{PE pb}$$

$$\begin{aligned} \text{s.t. } \sum_i x^i &\leq \sum_i \omega^i \\ U^j(x^j) &\geq \bar{U}^j \text{ for all } j \neq i \end{aligned}$$

for some given vector $\left(\bar{U}^j\right)_{j\neq i} \in \mathbb{R}^{I-1}$.

Varying the values of $\bar{U}^j$ for $j \neq i$ we obtain the set of Pareto efficient allocations.

Let the *Utility possibility set* be defined as

$$\mathbb{U} = \left\{ U \in \mathbb{R}^I \,\middle|\, U \leq \left(U^i(x^i)\right)_{i=1}^I, \text{ for some } x \in \mathbb{R}_+^{LI} \text{ such that } \sum_i x^i \leq \sum_i \omega^i \right\}.$$ [4]

Problem 1 *Show that* $\mathbb{U}$ *is bounded above and closed. Under which conditions is* $\mathbb{U}$ *also convex?*

The *Pareto utility frontier* is defined as

$$\mathbb{UP} = \left\{ U \in \mathbb{U} \,|\, \nexists U' \in \mathbb{U} \text{ such that } U' > U \right\}.$$

It contains the image of all Pareto optimal allocations in the space of utility levels.

Problem 2 *Show that* $\mathbb{UP} \subsetneq bdry(\mathbb{U})$.

There exist another formal characterization of Pareto efficiency.

[4]Note that $\mathbb{U}$ is possibly larger than

$$\mathbb{U}^0 = \left\{ U \in \mathbb{R}^I \,\middle|\, U = \left(U^i(x^i)\right)_{i=1}^I, \text{ for some } x \in \mathbb{R}_+^{LI} \text{ such that } \sum_i x^i \leq \sum_i \omega^i \right\},$$

the image of the feasible allocations in the space of utility levels; in particular $\mathbb{U}$ is unbounded below, which is not necessarily the case for $\mathbb{U}^0$.

Theorem 1 (Negishi) *(If) Suppose* $\mathbb{U}$ *is convex. Let* $\mathbf{x} \in X^I$ *be a Pareto efficient allocation. Then there exist a* $\alpha \in \mathbb{R}^I_+$, $\alpha \neq 0$, *such that*

$$\mathbf{x} = \arg\max \sum_{i \in I} \alpha^i U^i(x^i) \qquad \text{(Negishi pb)}$$

$$s.t. \sum_{i \in I} x^i \leq \sum_{i \subset I} \omega^i.$$

(Only if) Furthermore, any solution of the Negishi pb, for some $\alpha \in \mathbb{R}^I_+$, $\alpha \neq 0$, *is a Pareto efficient allocation.*

Proof. Consider a Pareto efficient allocation $\mathbf{x} \in X^I$, so that $U(\mathbf{x}) = \{U^i(x^i)\}_{i \in I} \in \mathbb{UP}$. $\mathbb{U} \subset \mathbb{R}^I$ is convex by assumption (in Problem 1 you are asked for conditions under which this is the case). Furthermore, $U(\mathbf{x}) \in bdry(\mathbb{U})$. The Supporting hyperplane theorem (see Theorem 7) then implies that there exists a $\alpha \in \mathbb{R}^I$, $\alpha \neq 0$, such that

$$\alpha U(\mathbf{x}) \geq \alpha U, \text{ for any } U \in \mathbb{U};$$

that is,

$$\begin{aligned} \alpha U(\mathbf{x}) &\in \arg\max \sum_{i \in I} \alpha^i U^i \\ \text{s.t. } U &\in \mathbb{U}. \end{aligned}$$

By the definition of $\mathbb{U}$:

$$\begin{aligned} \mathbf{x} &\in \arg\max \sum_{i \in I} \alpha^i U^i(x^i) \\ \text{s.t. } \sum_{i \in I} x^i &\leq \sum_{i \in I} \omega^i. \end{aligned}$$

Finally, $\mathbb{U}$ is unbounded below, which implies that $\alpha \in \mathbb{R}^I_+$. In fact, suppose by contradiction $\alpha^i < 0$ for some $i \in I$. Then there would exist a $\hat{U} \in \mathbb{U}$ with $\hat{U}^i < 0$ and small enough that $\alpha U(\mathbf{x}) < \alpha \hat{U}$. This proves the *if* part of the theorem.

The *only if* part is straightforward and hence left as a problem. ■

2.1.1 Characterization of Pareto efficient allocation

Both the PE pb and the Negishi pb are well-behaved convex maximization problems and hence first order conditions are necessary and sufficient (see Math Appendix). Recall also we assumed utility functions are strictly monotonic increasing (convince yourself that this implies that all constraints in either problem hold with equality).

The first order conditions (for an interior solution) of the PE pb are:

$$\begin{aligned} \nabla U^i &= \rho \\ \mu^j \nabla U^j &= \rho \text{ for all } j \neq i \\ \sum_i x^i &= \sum_i \omega^i \end{aligned}$$

where $\rho \in \mathbb{R}^L_+$ and $(\mu^j)_{j \neq i} \in \mathbb{R}^{I-1}_+$ and the Lagrange multipliers of, respectively, the feasibility constraint and the minimal utility constraints. Thus

$$\nabla U^i = \mu^j \nabla U^j \text{ for all } j \neq i$$

and utility gradients are co-linear for all agents. As a consequence, marginal rates of substitution are equalized across agents.

Problem 3 *Compare the Negishi and the PE pb. Show that an allocation* $\mathbf{x} \in X^I$ *is a solution of both problems iff*

$$\frac{\alpha^j}{\alpha^i} = \mu^j, \textit{ for any } j \neq i.$$

2.2 Competitive market equilibrium

An allocation mechanism is a rule which maps the preferences and endowments of each of the agents in the economy into an allocation. The allocation mechanism standing at the core of most of economics is that of *competitive markets*. But this is not the only possible mechanism. We now study in detail the *competitive equilibrium* concept, but we shall mention another allocation mechanism, the *jungle equilibrium*, introduced formally and studied in detail by Piccione and Rubinstein (2007).

At a *competitive equilibrium* agents trade in perfectly competitive markets, where:

prices are *linear*: the unit price p_l of each commodity l is fixed independently of level of individual trades and is the same for all agents;

prices are *non-negative*: this is typically justified under *free disposal*, that is, by the assumption that agents can freely dispose of any amount of any commodity;[5]

markets are complete: for each commodity l in X there is a market where the commodity can be traded.

Definition 3 *A* competitive equilibrium is an allocation $x \in X^I$ and a price $p \in \mathbb{R}^L_+$ such that:

i) *each agent $i \in I$ solves:*

$$\max_{x^i \in X} U^i(x^i) \qquad \text{(Consumer pb)}$$

$$\text{s.t. } px^i \leq p\omega^i$$

for given price $p \in \mathbb{R}^L_+$; and

ii) *markets clear (the allocation is feasible):*

$$\sum_i x^i \leq \sum_i \omega^i.$$

Remark 1 *As we have already mentioned,* competitive equilibrium *is hardly the only or even the most relevant allocation mechanism. Consider as an example the* jungle equilibrium*, which we introduce in the following.*

In the jungle*, allocations are determined by strength, whereby stronger agents can obtain the endowments of weaker agents. Formally, let S denote a binary relationship on the set I: iSj, for $i, j \in I$ is to be interpreted as "agent i is stronger than agent j."*

Definition 4 *A jungle equilibrium allocation is an allocation $\mathbf{x} \in X^I$ such that*

[5]We assumed utilities are strictly monotonic increasing and hence, as we shall later see, prices will be strictly positive at any equilibrium.

i) *the allocation, jointly with a vector of commodities $x^0 \in X$ not allocated to anyone, is feasible:*

$$x^0 + \sum_{i=1}^{I} x^i \leq \sum_{i=1}^{I} \omega^i;$$

ii) *stronger agents can obtain the endowments of weaker agents:*

$$\nexists i, j \in I \text{ such that } iSj \text{ and } U^i(x^i + x^j) > U^i(x^i) \text{ or } U^i(x^i + x^0) > U^i(x^i).$$

A few comments on the concept of *competitive equilibrium* and its components are in order.

Trade is voluntary, hence equilibrium allocations will satisfy individual rationality:

$$U^i(x^i) \geq U^i(\omega^i) \text{ for all } i = 1, ..., I.$$

Strong monotonicity implies that, at a competitive equilibrium:

budget constrants must hold with equality: $px^i = p\omega^i$; and

prices are strictly positive: $p \in \mathbb{R}^L_{++}$.

The solution to the Consumer pb can be represented by a demand function

$$x^i \left(p, \omega^i\right),$$

$x^i : \mathbb{R}^L_{++} \times \mathbb{R}^L_{++} \to X$. Let $z^i(p, \omega^i)$ denote agent $i \in I$'s excess demand: $z^i(p, \omega^i) = x^i(p, \omega^i) - \omega^i$. Finally, the aggregate excess demand is defined as

$$z(p, \omega) = \sum_{i \in I} z^i \left(p, \omega^i\right),$$

where $\omega = (\omega^i)_{i \in I}$, $z : \mathbb{R}^L_{++} \times \mathbb{R}^{LI}_{++} \to \mathbb{R}^{LI}$

It is convenient to identify and parametrize an economy with its endowment vector $\omega \in \mathbb{R}^{LI}_{++}$, keeping utility functions given.

Proposition 1 *For any economy $\omega \in \mathbb{R}^{LI}_{++}$ the aggregate excess demand $z(p, \omega)$ satisfies the following properties:*

smoothness: $z(p, \omega)$ *is* C^1;

homogeneity of degree 0*:* $z(\lambda p, \omega) = z(p, \omega)$ *,for any* $\lambda > 0$;

Walras Law: $pz(p, \omega) = 0$, $\forall p >> 0$;

lower boundedness: $\exists s$ *such that* $z_l(p, \omega) > -s$, $\forall l \in L$;

boundary property:

$$p^n \to p \neq 0, \text{ with } p_l = 0 \text{ for some } l, \Rightarrow \max\{z_1(p^n, \omega), ..., z_L(p^n, \omega)\} \to \infty.$$

The simple proof of this Proposition is left to the reader. [Hint: *lower boundedness* follows from demand being non-negative. The *boundary property* is written as confusingly as it is it is because if the prices of more than one commodity converge to 0, it is possible that only a subset of the corresponding excess demands explodes.]

2.2.1 Welfare

Let's first study the welfare properties of competitive equilibrium allocations.

Theorem 2 (First welfare) Consider an economy $\omega \in \mathbb{R}^{LI}_{++}$. All competitive equilibrium allocations are Pareto efficient.

Proof. Suppose $\mathbf{x} \in X^I$ is a competitive equilibrium allocation for some price $p \in \mathbb{R}^L_{++}$. Suppose by contradiction that $\mathbf{x} \in X^I$ is not Pareto efficient. Then there must be another allocation $\mathbf{y} \in X^I$ which is feasible and Pareto dominates $\mathbf{x}$. But since x^i is the optimal choice of consumer i at prices p and preferences are strongly monotone, $U^i(y^i) \geq U^i(x^i)$ implies $py^i \geq px^i$ for all $i \in I$ and also $py^i \geq p\omega^i$. Since $\mathbf{y} \in X^I$ Pareto dominates $\mathbf{x}$, for at least one i it must be that $U^i(y^i) > U^i(x^i)$ and hence, by strong monotonicity, $py^i > p\omega^i$. Summing the inequalities over i yields then $p\sum_i y^i > p\sum_i \omega^i$ which implies $p\sum_i y^i - \sum_i \omega^i > 0$, which in turn contradicts the feasibility of y since $p \in \mathbb{R}^L_{++}$. ■

Note that while monotonicity is crucial in the proof of the *First welfare* theorem, to go from $U^i(y^i) \geq U^i(x^i)$ to $py^i \geq px^i$, convexity of the Consumer pb is never used.

Theorem 3 (Second welfare) Consider an economy $\omega \in \mathbb{R}^{LI}_{++}$. For any Pareto efficient allocation $\mathbf{x} \in \mathbb{R}^{LI}_{++} \subset X^I$, there exist transfers $\mathbf{t} \in \mathbb{R}^{LI}$, with $\sum_i t^i = 0$, such that $\mathbf{x} \in \mathbb{R}^{LI}_{++}$ is a competitive equilibrium allocation for some prices $p \in \mathbb{R}^{L}_{++}$ of the economy $\omega + \mathbf{t} \in \mathbb{R}^{LI}_{++}$

Proof. First of all choose $\mathbf{t} \in \mathbb{R}^{LI}$ so that $t^i = x^i - \omega^i$ for all i. This is trivially possible since $\mathbf{x}$ is feasible. We then just need to find prices $p \in \mathbb{R}^{L}_{++}$ such that at these prices any agent $i \in I$, with endowment $x^i \in X$ will not trade. Recall that $\mathbf{x} = (x^i)_{i \in I}$ is Pareto efficient by assumption. The theorem is then an implication of the *Supporting hyperplane* theorem (Theorem 7). Consider the *better than* set for agent i: $B^i(x^i) = \{y^i \in X \,|U^i(y^i) > U^i(x^i)\}$. It is a convex set and $x^i \notin B^i(x^i)$. Construct the set

$$B(\mathbf{x}) = \{\sum_{i \in I} y^i \in X : y^i \in B^i(x^i), \text{ for all } i \in I\}.$$

By construction $B(\mathbf{x})$ is convex. Furthermore, $\mathbf{x}$ being Pareto efficient implies that $\sum_{i \in I} x^i \notin B(\mathbf{x})$.[6] The *supporting hyperplane* theorem implies that there exist a $p \in \mathbb{R}^L$ such that $p\mathbf{y} \geq p\mathbf{x}$, for any $\mathbf{y} \in B(\mathbf{x})$.

It remains to show that i) indeed $p \in \mathbb{R}^{L}_{++}$ and that ii) at such prices any agent $i \in I$, with endowment $x^i \in X$ will not trade, that is, we need to show that $py^i > px^i$, for any $y^i \in B^i(x^i)$.

We show i) first, proceeding by contradiction. Suppose first $p_l < 0$ for some $l = 1, ..., L$. We can then construct for any arbitrary $i \in I$ a vector y^i such that $y^i_{l'} = x^i_{l'}$, for any $l' \neq l$, and $y^i_l > x^i_l$ so that $y^i \in B^i(x^i)$ and $py^i < px^i$. Since $i \in I$ is arbitrary, this implies the desired contradiction with the implication of the *supporting hyperplane* theorem obtained above. Suppose now that $p_l = 0$ for some $l = 1, ..., L$. We can then construct for any arbitrary $i \in I$ a vector y^i such that $y^i_{l'} = x^i_{l'} - \epsilon$, for any $l' \neq l$, and $y^i_l > x^i_l$. By continuity of U^i we can in fact choose ϵ to be small enough and $y^i_l - x^i_l$ large enough so that $y^i \in B^i(x^i)$. But by construction then $py^i < px^i$. Since $i \in I$ is arbitrary, this again implies the desired contradiction with the implication of the *supporting hyperplane* theorem.

As for ii), note that for any $i \in I$, $px^i > 0$, since $x^i \in \mathbb{R}^{L}_{++}$ by assumption. As consequence, there exist a cheaper bundle $\hat{x}^i$, such that $p\hat{x}^i < px^i$. Consider now any $y^i \in B^i(x^i)$ and assume by contradiction $py^i \leq px^i$. Construct

[6] Note that, by strong monotonicity, if redistributing allocations so as to make one agent strictly better off is feasible, so it is to redistribute to make all agents strictly better off.

the bundle $\alpha y^i+(1-\alpha)\hat{x}^i$, for some $\alpha \in [0,1)$. Then $p\left(\alpha y^i+(1-\alpha)\hat{x}^i\right) < px^i$ and, for some $\tilde{\alpha}$ close enough to 1, $U^i\left(\tilde{\alpha} y^i+(1-\tilde{\alpha})\hat{x}^i\right) > U^i\left(x^i\right)$ by continuity (hence $\tilde{\alpha} y^i+(1-\tilde{\alpha})\hat{x}^i \in B^i(x^i)$). Let $\tilde{y}^i := \tilde{\alpha} y^i+(1-\tilde{\alpha})\hat{x}^i$. We have just shown that $p(\tilde{y}^i - x^i) = -\delta > 0$, for some δ. For any $\tilde{y}^i+\sum_{i'\in I, i\neq i} y^{i'} \in B(\mathbf{x})$, therefore, $p\left(\tilde{y}^i+\sum_{i'\in I, i\neq i} y^{i'}\right) \geq p\mathbf{x}$ by the *supporting hyperplane* theorem and hence $p\left(\sum_{i'\in I, i\neq i} y^{i'}\right) - p\left(\sum_{i'\in I, i\neq i} x^{i'}\right) \geq \delta > 0$. But consider $y^i = x^i + \epsilon \mathbf{1}$, for any $i \in I$ and some $\epsilon > 0$, where $\mathbf{1}$ is the L-dimensional 1 vector. Obviously $y^i \in B^i(x^i)$ by strong monotonicity and $\lim_{\epsilon \to 0} py^i = px^i$, for any $i \in I$. We therefore derived a contradiction to $p\left(\sum_{i'\in I, i\neq i} y^{i'}\right) - p\left(\sum_{i'\in I, i\neq i} x^{i'}\right) \geq \delta$ for any $y^{i'} \in B(x^{i'})$. ■

Note that, differently from the case of the *First welfare* theorem, convexity is crucial for the *Second welfare* theorem, to be able to apply the *separating hyperplane* theorem.

2.2.2 Existence

Competitive equilibrium prices are solutions of:

$$z(p;\omega) \equiv \sum_i x^i(p, p\cdot\omega^i) - \sum_i \omega^i = 0,$$

a system of L equations in L unknowns, the prices p. By Walras law

$$p \cdot \left(\sum_i x^i(p, p\omega^i) - \sum_i \omega^i \right) = 0, \text{ for all } p \in \mathbb{R}^L_+$$

and hence at most $L-1$ equations are independent (the market clearing equation for one market can be omitted without loss of generality). By homogeneity of degree 0 in p of $x^i(p, p\omega^i)$, for any $i \in I$, prices can always be normalized, e.g., restricted without loss of generality to

$$p \in \Delta^{L-1} \equiv \left\{ p \in \mathbb{R}^L_+ : \sum_l p_l = 1 \right\},$$

the L-simplex, a compact ad convex set. The equilibrium equations can thus be always reduced to $L-1$ equations in $L-1$ unknowns. Since equations are typically nonlinear, having number of unknowns less or equal than number of independent equations does not ensure a solution exists.

Existence proof 1: Trimmed simplex

Heuristic. Consider the following Lemma first.

Lemma 1 Let $z : \Delta^{L-1} \rightarrow R^L$ be a continuous function, such that $p{\cdot}z(p) = 0$ for all p. Then there exists p^* such that $z(p^*) \leq 0$.

Proof.

$$\text{Let } \varphi_l(p) = \frac{p_l + \max\{0, z_l(p)\}}{\sum_{j=1}^{L} [p_j + \max\{0, z_j(p)\}]}, \; l = 1, ..L$$

Note that Δ^{L-1} is convex and compact, and $\varphi : \Delta^{L-1} \rightarrow \Delta^{L-1}$. Hence by the *Brouwer Fixed Point theorem* there is a fixed point p:

$$p_l = \frac{p_l + \max\{0, z_l(p)\}}{\sum_{j=1}^{L} [p_j + \max\{0, z_j(p)\}]}, \; l = 1, ..L.$$

Then

$$z_l(p)p_l \sum_{j=1}^{L} [p_j + \max\{0, z_j(p)\}] = z_l(p)p_l + z_l(p) \max\{0, z_l(p)\}$$

and summing over l yields:

$$0 = \sum_l z_l(p) \max\{0, z_l(p)\} \Rightarrow z_l(p) \leq 0 \text{ for all } l = 1, ..., L.$$

■

This is not quite an existence proof because aggregate excess demand functions, differently from the the map $z : \Delta^{L-1} \rightarrow R^L$ in the Lemma, are not defined for prices on the boundary of Δ^{L-1}; that is, when the price of some commodity is zero. We need then to use a limit argument. Fix $\varepsilon > 0$ and let $\Delta_\varepsilon^{L-1} \equiv \left\{p \in \mathbb{R}_+^L : \sum_l p_l = 1, \, p_l \geq \varepsilon \text{ for all } l\right\}$ define a "trimmed" simplex. The aggregate excess demand is indeed well-defined on it: $z : \Delta_\varepsilon^{L-1} \rightarrow \mathbb{R}^L$. Consider an arbitrary continuous extension of the excess demand to the whole simplex; call it $z_\varepsilon : \Delta^{L-1} \rightarrow \mathbb{R}^L$. It is straightforward to construct this extension so that Walras law, $pz_\varepsilon(p) = 0$ is satisfied. Construct the corresponding map $\varphi_\varepsilon : \Delta^{L-1} \rightarrow \Delta^{L-1}$ by substituting z_ε for z. Consider now a sequence of "trimmed" simplexes, as $\varepsilon \rightarrow 0$ and the corresponding sequences of maps $\{z_\varepsilon\}$ and $\{\varphi_\epsilon\}$. It is straightforward to show that, for any

ε, φ_ε has a fixed point. Let it be denoted p_ε. We obtain then a corresponding sequence of fixed points $\{p_\varepsilon\}$.[7] The *boundary property* of the excess demand z implies that at least a fixed point of the map φ_ε must lie on the interior of the trimmed simplex Δ_ε^{L-1}, for ε small enough.

By construction of the map φ_ε, at a fixed point p_ε

$$z_{\varepsilon,l}(p_\varepsilon) < 0 \text{ iff } p_{\varepsilon,l} = 0,$$

which is impossible under strong monotonicity and hence:

$$z_\varepsilon(p_\varepsilon) = 0.$$

As $\varepsilon \to 0$ any of the fixed points in the interior of the trimmed simplex remains constant and therefore represents a competitive equilibrium price of the economy, a zero of the excess demand map z. ■

Problem 4 *The real sloppy point in this heuristic proof is the construction of the continuous extension. Does it always exists? In particular, does one always exist which satisfies Walras Law, $pz(p) = 0$? Any ideas about how to construct it?*

Existence proof 2: Debreu map

Proof. Let[8] $p \in \Delta^{L-1}$ and $z : \Delta^{L-1} \to \mathbb{R}^L$ denote the excess demand for given arbitrary $\omega \in \mathbb{R}^{LI}_{++}$. Consider the map, $f : \Delta^{L-1} \to 2^{\Delta^{L-1}}$ defined by:[9]

$$f(p) = \begin{cases} q \in \arg\max_{q\in\Delta^{L-1}} qz(p) & \text{if } p \in int\left(\Delta^{L-1}\right) \\ \{q \in \Delta^{L-1} : pq = 0\} & \text{if } p \in bdry\left(\Delta^{L-1}\right) \end{cases}$$

Observe the following:

1. $q_l = 0$ if $z_l(p) < \max\{z_1(p), \ldots, z_L p\}$ and hence, $f(p) \subset bdry\left(\Delta^{L-1}\right)$ for any $p \in int\left(\Delta^{L-1}\right)$ such that $z(p) \neq 0$,

[7] Fixed points are not necessarily unique. In this case the sequence is of sets of fixed points.

[8] The proof is taken, with minor changes, from Mas Colell et al. (1995), Proposition 17.C.1, p. 585-7.

[9] With the notation $2^{\Delta^{L-1}}$ it is meant the power set (the set of all subsets) of Δ^{L-1}. The map f can equivalently be said to be a correspondence from Δ^{L-1} into Δ^{L-1}.

2. $q_l = 0$ if $p_l > 0$ and furthermore, any $p \in bdry\left(\Delta^{L-1}\right)$ cannot satisfy $p \in f(p)$ as $pp > 0$ while $pq = 0$ for all $q \in f(p)$.

But then 2) implies that any fixed point $p \in f(p)$ must satisfy $p \in int\left(\Delta^{L-1}\right)$. In turn 1) implies that any $p \in int\left(\Delta^{L-1}\right)$ cannot satisfy $p \in f(p)$ if $z(p) \neq 0$. As a consequence, any $p \in f(p)$ must satisfy $p \in int\left(\Delta^{L-1}\right)$ and $z(p) = 0$. In other words, any fixed point of the map f is a competitive equilibrium price.

It remains to show that a fixed point $p \in f(p)$ exist. This is a consequence of Kakutani fixed point theorem, Theorem 9, if we can prove that i) Δ^{L-1} is a non-empty, compact, and convex set and ii) f is upper-hemi-continuous, non-empty and convex valued. i) is straightforward and hence we concentrate on ii).

Non-empty and convex valuedness. For any $p \in \Delta^{L-1}$, $f(p)$ is a face of the simplex Δ^{L-1},[10] hence non-empty and convex.

Upper-hemi-continuity. Consider sequences $\{p^n, q^n\} \in \Delta^{L-1} \times \Delta^{L-1}$ such that $p^n \to p$, $q^n \to q$, and $q^n \in f(p^n)$ for all n. We need to show that $q \in f(p)$. Consider the following three distinct cases: ii1) $p \in int\left(\Delta^{L-1}\right)$, ii2) $p \in bdry\left(\Delta^{L-1}\right)$ and $p^n \in bdry\left(\Delta^{L-1}\right)$, indeed in the same face of the simplex as p, for n large enough, ii3) $p \in bdry\left(\Delta^{L-1}\right)$ and $p^n \in int\left(\Delta^{L-1}\right)$ for n large enough. Note that as $p^n \to p$, p^n cannot be on a different face of the simplex as p for n large enough and as a consequence we can disregard this case. Case ii1) is straightforward: $p^n >> 0$ for n large enough and $z(p)$ is continuous and so the $\arg\max_{q \in \Delta^{L-1}} qz(p^n)$ is upper-hemi-continuous. Consider case ii2) and ii3). Let $p_l > 0$, without loss of generality. Case ii2) implies that for n large enough p^n is on the same face of the simplex as p. As a consequence, $q_l = 0$ by construction of the map f. This is enough to show that $q \in f(p)$in this case. Finally consider case ii3). Note that, as $p_l > 0$ and $p_l^n > 0$ for n large enough, $z_l p^n$ is bounded above for n large enough, as the budget set is bounded. On the contrary, for some $l' \in L$, $l' \neq l$, $p_{l'} = 0$, as $p \in bdry\left(\Delta^{L-1}\right)$ in this case. Then $z_{l'}(p^n) \to \infty$, for some $l' \in L$, $l' \neq l$, by the *boundary property* of $z(p)$ in Proposition 1. Therefore, for n large enough, $z_{l'}(p^n) > z_l(p^n)$ and hence $q_l^n = 0$. This concludes the proof of upper-hemi-continuity of f. ■

Problem 5 *Which steps of the proof of a) Negishi's theorem, b) First and Second Welfare theorem, c) Existence, relies crucially on i) strict monotonic-*

[10]In particular, for any fixed point p, $f(p) = \Delta^{L-1}$.

ity of preferences, ii) strict quasi-concavity of preferences, iii) strictly positive endowments?

2.2.3 Uniqueness

Existence of a competitive equilibrium can be proved under quite general conditions.[11] Equilibria are however unique only under very strong restrictions. Several examples of such restrictions are listed in the following, without any detail.

Pareto efficiency. If endowments are Pareto efficient, there exists a unique equilibrium which is autarchic: $x^i = \omega^i$ for all $i \in I$.

Aggregation. If preferences are identical and homothetic, then an aggregation result implies that the economy is equivalent to one with a single representative agent and hence the exists a unique equilibrium which is effectively autarchic.

Gross substitution. If the aggregate demand satisfies the *gross substitution property,*

$$p'_l > p_l \text{ and } p'_j = p_j \text{ for all } j \neq l \implies z_j(p') > z_j(p),$$

the *law of demand* holds at any equilibrium price and there exists a unique equilibrium. *Gross substitution* holds for instance for Cobb Douglas and CES utility functions under restrictions on the elasticity of substitutions across goods.

Problem 6 *Prove that indeed uniqueness holds in each of the above three environments, Pareto efficiency, aggregation, and gross substitution.*

2.2.4 Local uniqueness

Let an economy be parametrized by the endowment vector $\omega \in \mathbb{R}^{LI}_{++}$ keeping preferences $(u^i)_{i\in I}$ fixed. Furthermore, normalize $p_L = 1$ and eliminate the L-th component of the excess demand. Then

$$z : \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \to \mathbb{R}^{L-1}$$

[11] We shall leave this statement essentially unsubstantiated. General equilibrium theory, for more than half a century, has considered this as one of its main objectives.

represents an aggregate excess demand for an exchange economy $\omega = (\omega^i)_{i \in I} \in \mathbb{R}^{LI}_{++}$.

Definition 5 *A $p \in \mathbb{R}^L_{++}$ such that $z(p, \omega) = 0$ is* ***regular*** *if $D_p z(p, \omega)$ has rank $L - 1$.*

It is convenient to rely on standard notions from Linear algebra to better understand the concept of *regularity. For given* (p, ω), $D_p z(p, \omega)$ is an $L - 1 \times L - 1$ matrix. Its rank being $L - 1$ implies that the matrix spans the whole $\mathbb{R}^{L-1}$ space; that is,

$$\text{for any } z_0 \in \mathbb{R}^{L-1}, \text{ there exists a } p_0 \in \mathbb{R}^{L-1} \text{ such that } D_p z(p, \omega) p_0 = z_0.$$

Since we deal with non-linear maps $z : \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{++} \to \mathbb{R}^{L-1}$, these kind of arguments only hold locally, and z_0 needs to be restricted to an open ball around $z(p, \omega)$ and p_0 to an open ball around p.

Definition 6 *An economy $\omega \in \mathbb{R}^{LI}_{++}$ is regular if $D_p z(p, \omega)$ has rank $L - 1$ for any $p \in \mathbb{R}^{L-1}_{++}$ such that $z(p, \omega) = 0$.*

Definition 7 *An equilibrium price $p \in \mathbb{R}^{L-1}_{++}$ is* ***locally unique*** *if $\exists$ an open set P such that $p \in P$ and for any $p' \neq p \in P$, $z(p', \omega) \neq 0$.*

Proposition 2 *A regular equilibrium price $p \in \mathbb{R}^{L-1}_{++}$ is locally unique.*

Proof. Fix an arbitrary $\omega \in \mathbb{R}^{LI}_{++}$. Since $D_p z(p, \omega)$ has rank $L - 1$, by regularity of p, the Inverse function theorem - Local (see Math Appendix) applied to the map $z : \mathbb{R}^{L-1}_{++} \to \mathbb{R}^{L-1}$, directly implies local uniqueness of $p \in \mathbb{R}^{L-1}_{++}$. ■

Proposition 3 *Any economy ω in a full measure Lebesgue subset of $\mathbb{R}^{LI}_{++}$ is regular.*

We say then that regularity is a *generic* property in $\mathbb{R}^{LI}_{++}$ (or equivalently that it holds *generically*) if it holds in a full measure Lebesgue subset of $\mathbb{R}^{LI}_{++}$.

Proof. The statement follows by the Transversality theorem (see Math Appendix), if $z \pitchfork 0$. We now show that $z \pitchfork 0$. Pick an arbitrary agent $i \in I$. It will be sufficient to show that, for any $(p, \omega) \in \mathbb{R}^{L-1}_{++} \times \mathbb{R}^{LI}_{++}$ such that $z(p,\omega) = 0$, we can find a perturbation $d\omega^i \in \mathbb{R}^L$ such that $dz = D_{\omega^i} z(p,\omega) d\omega^i$, for any $dz \in \mathbb{R}^{L-1}$. Consider any perturbation $d\omega^i$ such that $d\omega^i_L + p d\omega^i_{-L} = 0$, for $d\omega^i_{-L} = (\omega^i_l)_{l=1}^{L-1}$. Any such perturbation, leaves each agent $i \in I$ demand unchanged and hence it implies $D_{\omega^i} z(p,\omega) d\omega^i = -d\omega^i_{-L}$, for any arbitrary $d\omega^i_{-L} \in \mathbb{R}^L$. ■

Proposition 4 *The set of equilibrium prices of an economy $\omega \in \mathbb{R}^{LI}_{++}$ is a smooth manifold (see Math Appendix) of dimension LI.*

2.2.5 Characterization of the structure of equilibria

The differential techniques exploited to study generic local uniqueness can be expanded to provide a general characterization of competitive equilibria as a manifold parametrized by endowments. This characterization implies an existence result. We sketch some of the analysis, just to provide the reader with the flavor of the arguments.

Definition 8 *The **index** $i(p,\omega)$ of a price $p \in \mathbb{R}^{L-1}_{++}$ such that $z(p,\omega) = 0$ is defined as*

$$i(p,\omega) = (-1)^{L-1} sign \left| D_p z(p,\omega) \right| .$$

The index $i(\omega)$ of an economy $(\omega^i)_{i \in I}$ is defined as

$$i(\omega) = \sum_{p : z(p,\omega) = 0} i(p,\omega).$$

Theorem 4 (Index) *For any regular economy $\omega \in \mathbb{R}^{LI}_{++}$, $i(\omega) = 1$.*

Proof. The theorem is a deep mathematical result whose proof is clearly beyond the scope of this class. Let it suffice to say that the proof relies crucially on the boundary property of excess demand. Adventurous reader might want to look at Mas Colell (1985), section 5,6, p. 201-15. ■

Corollary 1 *Any regular economy $\omega \in \mathbb{R}^{LI}_{++}$ has an odd number of equilibria.*

Problem 7 *Is it the case that any economy $\omega \in \mathbb{R}^{LI}_{++}$ which is not regular has an even number of equilibria? Can you support your argument with an example?*

Existence proof 3: Index theory and homothopy theory

Corollary 2 *Any economy* $\omega \in \mathbb{R}^{LI}_{++}$ *has at least one equilibrium price* $p \in \mathbb{R}^{L-1}_{++}$.

Proof. By contradiction. Suppose there exist an economy $\omega \in \mathbb{R}^{LI}_{++}$ with no equilibrium. Then, $\omega \in \mathbb{R}^{LI}_{++}$ is regular, by definition of regularity - a contradiction with previous corollary. ■

The existence result is then a corollary of the Index theorem. It is useful to study the simple case in which $L = 2$. In this case, then, $z : \mathbb{R}_{++} \to \mathbb{R}$. The boundary properties of the excess demand $z(p, \omega)$ imply that, for any $\omega \in \mathbb{R}^{2I}_{++}$,

$$\begin{aligned} z(p,\omega) &\to +\infty \text{ as } p \to 0 \\ z(p,\omega) &\to -L \text{ as } p \to \infty. \end{aligned}$$

As a consequence, an equilibrium exists by continuity of $z(p, \omega)$. Furthermore, suppose ω is regular, and let the prices p_j such that $z(p, \omega) = 0$ be ordered, so that $p_j < p_{j+1}$, $j = 1, 2, ...$ Then $\frac{\partial z(p,\omega)}{\partial p}|_{p=p_1} < 0$. Actually, $\left.\frac{\partial z(p,\omega)}{\partial p}\right|_{p=p_j} \begin{cases} < 0 \text{ for } j \text{ odd} \\ > 0 \text{ for } j \text{ even} \end{cases}$. As a consequence, $i(\omega) = 1$.

We can also try and give more intuitive sense of the arguments, off of the proof of the Index theorem, required for this approach to the existence question. To this end we need to use some construction used in homothopy theory (see Milnor (1965). Let $\omega \in \mathbb{R}^{LI}_{++}$ be an arbitrary regular economy. Pick an economy $\omega' \in \mathbb{R}^{LI}_{++}$ such that there exist a unique price $p \in \mathbb{R}^{L-1}_{++}$ such that $z(p, \omega) = 0$, and $D_p z(p)$ has rank $L-1$. One such economy can always be constructed by choosing $\omega' \in \mathbb{R}^{LI}_{++}$ to be a Pareto optimal allocation. In fact, [we can show that] generic regularity holds in the subset of economies with Pareto optimal endowments. Let $t\omega + (1-t)\omega'$, for $0 \leq t \leq 1$, represent a 1-dimensional subset of economies. Let $Z(p, t)$ be the map $Z : \mathbb{R}^{L-1}_{++} \times [0, 1] \to \mathbb{R}^{L-1}$ induced by $Z(p, t) = z(p, t\omega + (1-t)\omega')$ for given (ω, ω'). We say that $Z(p, t)$ is an *homotopy,* or that $z(p, \omega)$ and $z(p, \omega')$ are homotopic to each other. [We can show that] $DZ(p, t)$ has rank $L - 1$ in its domain. It follows from the Corollary of the Inverse function theorem - Global (see Math Appendix) that the set $(p, t) \in Z^{-1}(0)$, is a smooth manifold of dimension 1. [We can show that] prices p can, without loss of generality, be restricted to a compact set P such that $Z^{-1}(0)$ never intersects the boundary of P:

$Z^{-1}(0) \cap [bdry(P) \times [0,1]] = \varnothing$.[12] As a consequence $Z^{-1}(0)$ is a compact smooth manifold of dimension 1. By the Classification theorem (see Math Appendix), $Z^{-1}(0)$ is then homeomorphic to a countable set of segments in $\mathbb{R}$ and of circles S.[13] Regularity of $Z^{-1}(0)$ at the boundary, $t = 0$ and $t = 1$ and the property that $Z^{-1}(0) \cap [bdry(P) \times [0,1]] = \varnothing$ imply that at least one component of $Z^{-1}(0)$ is homeomorphic to a line with boundary at $t = 0$ and $t = 1$. It looks confusing, but it's easier with a few figures.

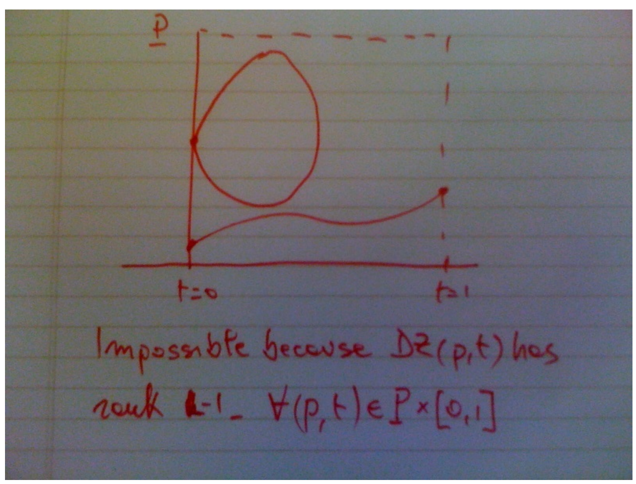

Figure 2.1: Characterization of $Z^{-1}(0)$ - impossible

[12] This is a consequence of the boundary conditions of excess demand systems. In other words, we could adopt the alternative normalization, restricting prices in the simplex Δ, a compact set, and show that equilibrium prices are never on $\partial\Delta$.

[13] Along a component of $Z^{-1}(0)$ (a line or a circle), a change in index occurs when the manifold folds.

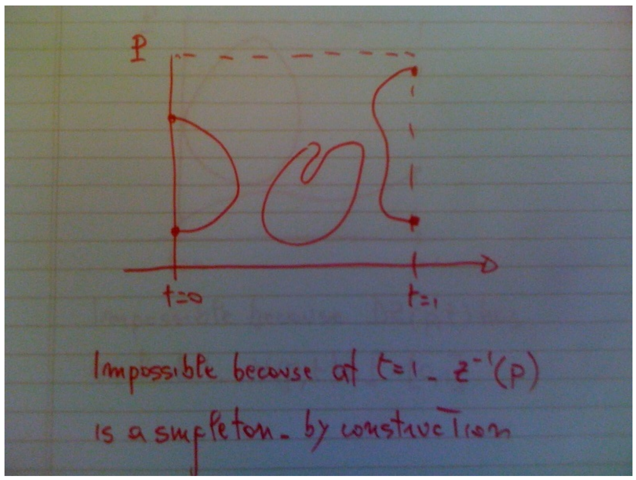

Figure 2.2: Characterization of $Z^{-1}(0)$ - impossible

2.3 Some useful math

2.3.1 Convexity and separation

Theorem 5 (Separating hyperplane) *Suppose $A, B \subset \mathbb{R}^N$ are convex, closed, and disjoint sets. Then there exist a $p \in \mathbb{R}^N$, $p \neq 0$, and a $c \in \mathbb{R}$ such that*

$$px \geq c, \text{ for any } x \in A; \text{ and } py \leq c, \text{ for any } y \in B.$$

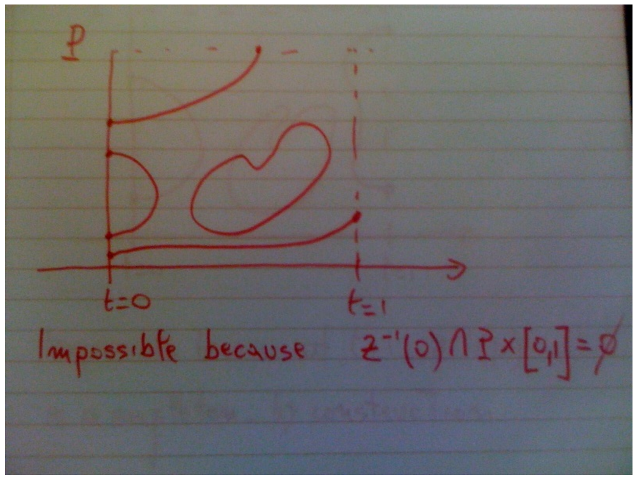

Figure 2.3: Characterization of $Z^{-1}(0)$ - impossible

Theorem 6 (Separating hyperplane; stronger version) *Let X be a finite dimensional vector space. Let K be a non-empty, compact and convex subset of X. Let M be a non-empty, closed and convex subset of X. Furthermore, let K and M be disjoint. Then, there exists $\hat{\pi} \in X \backslash \{0\}$ such that*

$$\sup_{\tau \in M} \hat{\pi}\tau < \inf_{\tau \in K} \hat{\pi}\tau.$$

Theorem 7 (Supporting hyperplane) *Suppose $B \subset \mathbb{R}^N$ is a convex set and suppose $x \notin int(B)$. Then there exist a $p \in \mathbb{R}^N$, $p \neq 0$, such that*

$$px \geq py \text{ , for any } y \in B.$$

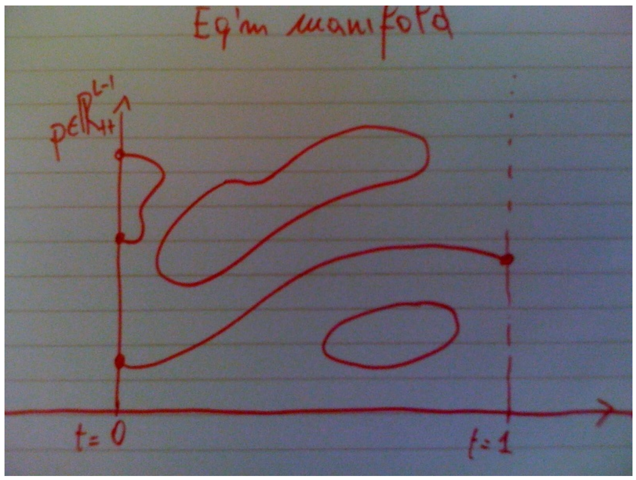

Figure 2.4: Characterization of $Z^{-1}(0)$ - impossible

2.3.2 Fixed point theorems

Theorem 8 (Brouwer) *Suppose $A \subset \mathbb{R}^N$ is non-empty, compact (closed and bounded), and convex set. Suppose $f : A \to A$ is a continuous function mapping A into itself. Then f has a fixed point in A, that is,*

$$\exists x \in A \text{ such that } x = f(x)$$

Theorem 9 (Kakutani) *Suppose $A \subset \mathbb{R}^N$ is non-empty, compact (closed and bounded), and convex set. Suppose $f : A \to A$ is a upper-hemi-continuous* *function correspondence*[14] *mapping* A *into itself and such that the set* $f(x) \subset A$ *is non-empty and convex for any* $x \in A$ *. Then* f *has a fixed point in* A*, that is,*

$$\exists x \in A \text{ such that } x \in f(x)$$

2.3.3 A primer on differential topology

Theorem 10 *(Inverse function theorem - Local). Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be* C^∞*. If* Df *has rank* n*, at some* $x \in \mathbb{R}^n$*, there exist an open set* $V \subseteq \mathbb{R}^n$ *and a function* $f^{-1} : V \to \mathbb{R}^n$ *such that* $f(x) \in V$ *and* $f^{-1}(f(z)) = z$ *in a neighborhood of* x*.*

Definition 9 *A subset* $X \subset \mathbb{R}^m$ *is a smooth manifold of dimension* n *if for any* $x \in X$ *there exist a neighborhood* $U \subset X$ *and a* C^∞ *function* $f : U \to \mathbb{R}^m$ *such that* Df *has rank* n *in the whole domain.*

Let $f(U) = V$. A smooth manifold of dimension n is then locally *parametrized* by a restriction of the function f^{-1} on the open set $V \cap \mathbb{R}^n \times \{0\}^{m-n}$, in the sense that f^{-1} maps $V \cap \mathbb{R}^n \times \{0\}^{m-n}$ onto U, a neighborhood of x on X.

Example 1 *An example of a* 1*-manifold of* $\mathbb{R}^2$ *is* $S = \left\{ x \in \mathbb{R}^2 \,\middle|\, (x_1)^2 + (x_2)^2 = 1 \right\}$*, the circle. An explicit parametrization for* S *can be constructed as follows. Seeing a restriction of* f^{-1}*on the open set* $V \cap \mathbb{R}^n \times \{0\}^{m-n}$ *as a map* $\phi_i : \mathbb{R}^n \to \mathbb{R}^m$*, the following four maps are sufficient to parametrize* S*:*

$$\begin{aligned}
\phi_1(x_1) &= \left(x_1, \sqrt{1-(x_1)^2} \right) \text{ if } x_2 > 0 \\
\phi_2(x_1) &= \left(x_1, -\sqrt{1-(x_1)^2} \right) \text{ if } x_2 < 0 \\
\phi_3(x_2) &= \left(\sqrt{1-(x_2)^2}, x_2 \right) \text{ if } x_1 > 0 \\
\phi_4(x_2) &= \left(-\sqrt{1-(x_2)^2}, x_2 \right) \text{ if } x_1 < 0
\end{aligned}$$

[14] See Mas Colell et al. (1995), Definition M.H.3, p. 950 for a definition of upper-hemi-continuous correspondence.

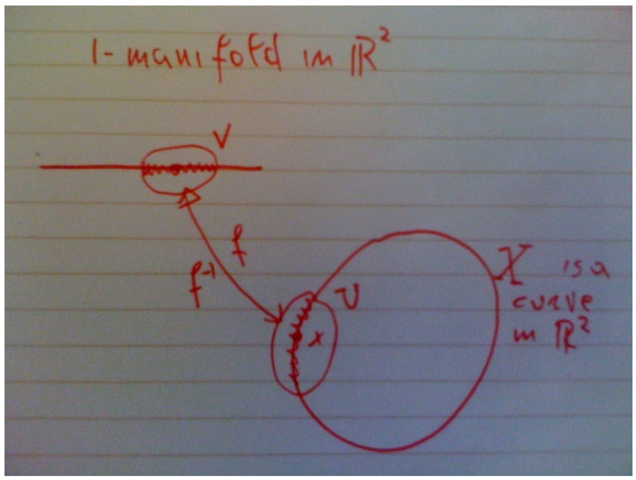

Figure 2.5: Parametrization of a 1-manifold X

Definition 10 *Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $m > n$, be C^∞. f is transversal to 0, denoted $f \pitchfork 0$, if Df has rank n for any $x \in \mathbb{R}^m$ such that $f(x) = 0$.*

Theorem 11 *(Transversality). Let $f : \mathbb{R}^m \to \mathbb{R}^n$, $m > n$, be C^∞ and transversal to 0, $f \pitchfork 0$. Decompose any vector $x \in \mathbb{R}^m_{++}$ as $x = [x_1, x_2]$, with $x_1 \in \mathbb{R}^{m-n}$, $x_2 \in \mathbb{R}^n$. Then $D_{x_2} f(x)$ has rank n for all x_1 in a Lebesgue measure-1 subset of $\mathbb{R}^{m-n}$.*

Definition 11 *A subset $X \subset \mathbb{R}^m$ is a smooth manifold with boundary of dimension n if for any $x \in X$ there exist a neighborhood $U \subset X$ and a C^∞ function $f : U \to \mathbb{R}^{m-1} \times \mathbb{R}_+$ such that Df has rank n in the whole domain.*

The boundary ∂X of X is defined by $\partial X = f^{-1}(\{0\}^{m-1} \times \mathbb{R}_+) \cap X$. It can be shown that, if $X \subset \mathbb{R}^m$ is a smooth manifold with boundary of dimension n, then ∂X is a smooth manifold (without boundary) of dimension $n-1$.

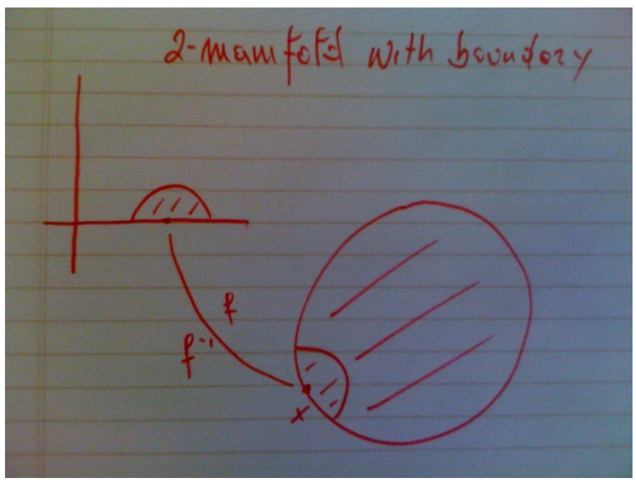

Figure 2.6: Parametrization of a 1-manifold X

Example 2 *An example of a smooth manifold with boundary of dimension 2 in* $\mathbb{R}^2$ *is* $\mathbf{S} = \{x \in \mathbb{R}^2 \,|\, (x_1)^2 + (x_2)^2 \leq 1\}$, *the sphere. A parametrization for can be constructed* $\mathbf{S}$, *by means of a series of maps* $\phi_i : \mathbb{R}^2 \to \mathbb{R}^2_+$, *along the lines of the parametrization of the circle, S. Furthermore,* $\partial\mathbf{S} = S$.

Theorem 12 (Inverse function theorem - Global) *Let* $f : \mathbb{R}^m \to \mathbb{R}^n$, $m \geq n$, *be* C^∞. *Suppose that* $D_x f$ *has rank n for any* $x \in \mathbb{R}^m$. *Then* $f^{-1}(0) = \{x \in \mathbb{R}^m \,|\, f(x) = 0\}$ *is a smooth manifold of dimension* $m - n$.

Corollary 3 *Let* $f : \mathbb{R}^n \times [0,1] \to \mathbb{R}^n$ *be* C^∞*. Suppose that* $D_x f$ *has rank* n *for any* $(x,t) \in \mathbb{R}^n \times [0,1]$*. Then* $f^{-1}(0) = \{(x,t) \in \mathbb{R}^n \times [0,1] \,|\, f(x,t) = 0\}$ *is a smooth* $1-$*manifold with boundary.*

Theorem 13 (Classification) *Every compact smooth* $1-$*manifold is homeomorphic to a disjoint union of countably many copies of segments in* $\mathbb{R}$ *and of* S*.*

Furthermore, $\partial\, f^{-1}(0) = \{x \in \mathbb{R}^n \,|\, f(x,0) = 0\} \cup \{x \in \mathbb{R}^n \,|\, f(x,1) = 0\}$.

2.3.4 References

The main reference is:

A. Mas-Colell (1985): *The Theory of General Economic Equilibrium: A Differentiable Approach*, Econometric Society Monograph, Cambridge University Press.

But, as Andreu told me once, "I do not hate students so much that I would give them this book to read." Similar comments hold, in my opinion, for

Y. Balasko (1988): *Foundations of the Theory of General Equilibrium*, Academic Press.

You are then left with:

A. Mas-Colell, M. Whinston, and J. Green (1995): *Microeconomic Theory*, Oxford University Press, ch. 17.D.

A. Mas-Colell, Four Lectures on the Differentiable Approach to General Equilibrium Theory, in A. A. Ambrosetti, F. Gori, and R. Lucchetti (eds.), *Lecture Notes in Mathematics*, No. 1330, Springer-Verlag, Berlin, 1986.

Chapter 3

Two-period economies

In a two-period pure exchange economy we study *financial market equilibria*. In particular, we study the *welfare* properties of equilibria and their implications in terms of *asset pricing*.

In this context, as a foundation for macroeconomics and financial economics, we study sufficient conditions for *aggregation*, so that the standard analysis of one-good economies is without loss of generality, sufficient conditions for the *representative agent* theorem, so that the standard analysis of single agent economies is without loss of generality.

The *No-arbitrage* theorem and the *Arrow theorem* on the decentralization of equilibria of state and time contingent good economies via financial markets are introduced as useful means to characterize financial market equilibria.

3.1 Arrow-Debreu economies

Consider an economy extending for 2 periods, $t = 0, 1$. Let $i \in \{1, ..., I\}$ denote agents and $l \in \{1, ..., L\}$ physical goods of the economy. In addition, the state of the world at time $t = 1$ is uncertain. Let $\{1, ..., S\}$ denote the state space of the economy at $t = 1$. For notational convenience we typically identify $t = 0$ with $s = 0$, so that the index s runs from 0 to S.

Define $n = L(S+1)$. The consumption space is denoted then by $X = \mathbb{R}^n_+$. Each agent is endowed with a vector $\omega^i = (\omega^i_0, \omega^i_1, ..., \omega^i_S)$, where $\omega^i_s \in X^I$, for any $s = 0, ...S$. Let $u^i : X \longrightarrow \mathbb{R}$ denote agent i's utility function. Let $prob = (prob_s)_{s=1}^S \in \Delta^{S-1}_{++}$ the strictly positive S-dimensional symplex. We will assume:

Assumption 1 *$\omega^i \in R^n_{++}$ for all i. Furthermore, u^i is continuous, strongly monotonic, strictly quasi-concave and smooth, for all i. Finally, u^i has a Von Neumann-Morgernstern representation:*

$$u^i(x^i) = u^i(x^i_0) + \sum_{s=1}^{S} prob_s u^i(x^i_s)$$

Suppose now that at time 0, agents can buy *contingent commodities.* That is, contracts for the delivery of goods at time 1 contingently to the realization of uncertainty. Denote by $x^i = (x^i_0, x^i_1, ..., x^i_S) \in X$ the vector of all such contingent commodities purchased by agent i at time 0, where $x^i_s \in \mathbb{R}^L_+$, for any $s = 0, ..., S$. Also, let $x = (x^1, ..., x^I) \in X^I$.

Let $\phi = (\phi_0, \phi_1, ..., \phi_S) \in \mathbb{R}^n_{++}$, where $\phi_s \in \mathbb{R}^L_+$ for each s, denote the *price of state contingent commodities*; that is, for a price ϕ_{ls} agents trade at time 0 the delivery in state s of one unit of good l.

Under the assumption that the markets for all contingent commodities are open at time 0, agent i's budget constraint can be written as[1]

$$\phi_0(x^i_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s(x^i_s - \omega^i_s) \leq 0 \tag{3.1}$$

Definition 12 *An Arrow-Debreu equilibrium is a $(x, \phi) \in X^I \times \mathbb{R}^n_{++}$ such that*

1. $x^i \in \arg\max u^i(x)$ *s.t.* $\phi_0(x_0 - \omega^i_0) + \sum_{s=0}^{S} \phi_s(x_s - \omega^i_s) \leq 0,$ *and*

2. $\sum_{i=1}^{I} x^i - \omega^i_s = 0$, *for any* $s = 0, 1, ..., S$

Observe that the dynamic and uncertain nature of the economy (consumption occurs at different times $t = 0, 1$ and states $s \in S$) does not manifests itself in the analysis: a consumption good l at a time t and state s is treated simply as a different commodity than the same consumption good

[1] We write the budget constraint with equality. This is without loss of generality under monotonicity of preferences, an assumption we shall maintain.

l at a different time t' or at the same time t but different state s'. This is the simple trick introduced in Debreu's last chapter of the *Theory of Value*. It has the fundamental implication that the standard theory and results of static equilibrium economies can be applied without change to our dynamic) environment. In particular, then, under the standard set of assumptions on preferences and endowments, an equilibrium exists and the *First* and *Second Welfare Theorems* hold.[2]

3.2 Financial market economies

Consider the 2-period economy just introduced. Suppose now contingent commodities are not traded. Instead, agents can trade in spot markets and in $j \in \{1, ..., J\}$ assets. An asset j is a promise to pay $a_s^j \geq 0$ units of good $l = 1$ in state $s = 1, ..., S$.[3] Let $a_j = (a_1^j, ..., a_S^j) \in \mathbb{R}_+^S$. To summarize the payoffs of all the available assets, define the $S \times J$ asset payoff matrix

$$A = \begin{pmatrix} a_1^1 & ... & a_1^J \\ ... & & ... \\ a_S^1 & ... & a_S^J \end{pmatrix}.$$

It will be convenient to define a_s to be the s-th row of the matrix. Note that it contains the payoff of each of the assets in state s.

Let $p = (p_0, p_1, ..., p_S)$, where $p_s \in \mathbb{R}_{++}^L$ for each s, denote the *spot price vector* for goods. That is, for a price p_{ls} agents trade one unit of good l in state s. Recall the definition of prices for state contingent commodities in Arrow-Debreu economies, denoted ϕ and note the difference (different commodity spaces are everything in the world of general equilibrium theory)! Let good $l = 1$ at each date and state represent the numeraire; that is, $p_{1s} = 1$, for all $s = 0, ..., S$.

Let x_{sl}^i denote the amount of good l that agent i consumes in good s. Let

[2]Having set definitions for 2-periods Arrow-Debreu economies, it should be apparent how a generalization to any finite T-periods economies is in fact effectively straightforward. Infinite horizon will be dealt with in the next chapter.

[3]The non-negativity restriction on asset payoffs is just for notational simplicity, essentially without loss of generality. Note also that each asset pays in units on good 1. We call such assets *numeraire assets*, for obvious reasons. This assumption instead not without loss of generality. We'll see this afterwards.

$q = (q_1, ..., q_J) \in \mathbb{R}^J_+$, denote the prices for the assets.[4] Note that the prices of assets are non-negative, as we normalized asset payoff to be non-negative.

Given prices $(p, q) \in \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ and the asset structure $A \in \mathbb{R}^{SJ}_+$, any agent i picks a consumption vector $x^i \in X$ and a portfolio $z^i \in \mathbb{R}^J$ to maximize presend discounted utility. s.t.

$$\begin{aligned} p_0(x^i_0 - \omega^i_0) &\leq -qz^i \\ p_s(x^i_s - \omega^i_s) &\leq A_s z^i, \text{ for } s = 1, ...S. \end{aligned}$$

Definition 13 *A Financial markets equilibrium is a* $(x, z, p, q) \in X^I \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ *such that*

1. $x^i \in \arg\max u^i(x) \ s.t. \ \begin{array}{c} p_0(x^i_0 - \omega^i_0) \leq -qz^i \\ p_s(x^i_s - \omega^i_s) \leq a_s z^i, \ \textit{for } s = 1, ...S \end{array};$

2. $\sum_{i=1}^{I} x^i - \omega^i_s \leq 0$, *for any* $s = 0, 1, ..., S$, *and* $\sum_{i=1}^{I} z^i = 0$.

Financial markets equilibrium is the equilibrium concept we shall care about. This is because i) Arrow-Debreu markets are perhaps too demanding a requirement, and especially because ii) we are interested in financial markets and asset prices q in particular. *Arrow-Debreu equilibrium* will be a useful concept insofar as it represents a benchmark (about which we have a wealth of available results) against which to measure *Financial markets equilibrium.*

Having paid our dues to precision, we shall now write budget constraints and feasibility conditions with equality, which is always the case under our stringent monotonicity assumptions.

Remark 2 *The economy just introduced is characterized by asset markets in zero net supply, that is, no endowments of assets are allowed for. It is straightforward to extend the analysis to assets in positive net supply, e.g., stocks. In fact, part of each agent i's endowment (to be specific: the projection of his/her endowment on the asset span,* $< A >= \{\tau \in R^S : \tau = Az, \ z \in R^J\}$*) can be represented as the outcome of an asset endowment,* z^i_w*:*

$$\omega^i_{s1} = w^i_{s1} + a_s z^i_w, \ \textit{for any } s \in S$$

and proceed straightforwardly by constructing the budget constraints and the equilibrium notion.

[4] Quantities will be column vectors and prices will be row vectors, to avoid the annoying use of transposes.

No Arbitrage

Before deriving the properties of asset prices in equilibrium, we shall invest some time in understanding the implications that can be derived from the milder condition of no-arbitrage. This is because the characterization of no-arbitrage prices will also be useful to characterize financial markets equilbria.

For notational convenience, define the $(S+1) \times J$ matrix

$$W = \begin{bmatrix} -q \\ A \end{bmatrix}.$$

Definition 14 *W satisfies the No-arbitrage condition if*

$$\textit{there does not exist a } z \in \mathbb{R}^J \textit{ such that } Wz > 0.$$[5]

The No-Arbitrage condition can be equivalently formulated in the following way. Define the span of W to be

$$< W >= \{\tau \in \mathbb{R}^{S+1} : \tau = Wz,\ z \in \mathbb{R}^J\}.$$

This set contains all the feasible wealth transfers, given asset structure A. Now, we can say that W satisfies the No-arbitrage condition if

$$< W > \bigcap \mathbb{R}_+^{S+1} = \{0\}.$$

Clearly, requiring that $W = (-q, A)$ satisfies the No-arbitrage condition is weaker than requiring that q is an equilibrium price of the economy (with asset structure A). By strong monotonicity of preferences, No-arbitrage is equivalent to requiring the agent's problem to be well defined. The next result is remarkable since it provides a foundation for asset pricing based only on No-arbitrage.

Theorem 14 *(No-Arbitrage theorem)*

$$< W > \bigcap \mathbb{R}_+^{S+1} = \{0\} \Longleftrightarrow \exists \hat{\pi} \in \mathbb{R}_{++}^{S+1} \textit{ such that } \hat{\pi} W = 0.$$

[5] Recall that $Wz > 0$ requires that all components of Wz are ≥ 0 and at least one of them > 0.

Note we are using the following definition: $\hat{\pi}W = 0$ implies $\hat{\pi}\tau = 0$ for all $\tau \in < W >$. Observe that there is no uniqueness claim on the $\hat{\pi}$, just existence. Importantly, $\hat{\pi}W = 0$ provides a pricing formula for assets:

$$\hat{\pi}W = \begin{pmatrix} ... \\ -\hat{\pi}_0 q^j + \hat{\pi}_1 a_1^j + ... + \hat{\pi}_S a_S^j \\ ... \end{pmatrix} z = \begin{pmatrix} ... \\ 0 \\ ... \end{pmatrix}_{Jx1} ;$$

a condition which must hold for any $z \in \mathbb{R}^J$, hence implying, after rearranging:

$$q^j = \pi_1 a_1^j + ... + \pi_S a_S^j, \text{ for } \quad \pi_s = \frac{\hat{\pi}_s}{\hat{\pi}_0} \text{ and any asset } j \in J. \tag{3.2}$$

Note how the positivity of all components of $\hat{\pi}$ was necessary to obtain (3.2).

Proof. $\Longrightarrow$ Define the simplex in $\mathbb{R}_+^{S+1}$ as $\Delta = \{\tau \in \mathbb{R}_+^{S+1} : \sum_{s=0}^S \tau_s = 1\}$. Note that by the No-arbitrage condition, $< W > \bigcap \Delta$ is empty. The proof follows crucially on the *separating hyperplane; stonger version*, stated as Theorem 6.

Let $X = \mathbb{R}_+^{S+1}$, $K = \Delta$ and $M =< W >$. Observe that all the required properties of the theorem hold. As a result, there exists $\hat{\pi} \in X \backslash \{0\}$ such that

$$\sup_{\tau \in <W>} \hat{\pi}\tau < \inf_{\tau \in \Delta} \hat{\pi}\tau. \tag{3.3}$$

We first show that $\hat{\pi} \in \mathbb{R}_{++}^{S+1}$. Suppose, on the contrary, that there is some s for which $\hat{\pi}_s \leq 0$. Then note that in (3.3), the RHS≤ 0. By (3.3), then, LHS < 0. But this contradicts the fact that $0 \in < W >$.

We still have to show that $\hat{\pi}W = 0$, or in other words, that $\hat{\pi}\tau = 0$ for all $\tau \in < W >$. Suppose, on the contrary that there exists $\tau \in < W >$ such that $\hat{\pi}\tau \neq 0$. Since $< W >$ is a subspace, there exists $\alpha \in \mathbb{R}$ such that $\alpha\tau \in < W >$ and $\hat{\pi}\alpha\tau$ is as large as we want. However, RHS is bounded above, which implies a contradiction.

$\Longleftarrow$ The existence of $\hat{\pi} \in \mathbb{R}_{++}^{S+1}$ such that $\hat{\pi}W = 0$ implies $< W > \bigcap \mathbb{R}_+^{S+1} = \{0\}$. By contradiction, suppose $\exists \tau^* \in < W >$ and such that $\tau^* \in \mathbb{R}_+^{S+1} \backslash \{0\}$. Since $\hat{\pi}$ is strictly positive, $\hat{\pi}\tau^* > 0$, the desired contradiction. ■

A few final remarks to this section.

Remark 3 *An asset which pays one unit of numeraire in state s and nothing in all other states (Arrow security), has price π_s; this is an immediate consequence of (3.2). Such asset is called Arrow security.*

Remark 4 *Is the vector $\hat{\pi}$ obtained by the No-arbitrage theorem unique? Notice how (3.2) defines a system of J equations and S unknowns, represented by π. Define the set of solutions to that system as*

$$R(q) = \{\pi \in \mathbb{R}^S_{++} : q = \pi A\}.$$

Suppose, the matrix A has rank $J' \leq J$ (that it, A has J' linearly independent column vectors and J' is the effective dimension of the asset space). In general, then $R(q)$ will have dimension $S - J'$. It follows then that, in this case, the No-arbitrage theorem restricts π to lie in a $S - J'$ dimensional set (or, equivalently, $\hat{\pi}$ in a $S - J' + 1$ dimensional set). If we had S linearly independent assets, the solution set has dimension zero, and there is a unique π vector that solves (3.2). The case of S linearly independent assets is referred to as "complete markets."

Remark 5 *Recall we assumed preferences are Von Neumann-Morgernstern:*

$$u^i(x^i) = u^i(x^i_0) + \sum_{s=1,\dots,S} prob_s u^i(x^i_s),$$

with $prob_s > 0$, for any $s \in S$, and $\sum_{s=1,\dots,S} prob_s = 1$. We never used this assumption until now. But in this case, let then $m_s = \frac{\pi_s}{prob_s}$. Then

$$q^j = E\left(mA_j\right) \tag{3.4}$$

In this representation of asset prices the vector $m \in \mathbb{R}^S_{++}$ is called Stochastic discount factor.

3.2.1 The stochastic discount factor

In the previous section we showed the existence of a vector that provides the basis for pricing assets in a way that is compatible with equilibrium, albeit milder than that. In this section, we will strengthen our assumptions and study asset prices in a full-fledged economy. Among other things, this will allow us to provide some economic content to the vector π

Recall the definition of Financial market equilibrium. Let $IMRS_s^i(x^i)$ denote agent i's marginal rate of substitution between consumption of the numeraire good 1 in state s and consumption of the numeraire good 1 at date 0:

$$IMRS_s^i(x^i) = \frac{\frac{\partial u^i(x_s^i)}{\partial x_{1s}^i}}{\frac{\partial u^i(x_0^i)}{\partial x_{10}^i}}$$

Let $IMRS^i(x^i) = (\ldots IMRS_s^i(x^i)\ldots) \in \mathbb{R}_+^S$ denote the vector of intertemporal marginal rates of substitution for agent i, an S dimentional vector. Note that, under the assumption of strong monotonicity of preferences, $IMRS^i(x^i) \in \mathbb{R}_{++}^S$.

By taking the First Order Conditions (necessary and sufficient for a maximum under the assumption of strict quasi-concavity of preferences) with respect to z_j^i of the individual problem for an arbitrary price vector q, we obtain that

$$q^j = \sum_{s=1}^{S} prob_s IMRS_s^i(x^i) a_s^j = E\left(IMRS^i(x^i) \cdot a^j\right), \tag{3.5}$$

for all $j = 1, ..., J$ and all $i = 1, ..., I$, where of course the allocation x^i is the equilibrium allocation. At equilibrium, therefore, the marginal cost of one more unit of asset j, q^j, is equalized to the marginal valuation of that agent for the asset's payoff, $\sum_{s=1}^S prob_s IMRS_s^i(x^i) a_s^j$.

Compare equation (3.5) to the previous equation (3.4). Clearly, at any equilibrium, condition (3.5) has to hold for each agent i. Therefore, in equilibrium, the vector of marginal rates of substitution of *any arbitrary* agent i can be used to price assets; that is any of the agents' vector of marginal rates of substitution (normalized by probabilities) is a viable stochastic discount factor m.

In other words, any vector $(\ldots prob_s IMRS_s^i(x^i)\ldots)$ belongs to $R(q)$ and is hence a viable π for the asset pricing equation (**??**). But recall that $R(q)$ is of dimension $S - J'$, where J' is the effective dimension of the asset space. The higher the the effective dimension of the asset space (intuitively said, the larger the set of financial markets) the more aligned are agents' marginal rates of substitution at equilibrium (intuitively said, the smaller are unexploited gains from trade at equilibrium). In the extreme case, when markets are complete (that is, when the rank of A is S) and the set $R(q)$ is a singleton, $IMRS^i(x^i)$ are equalized across agents i at equilibrium: $IMRS^i(x^i) = IMRS$, for any $i = 1, ..., I$.

Let $MRS^i_{ls}(x^i)$ denote agent i's marginal rate of substitution between consumption the good l and consumption of the numeraire good 1 in state $s = 0, 1, ..., S$:

$$MRS^i_{ls}(x^i) = \frac{\frac{\partial u^i(x^i_s)}{\partial x^i_{ls}}}{\frac{\partial u^i(x^i_s)}{\partial x^i_{1s}}};$$

let also $MRS^i_s(x^i) = (\ldots MRS^i_{ls}(x^i) \ldots) \in \mathbb{R}^L_+$ and $MRS^i(x^i) = (\ldots MRS^i_s(x^i) \ldots) \in \mathbb{R}^{LS}_+$.

Problem 8 *Write the Pareto problem for the economy and show that, at any Pareto optimal allocation, x, it is the case that*

$$\begin{aligned} IMRS^i(x^i) &= IMRS \\ MRS^i(x^i) &= MRS \end{aligned}$$

for any $i = 1, ..., I$. Furthermore, show that an allocation x which satisfies the feasibility conditions (market clearing) for goods and is such that

$$\begin{aligned} IMRS^i(x^i) &= IMRS \\ MRS^i(x^i) &= MRS \end{aligned}$$

for any $i = 1, ..., I$, is a Pareto optimal allocation.

We conclude that, when markets are Complete, equilibrium allocations are Pareto optimal. That is, the First Welfare theorem holds for Financial market equilibria when markets are Complete.

Problem 9 *(Economies with bid-ask spreads) Extend our basic two-period incomplete market economy by assuming that, given an exogenous vector $\gamma \in R^J_{++}$:*

the buying price of asset j is $q_j + \gamma_j$

while

the selling price of asset j is q_j

for any $j = 1, ..., J$. Write the budget constraint and the First Order Conditions for an agent i's problem. Derive an asset pricing equation for q_j in terms of intertemporal marginal rates of substitution at equilibrium.

3.2.2 Arrow theorem

The Arrow theorem is the fundamental decentralization result in financial economics. It states sufficient conditions for a form of equivalence between the Arrow-Debreu and the Financial market equilibrium concepts. It was essentially introduced by Arrow (1952). The proof of the theorem introduces a reformulation of the budget constraints of the *Financial market economy* which focuses on feasible wealth transfers across states directly, that is, on the span of A:

$$< A >= \left\{ \tau \in R^S \ : \ \tau = Az, \ z \in R^J \right\}.$$

Such a reformulation is important not only in itself but as a lemma for welfare analysis in *Financial market economies.*

Proposition 5 *Let* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ *represent a Financial market equilibrium of an economy with* $rank(A) = S$*. Then* $(x, \phi) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^n_{++}$ *represents an Arrow-Debreu equilibrium if* $\phi_s = \pi_s p_s$*, for any* $s = 1, ..., S$ *and some* $\pi \in R^S_{++}$*. The converse also holds. Let* $(x, \phi) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^n_{++}$ *represent an Arrow-Debreu equilibrium. Then* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ *represents a Financial market equilibrium of a complete market economy (that is, whose asset structure satisfies* $rank(A) = S$*) if*

$$\begin{aligned} \phi_s &= \pi_s p_s, \ \textit{for any } s = 1, ..., S, \ \textit{and some } \pi \in \mathbb{R}^S_{++} \\ q &= \sum_{s=1}^{S} prob_s IMRS^i_s(x^i) a_s. \end{aligned}$$

Proof. $\Longrightarrow$ Financial market equilibrium prices of assets q satisfy No-arbitrage. There exists then a vector $\hat{\pi} \in \mathbb{R}^{S+1}_{++}$ such that $\hat{\pi} W = 0$, or $q = \pi A$. The budget constraints in the financial market economy are

$$\begin{aligned} p_0 \left(x^i_0 - \omega^i_0 \right) + q z^i &= 0 \\ p_s \left(x^i_s - \omega^i_s \right) &= a_s z^i, \text{ for } s = 1, ...S. \end{aligned}$$

Substituting $q = \pi A$, expanding the first equation, and writing the constraints at time 1 in vector form, we obtain:

$$p_0\left(x_0^i - \omega_0^i\right) + \sum_{s=1}^{S} \pi_s a_s z^i = p_0\left(x_0^i - \omega_0^i\right) + \sum_{s=1}^{S} \pi_s p_s\left(x_s^i - \omega_s^i\right) = 0 \quad (3.6)$$

$$\begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \in \; < A > \quad (3.7)$$

But if $rank(A) = S$, it follows that $< A >= R^S$, and the constraint $\begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \in < A >$ is never binding. Each agent i's problem is then subject only to

$$p_0\left(x_0^i - \omega_0^i\right) + \sum_{s=1}^{S} \pi_s p_s\left(x_s^i - \omega_s^i\right) = 0,$$

the budget constraint in the Arrow-Debreu economy with

$$\phi_s = \pi_s p_s, \text{ for any } s = 1, ..., S.$$

$\Longleftarrow$ The converse is straightforward. By No-arbitrage

$$q = \sum_{s=1}^{S} prob_s IMRS_s^i(x^i) a_s.$$

and using $\pi_s = prob_s IMRS_s^i(x^i)$, for any $s = 1, ..., S$, proves the result. (Recall that, with Complete markets $IMRS^i(x^i) = IMRS$, for any $i = 1, ..., I$.) ■

3.2.3 Existence

We do not discuss here in detail the issue of existence of a financial market equilibrium when markets are incomplete (when they are complete, existence follows from the equivalence with Arrow-Debreu equilibrium provided by Arrow theorem). A sketch of the proof however follows.

The proof is a modification of the existence proof for Arrow-Debreu equilibrium. By Arrow theorem, fact, we can reduce the equilibrium system to an excess demand system for consumption goods; that is, we can solve out for the asset portfolios z^i's. An equilibrium will now be a zero of the excess demand function $z^{FM} : \mathbb{R}^S_{++} \times \mathbb{R}^{(L-1)(S+1)}_{++} \rightarrow \mathbb{R}^{L(S+1)-1}_{++}$

$$\sum_{i \in I} x^i(\pi, p) - \omega^i = z^{FM}(\pi, p) = 0.$$

Note that equations and unknowns match: in Financial Market economies (after Arrow Theorem is applied to them), the prices (π, p) are $S + L(S+1)$, but the normalizations (the budget constraints) are $S+1$ and hence we get to $L(S+1) - 1$ unknowns for the same number of equations. Note also that the count applies to Arrow-Debreu economies, where the prices ϕ are $L(S+1)$ and they become $L(S+1) - 1$ after the normalization (1 single budget constraint). We can then apply to $z^{FM}(\pi, p) = 0$ the techniques used to prove existence for Arrow-Debreu economies. The only conceptual problem with the proof is then that the boundary condition on the excess demand system might not be guaranteed as each agent's excess demand is restricted by $\begin{bmatrix} \cdot \\ \cdot \\ p_s\left(x^i_s - \omega^i_s\right) \\ \cdot \\ \cdot \end{bmatrix} \in < A >$. This is where the Cass trick comes in handy. It is in fact an important Lemma.

Cass trick. *For any Financial market economy, consider a modified economy where the constraint* $\begin{bmatrix} \cdot \\ \cdot \\ p_s\left(x^i_s - \omega^i_s\right) \\ \cdot \\ \cdot \end{bmatrix} \in < A >$ *is imposed on all agents* $i = 2, ..., I$ *but not on agent* $i = 1$. *Any equilibrium of the Financial Market economy is an equilibrium of the modified economy, and any equilibrium of the modified economy is a Financial market equilibrium.*

Proof. Consider an equilibrium of the modified economy in the statement. At equilibrium, $\sum_{i=1}^{I} p_s\left(x_s^i - \omega_s^i\right) = 0$. Therefore, $\sum_{i=2}^{I} p_s\left(x_s^i - \omega_s^i\right) = -p_s\left(x_s^1 - \omega_s^1\right)$. But $\begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \in < A >$, for any $i = 2, ..., I$, and hence $\sum_{i=2}^{I} p_s\left(x_s^i - \omega_s^i\right) \in < A >$. Since $\sum_{i=2}^{I} p_s\left(x_s^i - \omega_s^i\right) = -p_s\left(x_s^1 - \omega_s^1\right)$, it follows that $-p_s\left(x_s^1 - \omega_s^1\right) \in < A >$, and hence that $p_s\left(x_s^1 - \omega_s^1\right) \in < A >$. Therefore, the constraint $p_s\left(x_s^1 - \omega_s^1\right) \in < A >$ must necessarily hold at an equilibrium of the modified economy. In other words, the constraint $p_s\left(x_s^1 - \omega_s^1\right) \in < A >$ is not binding at a Financial market equilibrium. The equivalence between the modified economy and the Financial Market economy is now straightforward. ■

In the modified economy, now, agent 1 faces complete markets without loss of generality. His excess demand, therefore, will satisfy the boundary conditions; these properties will transfer than to the aggregate excess demand and the existence proof will proceed exactly as in the standard Arrow-Debreu economy.

3.2.4 Constrained Pareto optimality

Under Complete markets, the First Welfare Theorem holds for Financial market equilibrium. This is a direct implication of Arrow theorem.

Proposition 6 *Let* $(x, z, p, q) \in \mathbb{R}_{++}^{nI} \times \mathbb{R}^j \times \mathbb{R}_{++}^n \times \mathbb{R}_+^j$ *be a Financial market equilibrium of an economy with Complete markets (with* $rank(A) = S$*). Then* $x \in \mathbb{R}_{++}^{nI}$ *is a Pareto optimal allocation.*

However, under Incomplete markets (with $rank(A) < S$), Financial market equilibria are generically inefficient in a Pareto sense. That is, a planner could find an allocation that improves some agents without making any other agent worse off. Note that of course a Pareto optimal allocation is a Financial Market equilibrium (with no trade), independently of the asset matrix A in the economy. As a consequence, it is immediate that, even with Incomplete markets, equilibria are a most generically (not always) Pareto inefficient.

Theorem 15 *Let* $(x, z, p, q) \in \mathbb{R}_{++}^{nI} \times \mathbb{R}^j \times \mathbb{R}_{++}^n \times \mathbb{R}_+^j$ *be a Financial market* *equilibrium of an economy with Incomplete markets (with* $rank(A) < S$*). Then* $x \in \mathbb{R}^{nI}_{++}$ *is generically not a Pareto optimal allocation.*

Proof. From the proof of Arrow theorem, we can write the budget constraints of the Financial market equilibrium as:

$$p_0\left(x_0^i - \omega_0^i\right) + \sum_{s=1}^{S} \pi_s p_s^* \left(x_s^i - \omega_s^i\right) = 0 \tag{3.8}$$

$$\begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \in \; < A > \tag{3.9}$$

for some $\pi \in R^S_{++}$. Pareto optimality of $x$requires that there does not exist an allocation y such that

1. $u(y^i) \geq u(x^i)$ for any $i = 1, ..., I$ (strictly for at least one i), and
2. $\sum_{i=1}^{I} y^i - \omega_s^i = 0$, for any $s = 0, 1, ..., S$

Reproducing the proof of the First Welfare theorem, it is clear that, if such a y exists, it must be that $\begin{bmatrix} . \\ . \\ p_s\left(y_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \notin < A >$, for some $i = 1, ..., I$; otherwise the allocation y would be budget feasible for all agent i at the equilibrium prices. Generic Pareto sub-optimality of x follows then directly from the following Lemma.

Lemma 2 *Let* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ *be a Financial market equilibrium of an economy with* $rank(A) < S$*. For a generic set of economies, the constraints* $\begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} \in < A >$ *are binding for some* $i = 1, ..., I$*.*

■

Proof. We shall only sketch the proof here. Consider Financial market equilibria as the zeroes of the excess demand system for this economy, as defined earlier in this section (but making explicit the dependence on endowments $\omega \in \mathbb{R}^{nI}_{++}$): $z^{FM}(\pi, p, \omega) = 0$. Take any two distinct agents i and j and note that Pareto optimality requires that $IMRS^i(x^i, \omega) = IMRS^j(x^j, \omega)$, where once again we make explicit the dependence on endowments $\omega \in \mathbb{R}^{nI}_{++}$. Consider now the system

$$h(\pi, p, \omega) = \left[\begin{array}{c} z^{FM}(\pi, p, \omega) \\ IMRS^i(x^i, \omega) - IMRS^j(x^j, \omega) \end{array} \right] = 0.$$

Because of the normalizations, the system maps $\mathbb{R}^{n-1}_{++} \times \mathbb{R}^{nI}_{++}$ into $\mathbb{R}^{n}_{++}$ (recall that $n = L(S+1)$). Suppose we could show that, at any $(\pi, p, \omega) \in \mathbb{R}^{n-1}_{++} \times \mathbb{R}^{nI}_{++}$ such that $h(\pi, p, \omega) = 0$, $D_\omega h(\pi, p, \omega)$ has rank n. Then, the Transversality Theorem would immediately imply that $h(\pi, p, \omega) = 0$ has generically no solutions in $\omega \in \mathbb{R}^{nI}_{++}$. The proof that $D_\omega h(\pi, p, \omega)$ has rank n at equilibrium can be found in Magill-Shafer, ch. 30 in W. Hildenbrand and H. Sonnenschein (eds.), *Handbook of Mathematical Economics*, Vol. IV, Elsevier, 1991. ■

Pareto optimality might however represent too strict a definition of social welfare of an economy with frictions which restrict the consumption set, as in the case of incomplete markets. In this case, markets are assumed incomplete exogenously. There is no reason in the fundamentals of the model why they should be, but they are. Under Pareto optimality, however, the social welfare notion does not face the same contraints. For this reason, we typically define a weaker notion of social welfare, *Constrained Pareto optimality*, by restricting the set of feasible allocations to satisfy the same set of constraints on the consumption set imposed on agents at equilibrium. In the case of incomplete markets, for instance, the feasible wealth vectors across states are restricted to lie in the span of the payoff matrix. That can be interpreted as the economy's "financial technology" and it seems reasonable to impose the same technological restrictions on the planner's reallocations. The formalization of an efficiency notion capturing this idea follows.

Let $x^i_{t=1} = (x^i_s)^S_{s=1} \in \mathbb{R}^{SL}_{++}$; and similarly $\omega^i_{t=1} = (\omega^i_s)^S_{s=1} \in \mathbb{R}^{SL}_{++}$, $p_{t=1} = (p_s)^S_{s=1} \in \mathbb{R}^{SL}_{++}$. Let $g_{t=1}(\omega_{t=1}, \theta)$, mapping $\mathbb{R}^{SL}_{++} \times \mathbb{R}^J$ into $\mathbb{R}^{SL}_{++}$, denote the equilibrium map for $t = 1$ spot markets at when each agent $i = 1, ..., I$ has endowment $(\omega^i_{s1} + a_s\theta^i, \omega^i_{s2}, ..., \omega^i_{sL})$, for any $s \in S$.

Definition 15 *(Diamond, 1968; Geanakoplos-Polemarchakis, 1986) Let* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^{j} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{j}_{+}$ *be an Arrow-Debreu equilibrium of an economy whose consumption set at time* $t = 1$ *is restricted by*

$$x^i_{t=1} \in B(p_{t=1}) \subset \mathbb{R}^{SL}_{++}, \text{ for some set } B(p_{t=1}) \text{ and any } i = 1, ..., I.$$

In this economy, the allocation x *is Constrained Pareto optimal if there does not exist a* $(y, \theta) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^{j}$ *such that*

$$\begin{aligned} 1. \quad u(y^i) &\geq u(x^i) \text{ for any } i = 1, ..., I, \text{ strictly for at least one } i \\ 2. \quad \sum_{i=1}^{I} y^i_s - \omega^i_s &= 0, \text{ for any } s = 0, 1, ..., S \end{aligned}$$

and

$$3. \quad y^i_{t=1} \in B(g_{t=1}(\omega_{t=1}, \theta)), \text{ for any } i = 1, ..., I.$$

The constraint on the consumption set restricts only time 1 consumption allocations. More general constraints are possible but these formulation is consistent with the typical frictions we encounter in economics, e.g., on financial markets. It is important that the constraint on the consumption set depends in general on $g_{t=1}(\omega_{t=1}, \theta)$, that is on equilibrium prices for spot markets opened at $t = 1$ after income transfers to agents. It implicit identifies income transfers (besides consumption allocations at time $t = 0$) as the instrument available for Constrained Pareto optimality; that is, it implicitly constrains the planner implementing Constraint Pareto optimal allocations to interact with markets, specifically to open spot markets after transfers. On the other hand, the planner is able to anticipate the spot price equilibrium map, $g_{t=1}(\omega_{t=1}, \theta)$; that is, to internalize the effects of different transfers on spot prices at equilibrium. Consider first a degenerate case:

Proposition 7 *Let* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^{j} \times \mathbb{R}^{n}_{++} \times \mathbb{R}^{j}_{+}$ *be a Financial market equilibrium of an Arrow-Debreu economy whose consumption set at time* $t = 1$ *is restricted by*

$$x^i_{t=1} \in B \subseteq R^{SL}_{++}, \text{ for any } i = 1, ..., I$$

In this economy, the allocation x *is Constrained Pareto optimal.*

Crucially, markets are complete and B is independent of prices. The proof is then a straightforward extension of the First Welfare theorem combined with Arrow theorem.[6] Of course, using Arrow's theorem, this result implies the Constraint Pareto optimality of Financial market equilibrium allocations of economies with Complete markets as long as the constraint set B is exogenous.

But note that we can apply the definition of Constraint Pareto optimality also to Financial market equilibria with Incomplete markets. By Arrow's theorem Financial market economies with Incomplete markets are indeed Arrow-Debreu economies whose consumption set at time $t = 1$ is restricted by

$$x^i_{t=1} \in B(p_{t=1}) \subset \mathbb{R}^{SL}_{++}, \text{ for any } i = 1, ..., I;$$

where

$$B(p_{t=1}) = \left\{ x^i_{t=1} \in \mathbb{R}^{SL}_{++} \left| g_{t=1}(\omega_{t=1}, \theta) \left(x^i_{t=1} - \omega^i_{t=1} \right) \in < A > \right. \right\}.$$

Consider a weaker parametrization of the economy: rather than simply fixing utility functions $\{u^i\}_{i \in I}$ and having economies parametrized by endowments $\omega \in \mathbb{R}^{nI}_{++}$, we also parametrize utility functions by $\delta \in \mathbb{R}^{2I}$, letting $u^i(x) = v^i(x) + \delta^i_1 x + \delta^i_2 x^2$ for some well-behaved $v^i(x)$.

Proposition 8 *Let $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ be a Financial market equilibrium of an economy with Incomplete markets (with $rank(A) < S$). In this economy, the allocation x is, generically in $(\omega, \delta) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^{2I}$, not Constrained Pareto optimal.*[7]

Proof. Note first of all that, by construction, $p_s \in g_s(\omega_s, z)$. Following the proof of Pareto sub-otimality of Financial market equilibrium allocations, it then follows that if a Pareto-improving y exists, it must be that

[6] To be careful, we need to guarantee that monotonicity of preferences on $\mathbb{R}^{SL}_{++}$ results in monotonicity on $B \subseteq \mathbb{R}^{SL}_{++}$. This is the case e.g., if is a subspace in $\mathbb{R}^{SL}_{++}$ or in any case if it is an open set

[7] The genericity result is then weaker in this theorem than we are used to in the previous sections. We'll get back to this later, but we anticipate here that the parametrization of the utility functions is necessary to produce perturbatiopns away from homethtic utility functions, which have the property that spot prices are independent of the distribution of income across states.

$$\begin{bmatrix} \cdot \\ \cdot \\ p_s\left(y^i_s - \omega^i_s\right) \\ \cdot \\ \cdot \end{bmatrix} \notin < A >, \text{for some } i = 1, ..., I; \text{while} \begin{bmatrix} \cdot \\ \cdot \\ g_s(\omega_s, \theta)\left(y^i_s - \omega^i_s\right) \\ \cdot \\ \cdot \end{bmatrix} =$$

$A\theta^i$, for all $i = 1, ..., I$. Generic Constrained Pareto sub-optimality of x follows then directly from the following Lemma, which we leave without proof.[8]

Lemma 3 *Let* $(x, z, p, q) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^j \times \mathbb{R}^n_{++} \times \mathbb{R}^j_+$ *be a Financial market equilibrium of an economy with Incomplete markets (with* $rank(A) < S$*). For a generic set of economies* $(\omega, \delta) \in \mathbb{R}^{nI}_{++} \times \mathbb{R}^{2I}$*, the constraints*

$$\begin{bmatrix} \cdot \\ \cdot \\ g_s(\omega_s, z + dz)\left(y^i_s - \omega^i_s\right) \\ \cdot \\ \cdot \end{bmatrix} = A\left(z^i + dz^i\right), \text{ for some } dz \in R^{JI} \setminus \{0\} \text{ such}$$

that $\sum_{i \in I} dz^i = 0$*, are weakly relaxed for all* $i = 1, ..., I$*, strictly for at least one.*[9]

■

There is a fundamental difference between incomplete market economies, which have typically not Constrained Optimal equilibrium allocations, and economies with constraints on the consumption set, which have, on the contrary, Constrained Optimal equilibrium allocations. It stands out by comparing the respective trading constraints

$$g_s(\omega_s, \theta)(x^i_s - \omega^i_s) = A_s\theta^i, \text{ for all } i \text{ and } s, \qquad \textbf{vs.} \qquad x^i_{t=1} \in B, \text{ for all } i.$$

The trading constraint of the Incomplete market economy is determined at equilibrium, while the constraint on the consumption set is exogenous. Another way to re-phrase the same point is the following. A planner choosing (y, θ) will take into account that at each (y, θ) is typically associated a different trading constraint $g_s(\omega_s, \theta)(x^i_s - \omega^i_s) = A_s\theta^i$, for all i and s; while any agent i will choose (x^i, z^i) to satisfy $p_s(x^i_s - \omega^i_s) = A_s z^i$, for all s, taking as given the equilibrium prices p_s.

[8]The proof is due to Geanakoplos-Polemarchakis (1986). It also requires differential topology techniques.

[9]The Lemma implies that a Pareto improving allocation can be found locally around the equilibrium, as a *perturbation of the equilibrium.*

The constrained inefficiency due the dependence of constraints on equilibrium prices is sometimes called a *pecuniary externality.*[10] Several examples of such form of externality/inefficiency have been developed recently in macroeconomics. Some examples are:

- Thomas, Charles (1995): "The role of fiscal policy in an incomplete markets framework," *Review of Economic Studies*, 62, 449–468.

- Krishnamurthy, Arvind (2003): "Collateral Constraints and the Amplification Mechanism,"

Journal of Economic Theory, 111(2), 277-292.

- Caballero, Ricardo J. and Arvind Krishnamurthy (2003): "Excessive Dollar Debt: Financial Development and Underinsurance," *Journal of Finance*, 58(2), 867-94.

- Lorenzoni, Guido (2008): "Inefficient Credit Booms," *Review of Economic Studies,* 75 (3), 809-833.

- Kocherlakota, Narayana (2009): "Bursting Bubbles: Consequences and Causes,"

http://www.econ.umn.edu/˜nkocher/km_bubble.pdf.

- Davila, Julio, Jay Hong, Per Krusell, and Victor Rios Rull (2005): "Constrained Efficiency in the Neoclassical Growth Model with Uninsurable Idiosyncractic Shocks," mimeo, University of Pennsylvania.

Remark 6 *Consider an economy whose constraints on the consumption set depend on the equilibrium allocation:*

$$x^i_{t=1} \in B(x_{t=1}, z^*), \text{ for any } i = 1, ..., I$$

This is essentially an externality in the consumption set. It is not hard to extend the analysis of this section to show that this formulation introduces inefficiencies and equilibrium allocations are Constraint Pareto sub-optimal.

[10] The name is due to Joe Stiglitz (or is it Greenwald-Stiglitz?).

Corollary 4 *Let (x, z, p, q) represent a Financial market equilibrium of a 1-good economy ($L = 1$) with Incomplete markets (rank$(A) < S$). In this economy, the allocation x is Constrained Pareto optimal.*

Proof. *The constraint on the consumption set implied by incomplete markets, if $L = 1$, can be written*

$$(x_s^i - \omega_s^i) = A_s z^i.$$

It is independent of prices, of the form $x_{t=1}^i \in B$. ■

Remark 7 *Consider an alternative definition of Constrained Pareto optimality, due to Grossman (1970), in which constraints 3 are substituted by*

$$3'. \quad \begin{bmatrix} . \\ . \\ p_s\left(x_s^i - \omega_s^i\right) \\ . \\ . \end{bmatrix} = Az^i, \text{ for any } i = 1, ..., I$$

where p is the spot market Financial market equilibrium vector of prices. That is, the planner takes the equilibrium prices as given. It is immediate to prove that, with this definition of Constrained Pareto optimality, any Financial market equilibrium allocation x of an economy with Incomplete markets is in fact Constrained Pareto optimal, independently of the financial markets available (rank$(A) \leq S$).

Problem 10 *Consider a Complete market economy (rank$(A) = S$) whose feasible set of asset portfolios is restricted by:*

$$z^i \in Z \subsetneq R^J, \text{ for any } i = 1, ..., I$$

A typical example is borrowing limits:

$$z^i \geq -b, \text{ for any } i = 1, ..., I$$

Are equilibrium allocations of such an economy Constrained Pareto optimal (also if $L > 1$)?

Problem 11 *Consider a 1-good ($L = 1$) Incomplete market economy (rank$(A) < S$) which lasts 3 periods. Define an Financial market equilibrium for this economy as well as Constrained Pareto optimality. Are Financial market equilibrium allocations of such an economy Constrained Pareto optimal?*

Problem 12 *Extend our basic two-period complete market economy by assuming that, given an exogenous vector $\gamma \in R^J_{++}$:*

$$\text{the buying price of asset } j \text{ is } q_j + \gamma_j$$

while

$$\text{the selling price of asset } j \text{ is } q_j$$

for any $j = 1, ..., J$. 1) Suppose the asset payoff matrix is full rank. Do you expect financial market equilibria to be Pareto efficient? Carefully justify your answer. (I am not asking for a formal proof, though you could actually prove this.) How would you define Constrained Pareto efficiency in this economy? Do you expect financial market equilibria to be Constrained Pareto efficient? Carefully justify your answer. (I am really not asking for a formal proof.)

Problem 13 *Consider an economy with I agents, S states, a single commodity ($L = 1$), and a full set of Arrow securities: $A = I_S$ (the S-dimensional identity matrix). Agents can default in each state $s \in S$, after the state is realized. If they default they consume only a fraction α, $0 < \alpha < 1$, of their endowment. The remaining fraction, $1 - \alpha$, is first pooled across all defaulting agents and then re-distributed pro-rata to their creditors. i) Write down a constraint, Arrow security by Arrow Security (that is, state by state) on agents' portfolios which guarantees that they will not default. ii) Define a Constrained Pareto optimum for this economy; Will agents ever default at a Constrained Pareto optimum? iii) Is there anything fundamental about Arrow securities which drives your answer in ii)? iv) Is there anything fundamental about complete markets which drives your answer in ii)? Construct an economy with a single asset, a bond (an S-dimensional unit vector) to help answer this last question.*

Aggregation

Agent i's optimization problem in the definition of Financial market equilibrium requires two types of simultaneous decisions. On the one hand, the agent has to deal with the usual consumption decisions i.e., she has to decide how many units of each good to consume in each state. But she also has to make financial decisions aimed at transferring wealth from one state to the other. In general, both individual decisions are interrelated: the consumption and portfolio allocations of all agents i and the equilibrium prices for goods and assets are all determined simultaneously. The financial and the real sectors of the economy cannot be isolated. Under some special conditions, however, the consumption and portfolio decisions of agents can be separated. This is typically very useful when the analysis is centered on financial issue. In order to concentrate on asset pricing issues, most finance models deal in fact with 1-good economies, implicitly assuming that the individual financial decisions and the market clearing conditions in the assets markets determine the *financial equilibrium*, independently of the individual consumption decisions and market clearing in the goods markets; that is independently of the *real equilibrium* prices and allocations. In this section we shall identify the conditions under which this can be done without loss of generality. This is sometimes called "the problem of *aggregation*."

The idea is the following. If we want equilibrium prices on the spot markets to be independent of equilibrium on the financial markets, then the aggregate spot market demand for the L goods in each state s should must depend only on the incomes of the agents in this state (and not in other states) and should be independent of the distribution of income among agents in this state.

Theorem 16 ***Budget Separation.*** *Suppose that each agent i's preferences are separable across states, identical, homothetic within states, and von Neumann-Morgenstern; i.e. suppose that there exists an homothetic* $u : R^L \to R$ *such that*

$$u^i(x^i) = u(x^i_0) + \sum_{s=1}^{S} prob_s u(x^i_s), \text{ for all } i = 1, .., I.$$

Then equilibrium spot prices p^* *are independent of asset prices* q *and of the income distribution; that is, constant in* $\left\{ \omega^i \in R^{L(S+1)}_{++} \middle| \sum_{i=1}^{I} \omega^i \text{ given} \right\}$.

Proof. Normalize all spot prices of good 1: $p_{10} = p_{1s} = 1$, for any $s \in S$. The consumer's maximization problem in the definition of Financial market equilibrium can be decomposed into a sequence of spot commodity allocation problems and an income allocation problem as follows. *The spot commodity allocation problems.* Given the current and anticipated spot prices $p = (p_0, p_1, ..., p_S)$ and an exogenously given *stream of financial income* $y^i = (y^i_0, y^i_1, ..., y^i_S) \in R^{S+1}_{++}$ in units of numeraire, agent i has to pick a consumption vector $x^i \in R_+^{L(S+1)}$ to

$$\begin{array}{l} \max u^i(x^i) \\ \text{s.t.} \\ p_0 x_0^i = y_0^i \\ p_s x_s^i = y_s^i, \ for\ s = 1, ...S. \end{array}$$

Let the $L(S+1)$ demand functions be given by $x_{ls}^i(p, y^i)$, for $l = 1, ..., L$, $s = 0, 1, ...S$. Define now the indirect utility function for income by

$$v^i(y^i; p) = u^i(x^i(p, y^i)).$$

The Income allocation problem. Given prices (p, q), endowments ω^i, and the asset structure A, agent i has to pick a portfolio $z^i \in R^J$ and an income stream $y^i \in R_{++}^{S+1}$ to

$$\begin{array}{l} \max v^i(y^i; p) \\ s.t. \\ p_0 \omega_0^i - q z^i = y_0^i \\ p_s \omega_s^i + a_s z^i = y_s^i, \ for\ s = 1, ...S. \end{array}$$

By additive separability across states of the utility, we can break the consumption allocation problem into $S+1$ 'spot market' problems, each of which yields the demands $x_s^i(p_s, y_s^i)$ for each state. By homotheticity, for each $s = 0, 1, ...S$, and by identical preferences across all agents,

$$x_s^i(p_s, y_s^i) = y_s^i x_s^i(p_s, 1);$$

and since preferences are identical across agents,

$$y_s^i x_s^i(p_s, 1) = y_s^i x_s(p_s, 1)$$

Adding over all agents and using the market clearing condition in spot markets s, we obtain, at spot markets equilibrium,

$$x_s(p_s, 1) \sum_{i=1}^{I} y_s^i - \sum_{i=1}^{I} \omega_s^i = 0.$$

Again by homothetic utility,

$$x_s(p_s, \sum_{i=1}^{I} y_s^i) - \sum_{i=1}^{I} \omega_s^i = 0. \tag{3.10}$$

Recall from the consumption allocation problem that $p_s x_s^i = y_s^i$, for $s = 0, 1, ...S$. By adding over all agents, and using market clearing in the spot markets in state s,

$$\begin{aligned} \sum_{i=1}^{I} y_s^i &= p_s \sum_{i=1}^{I} x_s^i, \text{ for } s = 0, 1, ...S \\ &= p_s \sum_{i=1}^{I} \omega_s^i, \text{ for } s = 0, 1, ...S. \end{aligned} \tag{3.11}$$

By combining (3.10) and (3.11), we obtain

$$x_s(p_s, p_s \sum_{i=1}^{I} \omega_s^i) = \sum_{i=1}^{I} \omega_s^i. \tag{3.12}$$

Note how we have passed from the aggregate demand of all agents in the economy to the demand of an agent owning the aggregate endowments. Observe also how equation (3.12) is a system of L equations with L unknowns that determines spot prices p_s for each state s independently of asset prices q. Note also that equilibrium spot prices p_s defined by (3.12) only depend ω^i through $\sum_{i=1}^{I} \omega_s^i$. ■

The Budget separation theorem can be interpreted as identifying conditions under which studying a single good economy is without loss of generality. To this end, consider the income allocation problem of agent i, given equilibrium spot prices p^*:

$$\begin{aligned} &\max_{y^i \in \mathbb{R}_{++}^{S+1}, z^i \in \mathbb{R}^J} v^i(y^i; p) \\ \text{s.t. } y_0^i &= p_0 \omega_0^i - q z^i \\ y_s^i &= p_s \omega_s^i + a_s z^i, \text{ for } s = 1, ...S \end{aligned}$$

If preferences $u^i(x^i)$ are identical, homothetic within states, and von Neumann-Morgenstern, that is, if they satisfy

$$u^i(x^i) = u(x_0^i) + \sum_{s=1}^{S} prob_s u(x_s^i), \text{ with } u(x) \text{ homothetic, for all } i = 1, .., I$$

it is straightforward to show that indirect preferences $v^i(y^i; p^*)$ are also identical, and von Neumann-Morgenstern:

$$v^i(y^i; p) = v(y_0^i; p) + \sum_{s=1}^{S} prob_s v(y_s^i; p).$$

Note that homotheticity in $(y_0^i, y_1^i, ..., y_S^i)$ is guaranteed by the von Neumann-Morgenstern property. Let $w_0^i = p_0\omega_0^i$, $w_s^i = p_s\omega_s^i$, for any $s = 1, ..., S$; and disregard for notational simplicity the dependence of $v(y;p)$ on p. The income allocation problem can be written as:

$$\max_{y^i \in \mathbb{R}^{S+1}_{++}, z^i \in \mathbb{R}^J} v(y_0^i) + \sum_{s=1}^{S} prob_s v(y_s^i)$$

$$\begin{aligned} \text{s.t. } y_0^i - w_0^i &= -qz^i \\ y_s^i - w_s^i &= A_s z^i, \text{ for } s = 1, ...S \end{aligned}$$

which is homeomorphic to any agent i's optimization problem in the definition of Financial market equilibrium with $l = 1$. Note that y_s^i gains the interpretation of agent i's consumption expenditure in state s, while w_s^i is interpreted as agent i's income endowment in state s.

The representative agent theorem

A *representative agent* is the following theoretical construct.

Definition 16 *Consider a Financial market equilibrium (x, z, p, q) of an economy populated by $i = 1, ..., I$ agents with preferences $u^i : X \to R$ and endowments ω^i. A Representative agent for this economy is an agent with preferences $U^R : X \to R$ and endowment ω^R such that the Financial market equilibrium of an associated economy with the Representative agent as the only agent has prices (p, q).*

In this section we shall identify assumptions which guarantee that the Representative agent construct can be invoked without loss of generality. This assumptions are behind much of the empirical macro/finance literature.

Theorem 17 ***Representative agent.*** *Suppose preferences satisfy:*

$$u^i(x^i) = u(x_0^i) + \sum_{s=1}^{S} prob_s u(x_s^i), \text{ with homothetic } u(x), \text{ for all } i = 1, .., I.$$

Let p denote equilibrium spot prices. If $\begin{bmatrix} . \\ . \\ p_s\omega_s^i \\ . \\ . \end{bmatrix} \in < A >$, *then there exist* *a map* $u^R : R_+^{S+1} \to R$ *such that:*

$$
\begin{aligned}
\omega^R &= \sum_{i=1}^{I} \omega_s^i, \\
U^R(x) &= u^R(y_0) + \sum_{s=1}^{S} prob_s u^R(y_s), \text{ where } y_s = p \sum_{i=1}^{I} x_s^i, \; s = 0, 1, ..., S
\end{aligned}
$$

constitutes a Representative agent.

Since the Representative agent is the only agent in the economy, her consumption allocation and portfolio at equilibrium, $\left(x^R, z^R\right)$, are:

$$
\begin{aligned}
x^R &= \omega^R = \sum_{i=1}^{I} \omega^i \\
z^R &= 0
\end{aligned}
$$

If the Representative agent's preferences can be constructed independently of the equilibrium of the original economy with I agents, then equilibrium prices can be read out of the Representative agent's marginal rates of substitution evaluated at $\sum_{i=1}^{I} \omega^i$. Since $\sum_{i=1}^{I} \omega^i$ is exogenously given, equilibrium prices are obtained without computing the consumption allocation and portfolio for all agents at equilibrium, (x^*, z^*).

Proof. The proof is constructive. Under the assumptions on preferences in the statement, we need to show that, for all agents $i = 1, ..., I$, equilibrium asset prices q are constant in $\left\{\omega^i \in R_{++}^{L(S+1)} \,\middle|\, \sum_{i=1}^{I} \omega^i \text{ given}\right\}$.If preferences satisfy $u^i(x^i) = u(x_0^i) + \sum_{s=1}^{S} prob_s u(x_s^i)$, for all $i = 1, .., I$, with an homothetic $u(x)$, then by the Budget separation theorem, equilibrium spot prices p are independent of q and constant in $\left\{\omega^i \in R_{++}^{L(S+1)} \,\middle|\, \sum_{i=1}^{I} \omega^i \text{ given}\right\}$. Therefore, $\begin{bmatrix} . \\ . \\ p_s \omega_s^i \\ . \\ . \end{bmatrix} \in < A >$ can be written as an assumption on fundamentals, in particular on ω^i. Furthermore, we can restrict our analysis to the single good economy, whose agent i's optimization problem is:

$$\begin{aligned} & \max_{y^i \in \mathbb{R}^{S+1}_{++}, z^i \in \mathbb{R}^J} v(y^i_0) + \sum_{s=1}^{S} prob_s v(y^i_s) \\ \text{s.t. } y^i_0 - w_0 &= -qz^i \\ y^i_s - w_s &= A_s z^i, \text{ for } s = 1, ...S \end{aligned}$$

where $v(y)$ is homothetic.

We show next that $u^R(y) = v(y)$ and $\omega^R = \sum_{i=1}^{I} \omega^i_s$ constitute a Representative agent. By Arrow theorem,, we can write budget constraints as

$$\begin{aligned} y^i_0 - w^i_0 + \sum_{s=1}^{S} \pi_s \left(y^i_s - w^i_s\right) &= 0 \\ \begin{bmatrix} . \\ . \\ y^i_s - w^i_s \\ . \\ . \end{bmatrix} &\in \ < A > \end{aligned}$$

But, $\begin{bmatrix} . \\ . \\ w^i_s \\ . \\ . \end{bmatrix} \in < A >$ implies that there exist a z^i_w such that $\begin{bmatrix} . \\ . \\ w^i_s \\ . \\ . \end{bmatrix} = Az^i_w$.

Therefore, $\begin{bmatrix} . \\ . \\ w^i_s \\ . \\ . \end{bmatrix} \in < A >$ implies that $y^i_s = A_s \left(z^i + z^i_w\right)$, for any $s \in S$. We can then write each agent i's optimization problem in terms of (y^i_0, z^i), and the value of agent $i's$ endowment is $w^i_0 + \sum_{s=1}^{S} \pi_s w^i_s = w^i_0 + \sum_{s=1}^{S} \pi_s a_s z^i_w = w^i_0 + qz^i_w$.

The consumer i's problem becomes:

$$\begin{aligned}
& \max_{y^i \in \mathbb{R}_+^{S+1}} v(y_0^i) + \sum_{s=1}^{S} prob_s v(y_s^i), \\
& s.t. \\
y_0^i + \sum_{s=1}^{S} \pi_s y_s^i &= W^i \\
\begin{bmatrix} \cdot \\ \cdot \\ y_s^i \\ \cdot \\ \cdot \end{bmatrix} &\in \ < A >
\end{aligned}$$

where $W^i = w_0^i + q z_w^i$. Note that the solution is homogeneous of degree 1:

$$y^i(q, \alpha W^i) = \alpha y^i(q, W^i).$$

Hence

By the fact that preferences are identical across agents and by homotheticity of $v(y)$, then we can write

$$\begin{aligned}
y_0^i \left(q, W^i\right) &= \left(w_0^i + q z_w^i\right) y_0\left(q, 1\right) \\
y_s^i \left(q, W^i\right) &= \left(w_0^i + q z_w^i\right) y_s\left(q, 1\right), \text{ for any } s \in S
\end{aligned}$$

At equilibrium then

$$\begin{aligned}
y_0\left(q,1\right) \sum_{i \in I} \left(w_0^i + q z_w^i\right) &= y_0\left(q, \sum_{i \in I}\left(w_0^i + q z_w^i\right)\right) = \sum_{i \in I} w_0^i \\
y_s\left(q,1\right) \sum_{i \in I} \left(w_0^i + q z_w^i\right) &= y_s\left(q, \sum_{i \in I}\left(w_0^i + q z_w^i\right)\right) = A_s \sum_{i \in I} z_w^i, \text{ for any } s \in S
\end{aligned}$$

and prices q^* only depend on $\sum_{i=1}^{I} w_0^i$ and $\sum_{i=1}^{I} z_w^i$. ■

Make sure you understand where we used the assumption $\begin{bmatrix} . \\ . \\ p\omega_s^i \\ . \\ . \end{bmatrix} = \begin{bmatrix} . \\ . \\ w_s^i \\ . \\ . \end{bmatrix} \in < A >$. Convince yourself that the assumption is necessary in the proof.

The Representative agent theorem, as noted, allows us to obtain equilibrium prices without computing the consumption allocation and portfolio for all agents at equilibrium, (x, z).Let $w = \sum_{i=1}^{I} w^i$. Under the assumptions of the Representative agent theorem, let $w_0 = \sum_{i \in I} w_0^i$, and $w_s = \sum_{i \in I} w_s^i$, for any $s \in S$. Then

$$q = \sum_{s=1}^{S} prob_s MRS_s(w) A_s, \text{ for } MRS_s(w) = \frac{\frac{\partial v(w_s)}{\partial w_s}}{\frac{\partial v(w_0)}{\partial w_0}}$$

That is, asset prices can be computed from agents' preferences $u^R = v : R \to R$ and from the aggregate endowment $(w_0, ..., w_s, ...)$. This is called the *Lucas' trick* for pricing assets.

Problem 14 *Note that, under the Complete markets assumption, the span restriction on endowments,* $\begin{bmatrix} . \\ . \\ p\omega_s^i \\ . \\ . \end{bmatrix} \in < A >$*, for all agents* i*, is trivially satisfied. Does this assumption imply Pareto optimal allocations in equilibrium?*

Problem 15 *Assume all agents have identical quadratic preferences. Derive individual demands for assets (without assuming* $\begin{bmatrix} . \\ . \\ p\omega_s^i \\ . \\ . \end{bmatrix} \in < A >$*) and show that the Representative agent theorem is obtained.*

Another interesting but misleading result is the "weak" representative agent theorem, due to Constantinides (1982).

Theorem 18 *Suppose markets are complete ($rank(A) = S$) and preferences $u^i(x^i)$ are von Neumann-Morgernstern (but not necessarily identical nor homothetic). Let (x, z, p, q) be a Financial markets equilibrium. Then,*

$$\begin{aligned} \omega^R &= \sum_{i=1}^{I} \omega^i, \\ U^R(x) &= \max_{(x^i)_{i=1}^I} \sum_{i=1}^{I} \theta^i u^i(x^i) \quad s.t. \quad \sum_{i=1}^{I} x^i = x, \\ \text{where } \theta^i &= (\lambda_i)^{-1} \text{ and } \lambda_i = \frac{\partial u^i(x^i)}{\partial x_{10}^i} \end{aligned}$$

constitutes a Representative agent.

Clearly, then,

$$q = \sum_{s=1}^{S} prob_s MRS_s(w) A_s, \text{ for } MRS_s(w) = \frac{\frac{\partial U^R(w_s)}{\partial w_s}}{\frac{\partial U^R(w_0)}{\partial w_0}}.$$

Proof. Consider a Financial market equilibrium (x, z, p, q). By complete markets, the First welfare theorem holds and x^* is a Pareto optimal allocation. Therefore, there exist some weights that make x the solution to the planner's problem. It turns out that the required weights are given by

$$\theta^i = \left(\frac{\partial u^i(x^i)}{\partial x_{10}^i} \right)^{-1}.$$

This is left to the reader to check; it's part of the celebrated Negishi theorem. ■

This result is certainly very general, as it does not impose identical homothetic preferences, however, it is not as useful as the "real" Representative agent theorem to find equilibrium asset prices. The reason is that to define the specific weights for the planner's objective function, $(\theta^i)_{i=1}^I$, we need to know what the *equilibrium* allocation, x, which in turn depends on the whole distribution of endowments over the agents in the economy.

3.2.5 Asset pricing

Relying on the aggregation theorem in the previous section, in this section we will abstract from the consumption allocation problems and concentrate on one-good economies. This allows us to simplify the equilibrium definition as follows.

3.2.6 Some classic representation of asset pricing

Often in finance, especially in empirical finance, we study asset pricing representation which express asset returns in terms of *risk factors*. Factors are to be interpreted as those component of the risks that agents do require a higher return to hold.

How do we go from our basic asset pricing equation

$$q = E(mA)$$

to factors?

Single factor beta representation

Consider the basic asset pricing equation for asset j,

$$q_j = E(ma_j)$$

Let the return on asset j, R_j, be defined as $R_j = \frac{A_j}{q_j}$. Then the asset pricing equation becomes

$$1 = E(mR_j)$$

This equation applied to the risk free rate, R^f, becomes $R^f = \frac{1}{Em}$. Using the fact that for two random variables x and y, $E(xy) = ExEy + cov(x, y)$, we can rewrite the asset pricing equation as:

$$ER_j = \frac{1}{Em} - \frac{cov(m, R_j)}{Em} = R^f - \frac{cov(m, R_j)}{Em}$$

or, expressed in terms of excess return:

$$ER_j - R^f = -\frac{cov(m, R_j)}{Em}$$

Finally, letting

$$\beta_j = -\frac{cov(m, R_j)}{var(m)}$$

and

$$\lambda_\pi = \frac{var(m)}{Em}$$

we have the beta representation of asset prices:

$$ER^j = R^f + \beta_j \lambda_m \tag{3.13}$$

We interpret β_j as the "quantity" of risk in asset j and λ_m (which is the same for all assets j) as the "price" of risk. Then the expected return of an asset j is equal to the risk free rate plus the correction for risk, $\beta_j \lambda_m$. Furthermore, we can read (3.13) as a single factor representation for asset prices, where the factor is m, that is, if the representative agent theorem holds, her intertemporal marginal rate of substitution.

Multi-factor beta representations

A multi-factor beta representation for asset returns has the following form:

$$ER^j = R^f + \sum_{f=1}^{F} \beta_{jf} \lambda_{m_f} \tag{3.14}$$

where $(m_f)_{f=1}^F$ are orthogonal random variables which take the interpretation of *risk factors* and

$$\beta_{jf} = -\frac{cov(m_f, R_j)}{var(m_f)}$$

is the beta of factor f, the loading of the return on the factor f.

Proposition 9 *A single factor beta representation*

$$ER_j = R^f + \beta_j \lambda_m$$

is equivalent to a multi-factor beta representation

$$ER_j = R^f + \sum_{f=1}^{F} \beta_{jf} \lambda_{m_f} \quad \textit{with } m = \sum_{f=1}^{F} b_f m_f$$

In other words, a multi-factor beta representation for asset returns is consistent with our basic asset pricing equation when associated to a linear statistical model for the stochastic discount factor m, in the form of $m = \sum_{f=1}^{F} b_f m_f$.

Proof. Write $1 = E(mR_j)$ as $R_j = R^f - \frac{cov(m,R_j)}{Em}$ and then to substitute $m = \sum_{f=1}^{F}\ b_f m_f$ and the definitions of β_{jf}, to have

$$\lambda_{m_f} = \frac{var(m_f) b_f}{Em}$$

■

The CAPM

The CAPM is nothing else than a single factor beta representation of the following form:

$$ER^j = R^f + \beta_{jf}\lambda_{m_f}$$

where

$$m_f = a + bR^w$$

the return on the market portfolio, the aggregate portfolio held by the investors in the economy.

It can be easily derived from an equilibrium model under special assumptions.

For example, assume preferences are quadratic:

$$u(x_o^i, x_1^i) = -\frac{1}{2}(x^i - x^{\#})^2 - \frac{1}{2}\beta \sum_{s=1}^{S} prob_s (x_s^i - x^{\#})^2$$

Moreover, assume agents have no endowments at time $t = 1$. Let $\sum_{i=1}^{I} x_s^i = x_s$, $s = 0, 1, ..., S$; and $\sum_{i=1}^{I} w_0^i = w_0$. Then budget constraints include

$$x_s = R_s^w (w_0 - x_0)$$

Then,

$$m_s = \beta \frac{x_s - x^{\#}}{x_0 - x^{\#}} = \frac{\beta(w_0 - x_0)}{(x_0 - x^{\#})} R_s^w - \frac{\beta x^{\#}}{x_0 - x^{\#}}$$

which is the CAPM for $a = -\frac{\beta x^{\#}}{x_0 - x^{\#}}$ and $b = \frac{\beta(w_0 - x_0)}{(x_0 - x^{\#})}$.

Note however that $a = \frac{\beta x^{\#}}{x_0 - x^{\#}}$ and $b = \frac{\beta(w_0 - x_0)}{(x_0 - x^{\#})}$ are not constant, as they do depend on equilibrium allocations. This will be important when we study conditional asset market representations, as it implies that the CAPM is intrinsically a conditional model of asset prices.

Bounds on stochastic discount factors

Write the beta representation of asset returns as:

$$ER_j - R^f = \frac{cov(m, R_j)}{Em} = \frac{\rho(m, R_j)\sigma(m)\sigma(R_j)}{Em}$$

where $0 \leq |\rho(m, R_j)| \leq 1$ denotes the correlation coefficient and $\sigma(m)$, the standard deviation. Then

$$| \frac{ER_j - R^f}{\sigma(R_j)} | \leq \frac{\sigma(m)}{Em}$$

The left-hand-side is the *Sharpe-ratio* of asset j.

The relationship implies a lower bound on the standard deviation of any stochastic discount factor m which prices asset j. Hansen-Jagannathan are responsible for having derived bounds like these and shown that, when the stochastic discount factor is assumed to be the intertemporal marginal rate of substitution of the representative agent (with CES preferences), the data does not display enough variation in m to satisfy the relationship.

A related bound is derived by noticing that no-arbitrage implies the existence of a *unique* stochastic discount factor in the space of asset payoffs, denoted m_p, with the property that any other stochastic discount factor m satisfies:

$$m = m_p + \epsilon$$

where ϵ is orthogonal to m_p.

The following corollary of the No-arbitrage theorem leads us to this result.

Corollary 5 *Let (A, q) satisfy No-arbitrage. Then, there exists a unique $\tau^* \in < A >$ such that $q = A\tau^*$.*

Proof. By the No-arbitrage theorem, there exists $\pi \in R^S_{++}$ such that $q = \pi A$. We need to distinguish notationally a matrix M from its transpose, M^T. We write then the asset prices equation as $q^T = A^T \pi^T$. Consider π_p:

$$\pi_p^T = A(A^T A)^{-1} q.$$

Clearly, $q^T = A^T \pi_p^T$, that is, π_p^T satisfies the asset pricing equation. Furthermore, such π_p^T belongs to $< A >$, since $\pi_p^T = Az_p$ for $z_p = (A^T A)^{-1} q$. Prove uniqueness. ■

We can now exploit this uniqueness result to yield a characterization of the "multiplicity" of stochastic discount factors when markets are incomplete, and consequently a bound on $\sigma(m)$. In particular, we show that, for a given (q, A) pair a vector m is a stochastic discount factor if and only if it can be decomposed as a projection on $< A >$ and a vector-specific component orthogonal to $< A >$. Moreover, the previous corollary states that such a projection is unique.

Let $m \in R^S_{++}$ be any stochastic discount factor, that is, for any $s = 1, \ldots, S$, $m_s = \frac{\pi_s}{prob_s}$ and $q_j = E(mA_j)$, for $j = 1, ..., J$. Consider the orthogonal projection of m onto $< A >$, and denote it by m_p. We can then write any stochastic discount factors m as $m = m_p + \varepsilon$, where ε is orthogonal to any vector in $< A >$, in particular to any A_j. Observe in fact that $m_p + \varepsilon$ is also a stochastic discount factors since $q_j = E((m_p + \varepsilon)a_j) = E(m_p a_j) + E(\varepsilon a_j) = E(m_p a_j)$, by definition of ε. Now, observe that $q_j = E(m_p a_j)$ and that we just proved the uniqueness of the stochastic discount factors lying in $< A >$. In words, even though there is a multiplicity of stochastic discount factors, they all share the same projection on $< A >$. Moreover, if we make the economic interpretation that the components of the stochastic discount factors vector are marginal rates of substitution of agents in the economy, we can interpret m_p to be the economy's aggregate risk and each agents ε to be the individual's unhedgeable risk.

It is clear then that

$$\sigma(m) \geq \sigma(m_p)$$

the bound on $\sigma(m)$ we set out to find.

3.2.7 Production

Assume for simplicity that $L = 1$, and that there is a single type of firm in the economy which produces the good at date 1 using as only input the amount k of the commodity invested in capital at time 0.[11] The output depends on

[11]It should be clear from the analysis which follows that our results hold unaltered if the firms' technology were described, more generally, by a production possibility set $Y \subset \mathbb{R}^{S+1}$.

k according to the function $f(k;s)$, defined for $k \in K$, where s is the state realized at $t = 1$. We assume that

- $f(k;s)$ is continuously differentiable, increasing and concave in k,

- Φ, K are closed, compact subsets of $\mathbb{R}_+$ and $0 \in K$.

In addition to firms, there are I types of consumers. The demand side of the economy is as in the previous section, except that each agent $i \in I$ is also endowed with θ_0^i units of stock of the representative firm. Consumer i has von Neumann-Morgernstern preferences over consumption in the two dates, represented by $u^i(x_0^i) + \mathbb{E}u^i(x^i)$, where $u^i(\cdot)$ is continuously differentiable, strictly increasing and strictly concave.

Competitive equilibrium

Let the outstanding amount of equity be normalized to 1: the initial distribution of equity among consumers satisfies $\sum_i \theta_0^i = 1$. The problem of the firm consists in the choice of its production plan k..

Firms are perfectly competitive and hence take prices as given. The firm's cash flow, $f(k;s)$, varies with k. Thus equity is a different "product" for different choices of the firm. What should be its price when all this continuum of different "products" are not actually traded in the market? In this case the price is only a "conjecture." It can be described by a map $Q(k)$ specifying the market valuation of the firm's cash flow for any possible value of its choice k.[12] The firm chooses its production plan k so as to maximize its value. The firm's problem is then:

$$\max_k -k + Q(k) \tag{3.15}$$

When financial markets are complete, the present discounted valuation of any future payoff is uniquely determined by the price of the existing assets. This is no longer true when markets are incomplete, in which case the prices of the existing assets do not allow to determine unambiguously the value of any future cash flow. The specification of the price conjecture is thus more problematic in such case. Let k^* denote the solution to this problem.

At $t = 0$, each consumer i chooses his portfolio of financial assets and of equity, z^i and θ^i respectively, so as to maximize his utility, taking as given

[12] These price maps are also called *price perceptions*.

the price of assets, q and the price of equity Q. In the present environment a consumer's long position in equity identifies a firm's equity holder, who may have a voice in the firm's decisions. It should then be treated as conceptually different from a short position in equity, which is not simply a negative holding of equity. To begin with, we rule out altogether the possibility of short sales and assume that agents can not short-sell the firm equity:

$$\theta^i \geq 0, \ \forall i \tag{3.16}$$

The problem of agent i is then:

$$\max_{x_0^i, x^i, z^i, \theta^i} u^i\left(x_0^i\right) + \mathbb{E}u^i\left(x^i\right) \tag{3.17}$$

subject to (3.16) and

$$x_0^i = \omega_0^i + \left[-k + Q\right]\theta_0^i - Q\theta^i - q\, z^i \tag{3.18}$$

$$x^i(s) = \omega^i(s) + f(k;s)\theta^i + A(s)z^i, \ \forall s \in \mathcal{S} \tag{3.19}$$

Let $(x_0^i, x^i, z^i, \theta^i)$ denote the solution to this problem.

In equilibrium, the following market clearing conditions must hold, for the consumption good:[13]

$$\begin{aligned} \sum_i x_0^i + k &\leq \sum_i \omega_0^i \\ \sum_i x^i(s) &\leq \sum_i \omega^i(s) + f(k;s), \ \forall s \in \mathcal{S} \end{aligned}$$

or, equivalently, for the assets:

$$\sum_i z^i = 0 \tag{3.20}$$

$$\sum_i \theta^i = 1 \tag{3.21}$$

In addition, the equity price map faced by firms must satisfy the following consistency condition:

[13]We state here the conditions for the case of symmetric equilibria, where all firms take the same production and financing decision, so that only one type of equity is available for trade to consumers. They can however be easily extended to the case of asymmetric equilibria.

i) $Q(k^*) = Q$;

This condition requires that, at equilibrium, the price of equity conjectured by firms coincides with the price of equity, faced by consumers in the market: firms' conjectures are "correct" in equilibrium.

We also restrict out of equilibrium conjectures by firms, requiring that they satisfy:

ii) $Q(k) = \max_i \mathbb{E}\left[MRS^i f(k)\right]$, $\forall k$, where MRS^i denotes the marginal rate of substitution between consumption at date 0 and at date 1 in state s for consumer i, evaluated at his equilibrium consumption allocation (x_0^i, x^i).

Condition ii) says that for any k (not just at equilibrium!) the value of the equity price map $Q(k)$ equals the highest marginal valuation - across all consumers in the economy - of the cash flow associated to k. The consumers' marginal rates of substitutions $\overline{MRS}^i(s)$ used to determine the market valuation of the future cash flow of a firm are taken as given, unaffected by the firm's choice of k. This is the sense in which, in our economy, firms are competitive: each firm is "small" relative to the mass of consumers and each consumers holds a negligible amount of shares of the firm.

To better understand the meaning of condition ii), note that the consumers with the highest marginal valuation for the firm's cash flow when the firm chooses k are those willing to pay the most for the firm's equity in that case and the only ones willing to buy equity - at the margin - when its price satisfies ii). Given i) such property is clearly satisfied for the firms' equilibrium choice k. Condition ii) requires that the same is true for any other possible choice k: the value attributed to equity equals the maximum any consumer is willing to pay for it. Note that this would be the equilibrium price of equity of a firm who were to "deviate" from the equilibrium choice and choose k instead: the supply of equity with cash flow corresponding to k is negligible and, at such price, so is its demand.

In this sense, we can say that condition ii) imposes a consistency condition on the out of equilibrium values of the equity price map; that is, it corresponds to a "refinement" of the equilibrium map, somewhat analogous to bacward induction. Equivalently, when price conjectures satisfy this condition, the model is equivalent to one where markets for all the possible types of equity (that is, equity of firms with all possible values of k) are open, available for trade to consumers and, in equilibrium all such markets - except the one corresponding to the equilibrium k - clear at zero trade.[14]

It readily follows from the consumers' first order conditions that in equilibrium the price of equity and of the financial assets satisfy:

$$\begin{aligned} Q &= \max_i \mathbb{E}\left[MRS^i \cdot f(k)\right] \\ q &= \mathbb{E}\left[MRS^i \cdot A\right] \end{aligned} \tag{3.22}$$

The definition of competitive equilibrium is stated for simplicity for the case of symmetric equilibria, where all firms choose the same production plan. When the equity price map satisfies the consistency conditions i) and ii) the firms' choice problem is not convex. Asymmetric equilibria might therefore exist, in which different firms choose different production plans. The proof of existence of equilibria indeed requires that we allow for such asymmetric equilibria, so as to exploit the presence of a continuum of firms of the same type to convexify firms' choice problem. A standard argument allows then to show that firms' aggregate supply is convex valued and hence that the existence of (possibly asymmetric) competitive equilibria holds.

Proposition 10 *A competitive equilibrium always exist.*

Objective function of the firm

Starting with the initial contributions of Diamond (1967), Dreze (1974), Grossman-Hart (1979), and Duffie-Shafer (1986), a large literature has dealt with the question of what is the appropriate objective function of the firm when markets are incomplete.The issue arises because, as mentioned above, firms' production decisions may affect the set of insurance possibilities available to consumers by trading in the asset markets.

If agents are allowed infinite short sales of the equity of firms, as in the standard incomplete market model, a *small* firm will possibly have a *large* effect on the economy by choosing a production plan with cash flows which, when traded as equity, change the asset span. It is clear that the price

[14] An analogous specification of the price conjecture has been earlier considered by Makowski (1980) and Makowski-Ostroy (1987) in a competitive equilibrium model with differentiated products, and by Allen-Gale (1991) and Pesendorfer (1995) in models of financial innovation.

taking assumption appears hard to justify in this context, since changes in the firm's production plan have non-negligible effects on allocations and hence equilibrium prices. The incomplete market literature has struggled with this issue, trying to maintain a competitive equilibrium notion in an economic environment in which firms are potentially *large*.

In the environment considered in these notes, this problem is avoided by assuming that consumers face a constraint preventing short sales, (3.16), which guarantees that each firm's production plan has instead a negligible (infinitesimal) effect on the set of admissible trades and allocations available to consumers. Evidently, for price taking behavior to be justified a no short sale constraint is more restrictive than necessary and a bound on short sales of equity would suffice; see Bisin-Gottardi-Ruta (2009).

When short sales are not allowed, the decisions of a firm have a negligible effect on equilibrium allocations and market prices. However, each firm's decision has a non-negligible impact on its present and future cash flows. Price taking can not therefore mean that the price of its equity is taken as given by a firm, independently of its decisions. However, as argued in the previous section, the level of the equity price associated to out-of-equilibrium values of k is not observed in the market. It is rather *conjectured* by the firm. In a competitive environment we require such conjecture to be consistent, as required by condition ii) in the previous section. This notion of *consistency* of conjectures implicitly requires that they be *competitive*, that is, determined by a given pricing kernel, independent of the firm's decisions.[15] But which pricing kernel? Here lies the core of the problem with the definition of the objective function of the firm when markets are incomplete. When markets are incomplete, in fact, the marginal valuation of *out-of-equilibrium* production plans differs across different agents at equilibrium. In other words, equity holders are not unanimous with respect to their preferred production plan for the firm. The problem with the definition of the objective function of the firm when markets are incomplete is therefore the problem of aggregating equity holders' marginal valuations for *out-of-equilibrium* production plans. The different equilibrium notions we find in the literature differ primarily in the specification of a consistency condition on $Q(k)$, the price map which the firms adopts to aggregate across agents' marginal valuations.[16]

[15]Independence of the kernel is guarantee by the fact that $MRS^{i*}(s)$, for any i, is evaluated at equilibrium.

[16]A minimal consistency condition on $Q(k)$ is clearly given by i) in the previous section, which only requires the conjecture to be correct in correspondence to the firm's equilibrium

Consider for example the consistency condition proposed by Dreze (1974):

$$Q^D(k) = \mathbb{E}\left[\sum_i \theta^i MRS^i f(k)\right], \ \forall k \tag{3.23}$$

Such condition requires the price conjecture for any plan k to equal the pro rata marginal valuation of the agents who at equilibrium are the firm's equity holders (that is, the agents who value the most the plan chosen by firms in equilibrium). It does not however require that the firm's equity holders are those who value the most any possible plan of the firm, without contemplating the possibility of selling the firm in the market, to allow the new equity buyers to operate the production plan they prefer. Equivalently, the value of equity for out of equilibrium production plans is determined using the - possibly incorrect - conjecture that the firms' equilibrium shareholders will still own the firm out of equilibrium.

Grossman-Hart (1979) propose another consistency condition and hence a different equilibrium notion. In their case

$$Q^{GH}(k) = \mathbb{E}\left[\sum_i \theta_0^i MRS^i f(k)\right], \ \forall k$$

We can interpret such notion as describing a situation where the firm's plan is chosen by the initial equity holders (i.e., those with some predetermined stock holdings at time 0) so as to maximize their welfare, again without contemplating the possibility of selling the equity to other consumers who value it more. Equivalently, the value of equity for out of equilibrium production plans is again derived using the conjecture belief that firms' initial shareholders stay in control of the firm out of equilibrium.

Unanimity

Under the definition of equilibrium proposed in these notes, equity holders unanimously support the firm's choice of the production and financial decisions which maximize its value (or profits), as in (3.15). This follows from the fact that, when the equity price map satisfies the consistency conditions i) and ii), the model is equivalent to one where a continuum of types of equity

choice. Duffie-Shafer (1986) indeed only impose such condition and find a rather large indeterminacy of the set of competitive equilibria.

is available for trade to consumers, corresponding to any possible choice of k the representative firm can make, at the price $Q(k)$. Thus, for any possible value of k a market is open where equity with a payoff $f(k;s)$ can be traded, and in equilibrium such market clears with a zero level of trades for the values of k not chosen by the firms.

For any possible choice k of a firm, the (marginal) valuation of the firm by an agent i is

$$\mathbb{E}\ \left[MRS^i \cdot f(k)\right],$$

and it is always weakly to the market value of the firm, given by

$$\max_i \mathbb{E}\ \left[MRS^i \cdot f(k)\right].$$

Proposition 11 *At a competitive equilibrium, equity holders unanimously support the production k; that is, every agent i holding a positive initial amount θ_0^i of equity of the representative firm will be made - weakly - worse off by any other choice k' of the firm.*

Efficiency

A consumption allocation $(x_0^i, x^i)_{i=1}^I$ is *admissible* if:[17]

1. it is *feasible*: there exists a production plan k such that

$$\sum_i x_0^i \ + k \leq \sum_i \omega_0^i \tag{3.24}$$

$$\sum_i x^i(s) \leq \sum_i \omega^i(s) + f(k;s),\ \forall s \in \mathcal{S} \tag{3.25}$$

2. it is *attainable with the existing asset structure*: for each consumer i, there exists a pair (z^i, θ^i) such that:

$$x^i(s) = \omega^i(s) + f(k;s)\ \theta^i \ + A(s)z^i,\ \forall s \in \mathcal{S} \tag{3.26}$$

Next we present the notion of *efficiency* restricted by the *admissibility* constraints:

[17] To keep the notation simple, we state both the definition of competitive equilibria and admissible allocations for the case of symmetric allocations. The analysis, including the efficiency result ,extends however to the case where asymmetric allocations are allowed are admissible; see also the next section.

Constrained efficiency. *A competitive equilibrium allocation is constrained Pareto efficient if we can not find another admissible allocation which is Pareto improving.*

The validity of the First Welfare Theorem with respect to such notion can then be established by an argument essentially analogous to the one used to establish the Pareto efficiency of competitive equilibria in Arrow-Debreu economies.

First welfare theorem. *Competitive equilibria are constrained Pareto efficient.*

Modigliani-Miller

We examine now the case where firms take both production and financial decisions, and equity and debt are the only assets they can finance their production with. The choice of a firm's capital structure is given by the decision concerning the amount B of bonds issued. The problem of the firm consists in the choice of its production plan k and its financial structure B. To begin with, we assume without loss of generality that all firms' debt is risk free. The firm's cash flow in this context is then $[f(k;s) - B]$ and varies with the firm's production and financing choices, k, B. Equity price conjectures have the form $Q(k, B)$, while the price of the (risk free) bond is independent of (k, B); we denote it p. The firm's problem is then:

$$\max_{k,B} -k + Q(k, B) + p\ B \tag{3.27}$$

The consumption side of the economy is the same as in the previous section, except that now agents can also trade the bond. Let b^i denote the bond portfolio of agent i, and let continue to impose no-short sales contraints:

$$\begin{aligned} \theta^i &\geq 0 \\ b^i &\geq 0, \ \forall i. \end{aligned}$$

Proceeding as in the previous section, at equilibrium we shall require that

$$\begin{aligned} Q(k) &= \max_i \mathbb{E}\left[MRS^i \cdot [f(k) - B]\right], \forall k, \\ p &= \max_i \mathbb{E}\left[MRS^i\right] \end{aligned}$$

where MRS^i denotes the marginal rate of substitution between consumption at date 0 and at date 1 in state s for consumer i, evaluated at his equilibrium consumption allocation (x_0^i, x^i). Suppose now that financial markets are complete, that is $rank(A) = S$. At equilibrium then $MRS^i = MRS^*$, $\forall i$. Therefore, in this case

$$Q(k, B) + p\ B = \mathbb{E}\left[MRS^* \cdot f(k)\right], \forall k,$$

and the value of the firm, $Q(k, B) + p\ B$, is independent of B. This proves the celebrated

Modigliani-Miller theorem. *If financial markets are complete the financing decision of the firm, B, is indeterminate.*

It should be clear that when financial markets are not complete and agents are restricted by no-short sales constraints, the Modigliani-Miller theorem does not quite necessarily hold.